CHAPTER 5

FUZZY REVEALED PREFERENCE*

5.0 Introduction

Prof. P. A. Samuelson introduced the revealed preference theory in 1938\textsuperscript{[67]}. His hypothesis is: 'choice reveals preference', that is, if one is observed to choose \( x \), rejecting \( y \), he is declared to have 'revealed' his preference for \( x \) over \( y \)\textsuperscript{[68]} . The Weak Axiom of Revealed Preference (WARP), given by him and the Strong Axiom of Revealed Preference (SARP), given by Houthakker\textsuperscript{[34]} have been studied by many authors like Arrow\textsuperscript{[1]}, Gale\textsuperscript{[27]}, Peters\textsuperscript{[58]}, Sen\textsuperscript{[72]}, etc., as rationality conditions for the crisp case. But when the choice functions and the revealed preference relations are permitted to be fuzzy, these axioms do not guarantee rationality\textsuperscript{[11]}.

This chapter studies the conditions under which a fuzzy choice function is rational in the sense that it can be induced by a fuzzy revealed preference relation satisfying certain regularity conditions.

* Some of the ideas presented in this chapter have been used in a paper communicated to 'Fuzzy Sets and Systems', North Holland.
5.1 Preliminaries

5.1.1 Definition

Let $X$ be the universal set of alternatives. Let $H$ be the set of all nonempty, finite subsets of $X$. For any $S \in H$, the choice set $C(S)$ represents the chosen elements of $S$. For any pair $x, y \in X$, we define $R$, $P^-$ and $P^*$ as:

i. $x R y \iff x \in C(S)$, for some $S \in H$ and $y \in S$.

ii. $x P^- y \iff x \in C(S)$, and $y \in S - C(S)$, for some $S \in H$.

iii. $x P^* y \iff x_{i-1} P^- x_i \; \forall \; i = 1, 2, ..., n$;

for any sequence $(x_i), i = 0, 1, ..., n$ in $X$, such that $x_0 = x$, and $x_n = y$.

5.1.2 Definition

The Weak Axiom of Revealed Preference (WARP) says that,

'if $x P^- y$ then not $y R x$'

and the Strong Axiom of Revealed Preference (SARP) says that,

'if $x P^* y$ then not $y R x$'.

5.1.3 Definition

Let $X$ be the universal set of alternatives. Let $H$ be the set of all nonempty finite crisp subset of $X$ and $F$ be the set of all nonempty fuzzy subsets of $X$ with finite supports. A fuzzy choice function is a function, $f_c : H \rightarrow F$, such that for every $S \in H$, $\text{Supp} f_c(S) \subseteq S$, where $\text{Supp} f_c(S)$ is the support of the fuzzy set $f_c(S)$.

5.1.4 Definition

A fuzzy choice function $f_c$ is said to reveal a fuzzy preference relation $R$ if and only if,

$$R(x, y) = \max_{\{S : x, y \in S\}} f_c(S)(x),$$

where

$$f_c(S)(.) : f_c(S) \rightarrow [0, 1]$$

i.e., $f_c(S)(x)$ denotes the membership value of $x$ in $f_c(S)$.

Following example shows how a fuzzy preference relation $R$ is revealed from a choice function.

5.1.5 Example

Consider $X = \{x, y, z\}$. Let a choice function $f_c$ be defined as:

$$f_c\{x\}(x) = f_c\{y\}(y) = f_c\{z\}(z) = 1,$$

$$f_c\{x, y\}(x) = .5, \quad f_c\{x, y\}(y) = .8$$
\[ f_c(\{x, z\})(x) = .4, \quad f_c(\{x, z\})(z) = .6, \]
\[ f_c(\{y, z\})(y) = .5, \quad f_c(\{y, z\})(z) = .7, \]
\[ f_c(\{x, y, z\})(x) = .9, \quad f_c(\{x, y, z\})(y) = .7, \quad \text{and} \]
\[ f_c(\{x, y, z\})(z) = .3 \]

Then a fuzzy preference relation \( R \) is revealed from \( f_c \) as:
\[
R(x, y) = \max\{f_c(\{x, y\})(x), f_c(\{x, y, z\})(x)\}
\]
\[
= \max\{.5, .9\} = .9
\]

Clearly, \( R(x, x) = R(y, y) = R(z, z) = 1 \).

\( R \), for other pairs can also be found similarly.

5.1.6 Definition

The preference relation \( P^- \) is defined as:
\[
P^-(x, y) = \max_{\{x, y \in S | \{0, f_c(S)(x) - f_c(S)(y)\}}
\]
\( P^- \) is the strict (revealed) preference relation and is illustrated in the following example.

5.1.7 Example

Consider Example-5.1.5. The strict (revealed) preference relation \( P^- \) can be found as follows:
5.1.8 Definition

Let \( R \) be the fuzzy preference relation revealed by the choice function \( f_c \).

The image of \( f_c \) is the function,

\[
f_c^\wedge : H \to F, \quad \text{such that} \quad f_c^\wedge(S)(x) = \min_{y \in S} R(x, y),
\]

\( \forall S \in H \) and \( x \in S \).

The image of \( f_c \) is illustrated in the following example.

5.1.9 Example

Consider Example-5.1.5.

\[
f_c^\wedge(x, y)(x) = \min\{ R(x, x), R(x, y) \}
\]

\[
= \min\{ 1, .9 \} = .9
\]

\[
f_c^\wedge(x, y, z)(y) = \min\{ R(y, x), R(y, y), R(y, z) \}
\]

\[
= \min\{ .8, 1, .7 \} = .7
\]

\[
P^\wedge(x, y) = \max\{ 0, f_c^\wedge(x, y)(x) - f_c^\wedge(x, y)(y),
\]

\[
f_c^\wedge(x, y, z)(x) - f_c^\wedge(x, y, z)(y) \}
\]

\[
= \max\{ 0, .5 - .8, .9 - .7 \}
\]

\[
= \max\{ 0, -.3, .2 \} = .2
\]
5.1.10 Definition

The choice function \( f_c \) is called normal if and only if

\[
f_c(S)(x) = f_c^*(S)(x), \quad \forall \ S \in H \text{ and } x \in S.
\]

5.1.11 Remark

The choice function given in Example-5.1.5 is not normal.

5.1.12 Definition

An element \( x \) of \( S \) is called dominant in \( S \) if and only if

\[
f_c(S)(x) \geq f_c(S)(y), \quad \forall \ S \in H \text{ and } y \in S.
\]

5.1.13 Definition

Let \( S \in H \) and \( R \) be a fuzzy preference relation on \( X \). Then an element \( x \in S \) is said to be relation dominant in \( S \), with respect to \( R \) if and only if,

\[
R(x, y) \geq R(y, x), \quad \forall \ y \in S.
\]

5.1.14 Definition

A fuzzy choice function \( f_c \) is said to be rational if and only if the revealed preference relation \( R \) satisfies the following conditions:

i. \( f_c \) is normal
ii. $x$ is dominant in $S$ if and only if it is relation dominant in $S$ with respect to $R$, $\forall S \in H$ and $x \in S$.

iii. if $x$ is relation dominant in $S$ with respect to $R$, then

$$R(x, z) \geq R(y, x), \quad \forall x, y, z \in S.$$ 

iv. $R$ is a weak fuzzy ordering.

5.1.15 Definition

For every $x, y, z \in X$, $P^-(x, y) > 0$, if and only if there is a sequence $(x_i)$, $i = 0, 1, \ldots, n$ in $X$, such that $x_0 = x$, $x_n = y$ and $P^-(x_{i-1}, x_i) > 0$, for $i = 1, 2, \ldots, n$.

5.1.16 Note

This definition is an analog of Definition-5.1.1 for the crisp case. The Axioms of revealed preference can be fuzzified in many ways. Two fuzzy versions of WARP and SARP, given in [4] are as follow:

5.1.17 Definition

The first version of Weak Axiom of Fuzzy Revealed Preference (WAFRP-1) states that,
\[ P^-(x, y) > 0 \Rightarrow R(y, x) < 1, \ \forall \ x, y \in X, \]

and the first version of Strong Axiom of Fuzzy Revealed Preference (SAFRP-1) states that,

\[ P^*(x, y) > 0 \Rightarrow R(y, x) < 1, \ \forall \ x, y \in X. \]

5.1.18 Note

SAFRP-1 implies WAFRP-1.

5.1.19 Definition

For every \( x, y \in X \) and for every real number \( k \) such that \( 0 < k \leq 1 \),

\[ P^{**}(x, y) \geq k, \text{ if and only if there is a sequence } (x_i), \ i = 0, 1, \ldots, n, \ \text{in } X, \ \text{such that} \ x_0 = x \ \text{and} \ x_n = y; \ \text{and} \ P^-(x_{i-1}, x_i) \geq k, \ i = 1, 2, \ldots, n. \]

5.1.20 Definition

The second version of Weak Axiom of Fuzzy Revealed Preference (WAFRP-2) states that,

\[ P^{**}(x, y) > k \Rightarrow R(y, x) \leq 1-k ; \ \forall \ x, y \in X, \ \text{and for every real number } k \ \text{such that} \ 0 < k \leq 1. \]

The second version of Strong Axiom of Fuzzy Revealed Preference (SAFRP-2) states that,
\( P^{-*}(x, y) \geq k \Rightarrow R(y, x) \leq 1-k \; ; \; \forall x, y \in X, \) and for every real number \( k \) such that \( 0 < k \leq 1 \).

### 5.1.21 Note

Clearly SAFRP-2 implies WAFRP-2. Also SAFRP-2 implies SAFRP-1 and WAFRP-2 implies WAFRP-1. Since in the crisp case, for all \( x, y \in X \), the only admissible positive value of \( P^-(x, y) \) is 1 and the only admissible value of \( R(y, x) \), which is less than 1 is zero, both WAFRP-1 and WAFRP-2 reduce to WARP.

Similarly, SAFRP-1 and SAFRP-2 reduce to SARP for the crisp case. Hence these two versions are valid fuzzy versions of WARP and SARP.

### 5.2 A Necessary and Sufficient Condition for Rationality of the Choice Function

### 5.2.1 Proposition

SAFRP-1 (and hence WAFRP-1) does not imply rationality of the choice function.

Proof:

Consider Example-5.1.5.

\[ P^- (x, z) = .6 > 0 \] and \[ P^- (z, y) = .2 > 0. \]
Hence $P^*(x, y) > 0$.

Also, $R(y, x) = .8 < 1$.

We can see that for every pair $x, y \in X$,

$$P^*(x, y) > 0 \Rightarrow R(y, x) < 1.$$ 

Hence SAFRP-1 (and hence WAFRP-1) is satisfied. But $f_c$ given in the example is not normal (c.f. Remark-5.11). Since normality is the primary condition for rationality the result follows.

### 5.2.2 Proposition

SAFRP-2 (and hence WAFRP-2) does not imply rationality of the choice function.

**Proof:**

This can be proved using the following counter example.

Let $X = \{x, y, z\}$. Consider a fuzzy choice function on $X$ as:

$$f_c\{x\}(x) = f_c\{y\}(y) = f_c\{z\}(z) = 1,$$

$$f_c\{x, y\}(x) = f_c\{x, z\}(z) = f_c\{x, z\}(x) = 1,$$

$$f_c\{y, z\}(y) = f_c\{x, y\}(y) = f_c\{y, z\}(z) = .9$$

$$f_c\{x, y, z\}(x) = f_c\{x, y, z\}(y) = f_c\{x, y, z\}(z) = .9.$$
Then,
\[
R(x, y) = R(x, z) = R(z, x) = 1,
\]
\[
R(y, x) = R(y, z) = R(z, y) = .9,
\]
\[
P^-(x, y) = .1 \quad \text{and}
\]
\[
P^-(x, z) = P^-(z, x) = 0.
\]
Hence, \(f_c\) satisfies SAFRP-2, since there is no ordered triple \((x, y, z)\) and no real number \(k; 0 < k < 1\), such that \(P^-(x, z) < k\), but \(P^-(x, y) \geq k\) and \(P^-(y, z) \geq k\).

But, \(f_c\) is not normal since,
\[
f_c(\{x, y, z\})(x) = 1 \neq f_c(\{x, y, z\})(x).
\]
Hence SAFRP-2 does not imply rationality of \(f_c\).

5.2.3 Note

As these fuzzy versions of WARP and SARP do not imply rationality, they are not adequate for generating a fuzzy choice function back from its own fuzzy revealed preference relation.

In crisp revealed preference theory, a congruence axiom was introduced by Ritcher\textsuperscript{[63]}, named as strong congruence axiom (SCA). A weak version of the axiom
named as the weak congruence axiom (WCA) was given by Sen[^72], which is given below.

5.2.4 Definition

For any pair $x, y \in X$, $x \text{P}^w y$ if and only if there exists a sequence $(x_i)$, $i = 0, 1, \ldots, n$ in $X$, such that $x_0 = x$, $x_n = y$ and $x_{i-1} R x_i$; $i = 1, 2, \ldots, n$.

5.2.5 Definition

i. Strong Congruence Axiom (SCA) states that if $x \text{P}^w y$, then for any $S \in H$, such that $y \in C(S)$, we have $x \in C(S)$, and

ii. Weak Congruence Axiom (WCA) states that if $x \text{R} y$, then for any $S \in H$, such that $y \in C(S)$, we have $x \in C(S)$.

5.2.6 Note

WCA seems necessary and sufficient for an ordinal preference structure and it guarantees normality of the choice function for the crisp case. A fuzzy congruence condition is defined below, which is stronger than SAFRP-2 and it can be shown that this congruence condition is necessary and sufficient for rationality.

5.2.7 Definition

A fuzzy choice function $f_c$ is called congruent, if and only if it satisfies the following three congruency conditions:
i. \( f_c \) satisfies the first condition of fuzzy congruence (FC-1), if and only if for every \( S \in \mathcal{H} \) and \( x, y \in S \),
\[
[y \text{ is dominant in } S] \Rightarrow f_c(S)(x) = R(x, y).
\]

ii. \( f_c \) satisfies the second condition of fuzzy congruence (FC-2), if and only if for every \( S \in \mathcal{H} \) and \( x, y \in S \),
\[
[y \text{ is dominant in } S \text{ and } R(x, y) \geq R(y, x)] \Rightarrow x \text{ is dominant in } S.
\]

iii. \( f_c \) satisfies the third condition of fuzzy congruence (FC-3), if and only if for every \( S \in \mathcal{H} \) and \( x, y \in S \), and for every real number \( k \) such that \( 0 < k \leq 1 \),
\[
[f_c(S)(y) \geq k \text{ and } R(x, y) \geq k] \Rightarrow f_c(S)(x) \geq k.
\]

5.2.8 Lemma

\( FC-1 \Rightarrow \text{normality of } f_c \)

Proof :

Let \( f_c \) satisfies FC-1 and let \( f_c^* \) be the image of \( f_c \).

Then,
\[
f_c(S)(x) \leq f_c^*(S)(x), \forall S \in \mathcal{H} \text{ and } x \in S.
\]

If \( f_c \) is not normal, then, \( f_c(S)(x) < f_c^*(S)(x) \).
This implies, \( f_c(S)(x) < \min_{y \in S} R(x, y) \)

\[
\text{i.e., } \quad f_c(S)(x) < R(x, y), \quad \forall y \in S.
\]

If we take \( y \in S \) to be dominant in \( S \), this is a contradiction to FC-1.

Hence FC-1 implies normality.

5.2.9 Lemma

FC-1 \( \Rightarrow R \) is weakly reflexive.

Proof:

By Lemma-5.2.8,

\[
\begin{align*}
  f_c\{ x \}(x) &= f_c\{ x \}(x). \\
  f_c\{ x \}(x) &= R(x, x)
\end{align*}
\]

This implies

\[
R(x, x) > 0, \quad \text{since } f_c(S) \text{ is nonempty.}
\]

If possible let there exists \( x, y \in X \) such that \( R(x, x) < R(x, y) \).

Consider \( S = \{ x, y \} \).

Since \( f_c \) is normal,

\[
f_c(S)(x) = R(x, y).
\]

Similarly we get,

\[
f_c(S)(y) = R(y, x).
\]
Now let \( R(x, y) \geq R(y, x) \).

This \( \Rightarrow f_\varepsilon(S)(x) \geq f_\varepsilon(S)(y) \)
\[ \Rightarrow \] \( x \) is dominant in \( S \)
\[ \Rightarrow f_\varepsilon(S)(x) = R(x, x) \), \( \text{by FC-1} \)

FC-1 \( \Rightarrow \) if \( x \) is dominant in \( S \) then \( x \) is relation dominant in \( S \) with

respect to \( R \), \( \forall S \in H \) and \( x, y \in S \).

This implies, \( R(x, y) = R(x, x) \).

This is a contradiction.

Now, let \( R(x, y) < R(y, x) \).

This \( \Rightarrow f_\varepsilon(S)(x) < f_\varepsilon(S)(y) \)
\[ \Rightarrow \] \( y \) is dominant in \( S \)
\[ \Rightarrow f_\varepsilon(S)(y) = R(y, y) \), \( \text{by FC-1} \)
\[ \Rightarrow R(y, x) = R(y, y) = R(x, x) \).

\[ \therefore \] \[ f_\varepsilon(S)(x) = f_\varepsilon(S)(y) \)
\[ \Rightarrow \] \( R(y, x) = R(x, x) \)
\[ \Rightarrow \] \( R(y, x) < R(x, y) \), \( \text{by the assumption} \).
This is again a contradiction, showing that

\[ 0 < R(x, x) \geq R(x, y), \quad \forall x, y \in X. \]

Hence \( R \) is weakly reflexive.

5.2.10 Lemma

FC-1 \( \Rightarrow \) if \( x \) is dominant in \( S \) then,

\[ R(x, z) \geq R(y, z), \quad \forall S \in H; \quad x, y, z \in X. \]

Proof:

If possible let there exists \( S \in H \) and \( x, y, z \in S \), such that \( x \) is dominant in \( S \) but \( R(x, z) < R(y, z) \).

We have,

\[ f_c^c(S)(x) \leq R(x, z) \]

\[ \Rightarrow f_c(S)(x) \leq R(x, z), \]

as \( f_c \) is normal

\[ \Rightarrow R(x, x) \leq R(x, z), \]

as \( x \) is dominant in \( S \).

This is a contradiction as \( R \) is weakly reflexive.

Hence \[ R(x, z) \geq R(y, z). \]
5.2.11 Lemma

\[ \text{FC-1 } \Rightarrow \text{ if } x \text{ is dominant in } S \text{ then } x \text{ is relation dominant in } S \text{ with respect to } R, \forall S \in H \text{ and } x \in S. \]

Proof:

Let \( x \) is dominant in \( S \).

Since \( f_c(S)(y) \leq R(y, x) \), \( \forall y \in S \), as \( x \) is dominant in \( S \). If possible let there exists \( y \in S \), such that \( R(x, y) < R(y, x) \).

Then, \( R(y, x) > R(x, y) \).

\[ \Longrightarrow \text{ i.e., } R(y, x) \geq f_c(S)(x) \]

\[ \geq f_c(S)(y) \]

\[ = R(y, x) , \text{ which is a contradiction.} \]

Hence \( x \) is relation dominant in \( S \) with respect to \( R \).

5.2.12 Lemma

\[ \text{FC-2 } \Rightarrow \text{ if } x \text{ is relation dominant in } S \text{ with respect to } R, \text{ then } x \text{ is dominant in } S \text{ with respect to } R, \forall S \in H \text{ and } x \in S. \]
Proof:

Since $f_c$ satisfies FC-2,

if $y$ is dominant in $S$ and $R(x,y) \geq R(y,x)$, then $x$ is dominant in $S$.

Let $x$ is relation dominant in $S$ with respect to $R$.

Then, $R(x,y) \geq R(y,x), \quad \forall y \in S.$

This, $\Rightarrow f_c(S)(x) \geq f_c(S)(y), \quad \forall y \in S.$

$\Rightarrow x$ is dominant in $S$.

5.2.13 Lemma

FC-1 and FC-2 together imply that,

$[ R(x,y) \geq R(y,x) \text{ and } R(y,z) \geq R(z,y) ]$

$\Rightarrow R(x,z) \geq R(z,x), \quad \forall x,y,z \in X.$

Proof:

Let $R(x,y) \geq R(y,x)$ and $R(y,z) \geq R(z,y)$.

Consider $S = \{ x, y, z \}$.

If $y$ is dominant in $S$, then by FC-2,

$R(x,y) \geq R(y,x) \Rightarrow x$ is dominant in $S$.

Also, if $z$ is dominant in $S$, then
\[ R(y, z) \geq R(z, y) \Rightarrow y \text{ is dominant in } S. \]

Hence in all the cases, \( x \) is dominant in \( S \).

Then, by FC-1, \( x \) is relation dominant in \( S \) with respect to \( R \).

\[ \therefore \ R(x, z) \geq R(z, x). \]

5.2.14 Theorem

Fuzzy congruence is necessary and sufficient for rationality of a fuzzy choice function.

Proof:

Let the fuzzy choice function \( f_c \) is a fuzzy congruence. Then by Lemma-5.2.8, \( f_c \) is normal, by Lemma-5.2.11, \( f_c \) satisfies condition-(ii) of Definition-5.1.14 and by Lemma-5.2.12, \( f_c \) satisfies condition-(iii) of Definition-5.1.14. So it remains to prove that \( R \) is a weak fuzzy ordering.

By Lemma-5.2.9, \( R \) is weakly reflexive.

If \( R \) is not weakly complete, then, there exists \( x, y \in X \), such that

\[ R(x, y) + R(y, x) = 0 \]

\[ \Rightarrow R(x, y) = R(y, x) = 0 \]

\[ \Rightarrow f_c\{x, y\} = \phi, \text{ which is not permissible.} \]

Hence \( R \) is weakly complete.
To prove transitivity,

Let \( x, y, z \in X \), be such that

\[
R(x, z) < \min\{R(x, y), R(y, z)\} = k \ (\text{say})
\]

Then,

\[
R(x, z) < R(x, y) \quad \text{and} \quad R(x, z) < R(y, z)
\]

\[
\therefore \quad R(x, z) < k \quad \text{and} \quad R(x, y) \geq k
\]

Let \( S = \{x, y, z\} \)

\[
f_c(S)(x) = \min\{R(x, x), R(x, y), R(x, z)\} = R(x, z)
\]

\[
= f_c(S)(x), \quad \text{by the normality of } f_c.
\]

Let \( y \) be dominant in \( S \). Then by FC-1,

\[
f_c(S)(x) = R(x, y) \geq k.
\]

Since \( R(x, y) \geq k \) and \( f_c(S)(x) = R(x, z) < k \), FC-3 is contradicted. Hence \( x \) must be dominant in \( S \). But then Lemma-5.2.10 is contradicted, since \( R(x, z) < R(y, z) \). This along with Lemma-5.2.13 ensures the transitivity of \( R \). Thus \( R \) is a weak fuzzy ordering, which completes the proof of sufficiency.

For the necessity part, assume that \( f_c \) is rational.

If \( f_c \) does not satisfy FC-1, then there exists \( S \in \mathcal{H} \) and \( x, y \in S \) such that \( y \) is dominant in \( S \) but \( f_c(S)(x) < R(x, y) \).
Let \( f^*_c(S)(x) = \min_{y \in S} R(x, y) \)

\[ = R(x, z), \text{ for some } z \in S \]

Therefore by normality,

\[ f^*_c(S)(x) = f_c(S)(x) \]

\[ = R(x, z) \]

\[ < R(x, y). \]

If \( R(x, y) \leq R(z, z) \), then the first condition of B-transitivity will be violated.

Hence, \( R(x, y) > R(y, z) \).

But this contradicts the condition-(iii) of Definition-5.1.14 (since \( y \) is dominant in \( S \) by condition-(ii) of Definition-5.1.14). Hence \( f_c \) satisfies FC-1.

Now, let \( S \in H \) and \( x, y \in S \) be such that \( y \) is dominant in \( S \) and \( R(x, y) \geq R(y, x) \).

Since \( y \) is dominant in \( S \),

\[ R(y, z) \geq R(z, y), \forall \ z \in S, \text{ By Lemma-5.2.11}. \]

Also by the second condition of B-transitivity,

\[ R(x, z) \geq R(z, x), \forall \ z \in S, \text{ (c.f. Definition-1.3.23)}. \]

i.e., \( x \) is relation dominant in \( S \) with respect to \( R \) and hence \( x \) is dominant in \( S \).

Thus \( f_c \) satisfies FC-2.
Finally, let $S \in H$ and $x, y \in S$ be such that

$$f_c(S)(y) \geq k \quad \text{and} \quad R(x, y) \geq k.$$  

$$f_c(S)(y) \geq k \Rightarrow R(y, z) \geq k , \forall \ z \in S.$$  

The transitivity of $R$ \quad $\Rightarrow$ $\ R(x, z) \geq \ \min \{ R(x, y), R(y, z) \}$  

$\Rightarrow$ $R(x, z) \geq k, \forall \ z \in S$  

$\Rightarrow$ $f_c^*(S)(x) \geq k$  

$\Rightarrow$ $f_c(S)(x) \geq k$, by normality  

$\Rightarrow$ $f_c$ satisfies FC-3, which completes the proof.  

In the next chapter we develop a new measure for the welfare of individuals in a society using fuzzy methods, particularly using fuzzy preference relations.