CHAPTER 2
LITERATURE REVIEW

2.1 General

One of the important steps to develop accurate analysis of structures made of laminated composite materials is to select a proper structural theory for the problem. An overview of the literature on laminate plate theories is included here. The analyses of composite plates have been based on one of the following approaches.

1. Equivalent Single Layer Theories (2D)
   a) Classical Laminated Plate Theory (CLPT)
   b) Shear Deformation Laminate Theories

2. Layer Wise Theories

3. Three Dimensional Elasticity Theories.

2.2 Equivalent Single Layer Theories (ESL Theories)

The ESL theories are those in which a heterogeneous laminate plate is treated as a statically equivalent single layer having a complex constitutive behaviour, reducing the 3D continuum problem to a 2D problem. In these theories the displacement or stress components are expanded as a linear combination of the thickness coordinate and undetermined functions of positions in the reference surface. In the stress based theories the governing equations are derived either using virtual work principles or integrating the 3D stress equilibrium equations through the thickness of the laminate yielding an equivalent single layer plate theory. In the theories based on displacement...
expansions, the principle of virtual displacements is used to derive equations of equilibrium. This particular study is confined to the development of displacement based higher order models for laminated composite plates and functionally graded plates.

2.2.1 Classical Laminated Plate Theory (CLPT)

The simplest ESL laminate theory is the Classical Laminated Plate Theory (CLPT), which is an extension of the Kirchhoff Classical Plate Theory to laminated composite plates. This theory is based on the assumptions that the laminate is thin and the deflections of laminate are small. Also it is assumed that the normal to the laminate mid surface remains straight, inextensible and normal during deformation. The displacement fields used for this formulation is:

\[ u(x, y, z, t) = u_0(x, y, t) - z \frac{\partial w_0}{\partial x} \]  
\[ v(x, y, z, t) = v_0(x, y, t) - z \frac{\partial w_0}{\partial y} \]  
\[ w(x, y, z, t) = w_0(x, y, t) \]

where \( u_0, v_0, w_0 \) are the displacements along \( x, y, z \) coordinates directions respectively on a point on the mid plane \((z=0)\).


CLPT when applied to isotropic or orthotropic plates underpredict the lateral deflections and stresses, and overpredict buckling loads, the error being more in the case of orthotropic plates compared to isotropic plates. Because of this inherent
inadequacy of the CLPT by not considering the transverse shear deformation in the formulation, efforts have been made to develop refined theories which would overcome these limitations.

2.2.2 Shear Deformation Theories

It is well known that the CLPT is inadequate in modeling thick laminates as the transverse shear deformation is not accounted in the formulation. Since laminated composite materials are often very flexible in shear, the transverse shear strains must be taken into account if an accurate representation of the behavior of the laminated plate is to be achieved. The most widely used theory is the first order shear deformation theory which is based on the displacement field:

\[
\begin{align*}
  u(x, y, z, t) &= u_0(x, y, t) + z\phi_x(x, y, t) \\
  v(x, y, z, t) &= v_0(x, y, t) + z\phi_y(x, y, t) \\
  w(x, y, z, t) &= w_0(x, y, t)
\end{align*}
\]  

(2.2a) (2.2b) (2.2c)

where \( u_0, v_0, w_0 \) are the displacements along \( x, y, z \) directions respectively \( \phi_x \) and \( \phi_y \) denote rotations about the \( y \) and \( x \) axis respectively.

Thick plate studies were first initiated in mid 1940s and early 1950s pioneered by Reissner [1944,1945] who proposed the simplest thick plate theory by introducing the effect of transverse shear deformation through a complementary energy principle. Mindlin [1951] presented a first order shear deformation theory in which shear deformation is accounted in conjunction with shear correction factors. In this theory, normality assumption of CLPT is modified in such a way that normal to the undeformed mid-plane remains straight and unstretched in length but not necessarily normal to the deformed mid plane. This assumption implies a non-zero transverse shear strain, but it also leads to the statical violation of zero shear stress of free surfaces since
the shear stress becomes constant through the plate thickness. To compensate for this error, correction factor was proposed. This theory was employed by Whitney and Pagano [1970] to study the vibration and bending of anisotropic plates and by Liew et al [1993] to analyse a thick plate with a maximum of 20 percentage of thickness to width ratio. These theories were extended to laminates by Yang et al [1966], Reissner [1979], Wang and Chow [1977]. Pryor and Barker [1971] presented FE model based on FSDT for cross ply symmetric and unsymmetric laminates. Hinton [1975] and Reddy and Chao [1981] also developed FE models based on FSDT for laminated composite plates. FSDT yields a constant value of transverse shear strain through the thickness of the plate and thus requires shear correction factors. The shear correction factors are dimensionless quantities introduced to account for the discrepancy between the constant state of shear strains in FSDT and the parabolic distribution of shear strains in the elasticity theory. The shear correction factor depends upon various factors such as laminate properties, ply layer, orientation of fibres and boundary conditions. Whitney [1973], Chatterjee and Kulkarni [1979] and Vlachoutsis [1992] presented study on shear correction factors and concluded that shear correction factors are different for isotropic plates and laminates.

To achieve a reliable analysis and safe design the proposal and developments of models using higher order shear deformation theories have been considered. Here the through thickness distribution of the displacement functions are assumed to be higher order polynomials of thickness coordinate. In principle, it is possible to expand the displacement field in terms of thickness coordinate up to any desired degree. However, due to algebraic complexity and computational effort involved with higher order theories in return for marginal gain in accuracy, theories higher than third order have
not been popular. The reason for expanding the displacements up to the cubic term in the thickness coordinate is to have quadratic variation of the transverse shear strain and stress through each layer. This avoids the need for shear correction factors used in FSDT.

Hildebrand et al [1949] pioneered such an approach. Lo et al. [1977a,1977b] reviewed the pioneering work on the field and formulated a theory which accounts for the effects of transverse shear deformation, transverse strain and non-linear distribution of the in-plane displacements with respect to thickness coordinate. Third order theories have been proposed by Reddy [1984a,1984b,1990a,1993], Librescu [1975], Schmidt [1977], Krishnamurty [1977], Levinson [1980], Seide [1980], Murthy [1981], Bhimaraddi and Stevens [1984], Mallikarjuna and Kant [1993], Kant and Pandya [1988], and Phan and Reddy [1985]. Excellent reviews of refined theories of laminated composite plates have been presented by Reddy [1990b], Noor and Burton [1990a], Bert [1984].

Review articles such as Kant and Swaminathan [2000], Reddy and Arciniega [2004] and Mittelstedt and Becker [2004] covered much of the previous research on laminated plate theories in the past decades. From these surveys, mainly related to static analysis, one can find the need for and great interest in the development of computational efficient numerical tools to study buckling and dynamics of multi laminated composite plate-shell structures. Very few research publications deal with the buckling of multi laminated plate-shell structures using higher order displacement fields, Phan and Reddy [1985] and Reddy and Phan [1985], used a higher order shear deformation theory to analyse laminated anisotropic plates for deflections, stresses, buckling load and natural frequencies. They used a displacement field, which accounts
for layer wise parabolic distribution of transverse shear stress, but imposing the
condition that these stresses vanish on the plate top and bottom surfaces.

Other studies using the same displacement field are due to Reddy and Khdeir
[1989] and Khdeir [1989], where closed form solutions are derived and compared with
CLPT and HSDT theories for unsymmetric cross ply rectangular composite laminates
for buckling and free vibration under several boundary conditions. Finite element
higher order discrete models have been developed and discussed for vibration and/or
buckling by Senthilnathan et al.[1988], Kant et al. [1988], Putcha and Reddy [1986],
node rectangular element has been developed for the buckling analysis of multi
laminated plates by Ghosh and Dey [1992,1994]. This model assumes a parabolic
distribution of the transverse shear stresses and the non-linearity of the in-plane
displacements across the thickness. The geometric stiffness matrix is developed using
in-plane stresses. Finite element formulation with mixed variables for stress analysis of
composites was developed by Kant et al.[2007]. Comprehensive overviews of buckling
and vibration of composite plate-shell structures are given in Noor and Peters [1994],
Leissa [1987a,1987b], Kapania and Raciti [1989a,1989b] and Palazotto and Dennis
[1992]. Jose et al.[1999] developed a higher order discrete model to study buckling and
dynamic behaviour of laminated composites. A Zeroth order Shear Deformation
Theory has been derived by Shimpi [1999], which predicts accurate results for both
thick and thin isotropic plates. ZSDT approach utilises only physically meaningful
entities eg. lateral deflection and shear forces right from formulation stage. Most of the
higher order plate theories enforces traction free boundary conditions at the plate faces.
Leung et al. [2003] developed an unconstrained third order theory for the stress analysis of symmetrically laminated plates.

All the higher order theories available in the literature are third order theories. However, Matsunaga [1997a, 1997b, 2000] proposed a ninth order theory in which the in-plane displacement fields consist of ninth order polynomial in global thickness coordinate $z$ whereas the transverse deflection is represented by eighth order polynomial of global $z$. Numerical results showed that higher order shear deformations surely have an important effect on natural frequencies and buckling stresses. Wu and Chen [2005] developed finite element formulations using third order global-local theory to predict thermal and mechanical behaviour of composite plates. These theories predict shear stress components without post processing methods. Later, a fifth order global-local theory has been proposed by Wu and Chen [2007] for the analysis of detailed response of multilayered plates.

Although most of the higher order theories begin with the displacement field, or its special forms, the governing equations differ considerably from each other. One source of difference is the method of deriving them. Other sources of difference include the choice of variables, accounting for various degrees of geometric non linearity, inclusion of transverse normal stress, and satisfaction of specified (zero or non zero) transverse shear stress conditions on the plate surfaces.

2.3 Layer Wise Theories

In contrast to the ESL theories, the layer wise theories are developed by assuming that the displacement field exhibits only $C^0$- continuity through the laminate thickness. Thus the displacements components are continuous through the laminate
thickness but the derivatives of the displacement with respect to the thickness coordinate may be discontinuous at various points through the thickness, thereby allowing for the possibility of continuous transverse stresses at interfaces. In layer wise models the continuity of displacements as well as transverse stresses are satisfied at the interface level.

Mau [1973] used the FSDT with layer wise definition of the generalized displacements. By introducing the interlayer shear stresses as Lagrange multipliers to satisfy the continuity of displacements at the layer interfaces, the governing equations were obtained by minimizing the modified potential energy functional. Based on elasticity solution for a laminated beam, Ren [1986a] developed a plate theory with the corresponding shear stress distribution. This theory has seven constants and has been shown to be very accurate. To obtain a laminated plate theory for accounting the variation of the zig-zag form of in-plane displacement components along the thickness direction, Murakami [1986] proposed to add a zigzag-shaped $C^0$ function to in-plane displacement components of arbitrary global displacement theories. By combining both zig-zag shape function and Legendre polynomials to approximate in-plane displacement fields, Murakami and Toledano [1987] developed a higher-order zig-zag theory. To assess performance of this theory, the cylindrical bending problems of laminated plates was analyzed. Carrera [2003,2004] also discussed, with the help of numerical examples, the use of the zig-zag function proposed by Murakami [1986] in the global displacement theories and a conclusion has been drawn that introduction of zig-zag function is more effective than improvement of order number in global displacement theories. In addition, Demasi [2005] proposed a multilayered plate element and indicated that this approach is unable to \textit{a priori} satisfy inter laminar
continuity for transverse stresses. Therefore, this approach can not compute transverse
stresses directly from constitutive equations. These discrete layer / zig-zag theories
require a large number of degrees of freedom to represent through thickness
displacement functions. This significantly affects the computational efficiency of the
analysis procedure. The imposition of certain conditions of continuity at the layer
interfaces can be used to reduce the total number of unknown displacement parameters.
Di Sciuva [1986] proposed a linear zig-zag model which can guarantee the continuity
of transverse shear stresses at interfaces, but unable to satisfy the conditions of zero
transverse shear stresses on the upper and lower surfaces. Subsequently, a cubic zig-zag
model has been developed by Di Sciuva [1992,1995]. Compared to the linear zig-zag
model, the cubic zig-zag model can improve the accuracy of transverse shear stresses.
Reddy [1987] proposed a layer wise model in terms of one dimensional Lagrangian
finite elements. This model is very general to accommodate any number of layers,
distribution of layers and order of interpolation. Savithri [1991], Savithri and Varadan
[1990,1992] developed a refined theory for linear and non linear analysis of thick
laminated plates. This theory incorporates realistic non linear variations of in-plane
displacements through the thickness and sudden changes of slope at the interfaces and
at the same time retaining the same number of undetermined constants as in FSDT
theory. Even though the layer wise models are satisfying the shear stress continuity at
the interface, but are unable to calculate transverse shear stresses accurately directly
from constitutive equations. To obtain accurate transverse stresses, the equilibrium
equation method is usually adopted.
2.4 Three Dimensional Elasticity Theories

Various investigators have reported their contributions based on 3D elasticity solution for plates. The fundamental objective in the development of elasticity solution is to obtain accurate results of global and local quantities. These solutions are serving as benchmark solution for approximate theories.

2.5 Functionally Graded Plates

Functionally Graded Materials (FGM) is a class of composites that has a gradual variation of material properties from one surface to another. In general, all the multi-phase materials, in which the material properties are varied gradually in a predetermined manner, fall into the category of FGMs. The concept of FGM was proposed in 1984 by the material scientist in the Sendai area of Japan [Koizumi 1993]. Plate models for the FGM have been studied with analytical and numerical methods. Various approaches have been developed to establish the appropriate analysis of the FG plates. The model based on classical plate theory of Kirchhoff was applied by Chi and Chung [2006a, 2006b]. They developed the analytical solution for simply supported FG plates subjected to mechanical loads. A finite element formulation based on CPT was studied by He et al. [2001] to control the shape and vibration of the FG plate with integrated piezo electric sensors and actuators. Within the scope of CPT, Javaheri and Eslami [2002] presented the buckling analysis of FG plates under in-plane compression and Samsam et al. [2005] presented an analysis of buckling behaviour of rectangular FG plates with geometrical imperfections. In practice, these models are not used for thick plates which have an important contribution of the shear deformation energy. In order to take into account the effects of gradual change of material properties, the FSDT and HSDT theories have been used in the analysis. However, since in the FSDT the transverse shear strains are assumed to be constant in the thickness direction, shear correction factors have to be incorporated to adjust the transverse shear stiffness for studying the static of dynamic problems of plate. Praveen and Reddy [1998] examined the non linear static and dynamic responses of FG plates using the FSDT and Von-Karman’s strain. Croce and Venini [2004] formulated a hierarchic family of finite
elements according to the Reissner-Mindlin theory. Trung et al [2008] presented a FSDT model for functionally graded material and numerical solutions of the static analysis were compared with available solutions. Based on FSDT Sahraee [2009] presented the bending analysis of FG thick circular sector plates. It is assumed that the non-homogeneous mechanical properties of plate, graded through the thickness, are described by a power function of the thickness coordinate. Reddy [2000] presented a third order shear deformation theory for the static and dynamic analysis of FG plates. Cheng and Batra [2000a] used the theory of Reddy for studying the buckling and steady state vibrations of a simply supported functionally graded polygonal plate. Also, Cheng and Batra [2000b] have related the deflections of FG plates given by FSDT and third order shear deformation theories to that of an equivalent homogeneous Kirchhoff plate. The static response of FG plate has been investigated by Zenkour [2004,2006] using a generalized shear deformation theory. Furthermore, Zenkour[2005a,2005b] presented the analysis of FG sandwich plates for the deflection, stresses, buckling and free vibration using Classical Plate Theory. In order to analyse the complete effects of higher order deformations on the natural frequencies and buckling stresses of FG plates various orders of the expanded higher order shear deformation theories have been presented by Matsunaga [2008]. Reddy and Chin [1998] have studied the dynamic thermoelastic response of functionally graded cylinders and plates. Najafizadeh and Eslami [2002] presented the buckling analysis of radially loaded solid circular plate made of functionally graded material.

For the thick FG plates whose thickness is not negligible when compared to the side length, 3D models for static and dynamic problems can be used. Cheng and Batra [2000c] studied 3D thermo mechanical deformations of the FG plates.
Elishakoff et al [2005] used the Ritz method to derive the 3D governing equations for the all-round clamped FG plates. An approximate solution developed by Ramirez et al. [2006] for the static analysis of 3D, anisotropic, elastic plates composed of FGM is obtained by using a discrete layer approach. A 3D solution has been presented for free and forced vibration of simply supported rectangular plates by Vel and Batra [2004]. The effective material property at a point is estimated by either the Mori – Tanaka [1973] or the self consistent schemes [Reiter et al. [1997]].

2.6 Concluding Remarks

A comprehensive review is made in this chapter on the various theories applicable to laminated composite plates. Although different equivalent single layer higher order theories have been developed to improve stress distributions, it remains that none of the theories can give good interlaminar stresses. Advancement over the mechanics of functionally graded plates is also reviewed. As a part of design process it is necessary to predict accurately displacements, normal and transverse stresses which contribute significantly to delamination and buckling criteria to establish the load and performance capacities of multi laminated structures. The 3D elasticity solutions are computationally complex and thus the review emphasizes the need for further development of efficient tools for flexure and buckling analysis of laminated composite plates and Functionally Graded Plates.