Chapter – 1

Introduction

1.1 Introduction

The chapter attempts to give an introduction to the topic of study and a brief survey of the contributions made by earlier works on the subject matter presented in the thesis. At the end of the chapter, a chapterwise summary of the thesis has also been given.

1.2 The Gaussian Hypergeometric Function and its Generalizations

John Wallis, in his work Arithmatica Infinitorum in 1655, first used the term ‘hypergeometric’ (from the Greek word νπερ, above or beyond) to denote any series which was beyond the ordinary geometric series $1 + x + x^2 + \cdots$. In particular, he studied the series $1 + a + a(a + 1) + a(a + 1)(a + 2) + \cdots$

Because of the many relations connecting the special functions to each other and to the elementary functions, it is natural to enquire whether more general functions can be developed so that the special functions and elementary functions are merely specializations of these general functions. General functions of this nature have in fact been developed and are collectively referred to as functions of the hypergeometric type. There are several varieties of these functions, but the most common are the hypergeometric functions.

Some important results concerning the hypergeometric function had been developed earlier by Euler and others, but it was famous German mathematician C.F. Gauss who in 1812, studied the following infinite series which is generalization of the elementary geometric series and popularly known as Gauss series or more precisely Gauss hypergeometric series
\[ \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n = 1 + \frac{a \cdot b}{1 \cdot c} x + \frac{a(a + 1) \cdot b(b + 1)}{1.2 \cdot c(c + 1)} x^2 + \ldots \]

...(1.2.1)

where

\[(a)_n = \prod_{k=1}^{n} (a + k - 1) = a(a + 1)(a + 2) \ldots (a + n - 1)\]

is the Factorial function, or if \(a > 0\) then \((a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}\) (where \(\Gamma\) is Euler’s Gamma function), obviously \((a)_0 = 1\) and \((a)_n = n!\).

Gauss represented this series by the symbol \(\mathbf{2F1}(a, b; c; x)\) and called it the hypergeometric function. Here \(x\) is a real or complex variable, \(a, b\) and \(c\) are parameters having real or complex values and \(\neq 0, -1, -2, \ldots\).

\[ \mathbf{2F1}(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n \]

...(1.2.2)

If \(c\) is zero or a negative integer, the function \(\mathbf{2F1}(a, b; c; x)\) is not defined unless one of the parameters \(a\) or \(b\) is also a negative integer such that \(-c < -a\). If either of the parameters \(a\) or \(b\) is a negative integer, say \(-r\), then in this case (1.2.1) reduces to the hypergeometric polynomial defined by

\[ \mathbf{2F1}(-r, b; c; x) = \sum_{n=0}^{\infty} \frac{(-r)_n(b)_n}{(c)_n n!} x^n , \quad -\infty < x < \infty \]

...(1.2.3)

The series given by (1.2.1) is convergent when \(|x| < 1\) and when \(|x| = 1\), provided that \(\text{Re } (c - a - b) > 0\) and also when \(x = -1\), provided that \(\text{Re } (c - a - b) > -1\).

In (1.2.1) if we replace \(x\) by \(\frac{x}{b}\) and let \(b \to \infty\), then on taking into account the formula

\[ \lim_{b \to \infty} \frac{(b)_n}{b^n} x^n = x^n , \]

...(1.2.4)

we arrive at the following well-known Kummer’s series

\[ \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} x^n = 1 + \frac{a}{c} x + \frac{a(a + 1)}{c(c + 1)} \frac{x^2}{2!} + \ldots \]

...(1.2.5)
The above series is represented by the symbol \( \text{F}_1(a; c; x) \) and known as Confluent Hypergeometric Function. The series given by (1.2.5) is absolutely convergent for all values of \( a, c \) and \( x \), real or complex, excluding \( c = 0, -1, -2, \ldots \).

Gauss’s hypergeometric function \( \text{F}_2 \) and its confluent form \( \text{F}_1 \) form the core of special functions and include as special cases most of the commonly used functions. Thus \( \text{F}_2 \) includes as its special cases Legendre functions, the incomplete beta function, the complete elliptic functions of the first and second kinds and most of the classical orthogonal polynomials. On the other hand, the confluent hypergeometric function \( \text{F}_1 \) includes as its special cases Bessel functions, parabolic cylinder functions, Coulomb wave functions, etc. Whittaker functions are also a slightly modified form of confluent hypergeometric functions. On account of their usefulness, the functions \( \text{F}_2 \) and \( \text{F}_1 \) have already been explored to a considerable extent by a number of eminent mathematicians like C. F. Gauss, E. E. Kummer, L. J. Slater, R. Mellin and E. W. Barnes.

Hypergeometric function \( \text{F}_2 \) has been generalized by various mathematicians, mainly in three ways:

(i) Increasing the number of parameters,
(ii) Increasing the number of variables, and
(iii) Increasing the number of parameters as well as variables.

The most known generalization of first kind is the generalized hypergeometric function defined by the series [133, p.73]:

\[
p_{\text{F}}q \left[ \begin{array}{c} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \\ x \end{array} \right] = p_{\text{F}}q \left[ (a_p); (b_q); x \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{x^n}{n!}
\]

...(1.2.6)

where \( p \) and \( q \) are positive integers or zero (interpreting an empty product as 1) and we assume that the variable \( x \), the numerator parameters \( a_1, \ldots, a_p \) and the denominator parameters \( b_1, \ldots, b_q \) take on complex values, provided that \( b_j \neq 0, -1, -2, \ldots; j = 1, 2, \ldots, q \).

An application of the elementary ratio test to the power series on the right hand side of (1.2.6) shows that

(i) If \( p \leq q \), the series converges for all finite \( x \),
(ii) If \( p = q + 1 \), the series converges for \( |x| < 1 \) and diverges for \( |x| > 1 \).
(iii) Furthermore, with \( p = q + 1 \), the series (1.2.6) is

(a) Absolutely convergent on the circle \( |x| = 1 \), if \( \text{Re} (w) > 0 \), where
\[ w = \sum_{k=1}^{q} b_k - \sum_{k=1}^{p} a_k \]

\[ \cdots (1.2.7) \]

(b) Conditionally convergent for \(|x| = 1, x \neq 1\), if \(-1 < Re(w) \leq 0\), and

(c) Divergent for \(|x| = 1\) if \(Re(w) \leq -1\).

(iv) If \(p > q + 1\), the series never converges except when \(x = 0\), and the function is only defined when the series terminates. A comprehensive account of \(_2F_1\) and \(_pF_q\) functions can be found in the standard works by Slater [150], Exton [67] and Rainville [43].

In attempt to give meaning to \(_pF_q\) in the case when \(p > q + 1\), Mac Robert [245] and Meijer [38] introduced and studied in detail, the two special functions which are well-known in the literature as the E-function and the G-function, respectively. A detailed account of the G-function is given in the works by Luke [270] and Mathai and Saxena [25]. The E- and G-functions include wide variety of special functions as their particular cases. Though E- and G-functions are quite general in character, but still many functions like Wright’s generalized hypergeometric function [45], Wright’s generalized Bessel function [44], Mittag-Leffler function and several other functions do not form their special cases.

A generalization of \(_pF_q\) was given by Wright [45, p.287] in the following form:

\[ p\psi_q \left[ \begin{array}{c} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) ; \\
(b_1, \beta_1), \ldots, (b_q, \beta_q) ; 
\end{array} \right] _x = p\psi_q \left[ \begin{array}{c} (a_j, \alpha_j) _{1,p} ; \\
(b_j, \beta_j) _{1,q} ; 
\end{array} \right] _x = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + \alpha_j r)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j r)} \frac{x^r}{r!} \]

\[ \cdots (1.2.8) \]

where \(\alpha_j\) and \(\beta_j\) \((i = 1, \ldots, p ; j = 1, \ldots, q)\) are real and positive, and

\[ 1 + \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j > 0 \]

### 1.3 The Fox’s H-Function and its Generalization

In 1961, Charles Fox [34] introduced a more general function which is well-known in the literature as Fox’s H-function or simply the H-function. This function has been defined and
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represented by means of the following Mellin-Barnes type of contour integral (see, e.g. [83, p.10]):

\[
H_{p,q}^{m,n} \left\{ x \begin{pmatrix} (a_j, \alpha_j)_{1,p} \end{pmatrix} \right\} = H_{p,q}^{m,n} \left\{ x \begin{pmatrix} (a_1, \alpha_1), ..., (a_p, \alpha_p) \end{pmatrix} \right\} = \frac{1}{2\pi i} \int_L \theta(s)x^s ds
\]

...(1.3.1)

where \( \omega = \sqrt{-1}, x \neq 0 \) and

\[
\theta(s) = \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j s) \prod_{j=1}^n \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + \beta_j s) \prod_{j=n+1}^p \Gamma(a_j - \alpha_j s)}
\]

...(1.3.2)

Here \( m, n, p \) and \( q \) are non-negative integers satisfying \( 0 \leq n \leq p, 1 \leq m \leq q \). Also \( \alpha_j (j = 1, ..., p) \) and \( \beta_j (j = 1, ..., q) \) are assumed to be positive quantities for standardization purpose. Also \( \alpha_j (j = 1, ..., p) \) and \( b_j (j = 1, ..., q) \) are complex numbers such that

\[
\alpha_i (b_h + \nu) \neq \beta_h (a_i - 1 - \eta)
\]

...(1.3.3)

for \( \nu, \eta = 0, 1, 2, ..., h = 1, ..., m; i = 1, ..., n \).

L is contour separating the points

\[
s = \left( \frac{b_h + \nu}{\beta_h} \right); (h = 1, ..., m; \nu = 0, 1, 2, ...)
\]

which are the poles of \( \Gamma(b_h - \beta_h s); (h = 1, ..., m) \), from the points

\[
s = \left( \frac{a_i - 1 - \eta}{\alpha_i} \right); (i = 1, ..., n; \eta = 0, 1, 2, ...)
\]

which are the poles of \( \Gamma(1 - a_i + \alpha_i s); i = 1, ..., n \). The contour L exists on account of (1.3.3).

The series representation of the H-function is

\[
H_{p,q}^{m,n} [x] = \sum_{h=1}^{m} \sum_{r=0}^{\infty} \prod_{j=1}^{m} \Gamma(b_j - \beta_j \xi_{h,r}) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j \xi_{h,r}) \frac{(-1)^r (x) \xi_{h,r}}{r! \beta_h}
\]

...(1.3.4)

where

\[
\xi_{h,r} = \frac{b_h + r}{\beta_h}
\]

...(1.3.5)

also,
\[ |\arg x| < \frac{1}{2} A\pi \]

...(1.3.6)

\[
A = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j > 0
\]

...(1.3.7)

The detailed discussion of asymptotic expansions of the H-function, some of its properties and special cases can be referred to the books [24, 34].

Moreover, some special functions provide direct solutions of several ordinary and fractional order differential equations occurring in the boundary value problems of diverse fields of engineering. Such functions are Mittag-Leffler function, Agarwal function, Erdélyi function, Robotnov-Hartley F-function, Lorenzo-Hartley R- and G-functions [249, p.3, Eq.(12); p.15, Eq. (101)], Miller-Ross E, C, and S-functions [147, p.49, Eq. (3.10)], reduced Green function due to Mainardi, Luchko and Pagnini [52]. Gupta [133] and Gupta and Soni [140] have established relationships between these highly useful named functions and Fox’s H-function. The relationships are

\[
\frac{1}{at} H^{2,1}_{3,3} \left\{ x \left( \frac{1}{\alpha}, \frac{1}{\alpha}, (1, \beta), (1, \rho) \right) \right\} = K_{\alpha,\beta}(t),
\]

[Reduced Green function] ...(1.3.8)

Where

\[
\rho = (\alpha - \theta)/2\alpha,
\]

\[
\frac{t^{rq-v-1}}{\Gamma(r)} H^{1,1}_{1,2} \left\{ -at \left( \frac{1}{\alpha}, (1, 1) \right) \right\} = G_{q,v,r}[a, t]
\]

[Lorenzo-Hartley G-function] ...(1.3.9)

\[
t^u H^{1,1}_{1,2} \left\{ -at \left( 0,1 \right) \right\} = E_t[u, a]
\]

[Miller-Ross E\_\text{-}function] ...(1.3.10)

\[
t^u H^{1,1}_{1,2} \left\{ a^2 t^2 \left( 0,1 \right) \right\} = C_t[u, a]
\]

[Miller-Ross C\_\text{-}function] ...(1.3.11)
\[ t^v H_{1.2}^{1,1} \left\{ \frac{a^2 t^2}{(1, 1), (-v, 2)} \right\} = S_t[v, a] \]

[Miller-Ross S-t-function] \hspace{1cm} \ldots(1.3.12)

\[ t^{q-v-1} H_{1.2}^{1,1} \left\{ -at^q \frac{(0,1)}{(0,1), (1 + v - q, q)} \right\} = R_{q,v}[a, t] \]

[Lorenzo-Hartley R-function] \hspace{1cm} \ldots(1.3.13)

Since Mittag-Leffler function, Agarwal function, Erdélyi function, Robotnov-Hartley F-function are all special cases of the R-function [249, p.16, table 1], these functions can be expressed easily in terms of the H-function.

The importance of the study of Fox’s H-function lies in the fact that all the special functions mentioned in the earlier paragraphs follow as its particular cases, so that each of the formula developed for the H-function becomes a key formula from which a considerably large number of relations for other special functions can be deduced by suitably specializing the parameters of the H-function involved therein. A good collection of the work done on the H-function can also be seen in the two books referred above (see also [154]).

The H-function defined by (1.3.1) contains as particular cases most of the special functions as mentioned above, but it does not contain some of the important functions such as polylogarithm of a complex order, the exact partition function of the Gaussian model in statistical mechanics, etc.

Motivated by some further examples of the use of Feynman integrals which arise in perturbation calculations of the equilibrium properties of magnetic model of phase transitions, a generalization of Fox’s H-function was proposed by Inayat-Hussain. This function is known as \(\overline{H}\)-function. Details of this function are recorded in the Appendix II of this thesis.

Saxena [267, 268] has also studied I-function which is more general than Fox’s H-function. The \(\overline{H}\)-function cannot be obtained from the I-function. Al-Musallam and Tuan [54, 55] have introduced and studied the H-function with complex parameters. They have established necessary and sufficient conditions, absolute convergence of Mellin-Barnes integral (defining their H-function) and sufficient conditions for computation of the H-function by means of residues.
1.4 The I-Function

V. P. Saxena [268] introduced a new function in the literature namely the “I-function” which provides the generalization of Fox’s H-function. This newly defined function contains the polylogarithms, the exact partition of Gaussian free energy model from statistical mechanics, Feynmann integrals and functions useful in testing hypothesis from statistics as special cases.

The I-function is defined and represented by [268]:

$$I[z] = \frac{1}{2\pi \omega} \int_{L} \phi(\xi) z^{\xi} d\xi$$

where

$$\phi(\xi) = \frac{\prod_{j=1}^{m} \Gamma(b_{j} - \beta_{j} \xi) \prod_{k=1}^{n} \Gamma(1 - a_{j} + \alpha_{j} \xi)}{\prod_{i=1}^{r} \Gamma(1 - b_{ji} + \beta_{ji} \xi) \prod_{k=1}^{p} \Gamma(a_{ji} - \alpha_{ji} \xi)}$$

and $\omega = \sqrt{-1}$.

$p_{i}(i = 1,2, ..., r), q_{i}(i = 1,2, ..., r), m, n$ are integers satisfying $0 \leq n \leq p_{i}, 0 \leq m \leq q_{i}$ ($i = 1,2, ..., r)$; $r$ is finite, $a_{j}, \beta_{j}, \alpha_{ji}, \beta_{ji}$ are real and positive and $a_{j}, b_{j}, a_{ji}, b_{ji}$ are complex numbers such that

$$a_{j}(b_{h} + v) \neq \beta_{j}(a_{j} - 1 - k)$$

for $v, k = 0,1, ... ; h = 1,2, ..., m ; i = 1,2, ..., r$.

$L$ is a contour running from $\gamma - \omega \infty$ to $\gamma + \omega \infty$ ($\gamma$ is real) in the complex $\xi$ plane such that the points

$$\xi = \frac{(a_{j} - 1 - p_{j})}{a_{j}} ; j = 1,2, ..., n ; v = 0,1, ... \text{ and } \xi = \frac{(b_{j} + p_{j})}{b_{j}} ; j = 1,2, ..., m ; v = 0,1, ... \text{ lie on the left hand and right hand sides of } L, \text{ respectively.}$$

For the I-Function, defined by (1.4.1), there are three different $L$ paths of integration:

a) $L$ is a contour which runs from $\gamma - \omega \infty$ to $\gamma + \omega \infty$ ($\gamma$ is real), so that all poles of $\Gamma(b_{j} - \beta_{j} \xi) ; j = 1,2, ..., m$ are to the right and the poles of $\Gamma(1 - a_{j} + \alpha_{j} \xi) ; j = 1,2, ..., n$ to the left of $L$, the integral converges if $p + q < 2(m + n)$ and $|\arg z| < \left( m + n - \frac{1}{2}(p + q) \right)$ for all $i = 1,2, ..., r$.

b) $L$ is a loop starting and ending at $\gamma + \omega \infty$ and encircling all poles of $\Gamma(b_{j} - \beta_{j} \xi) ; j = 1,2, ..., m$, one in the negative direction, but none of the poles of $\Gamma(1 - a_{k} + \alpha_{k} \xi) ; k = 1,2, ..., r$, respectively.
$1, 2, \ldots, n$. The integral converges if $q_i \geq 1$ and either $p_i < q_i$ or $p_i = q_i$ and $|z| < 1$

for all $i = 1, 2, \ldots, r$.

c) $L$ is a loop starting and ending at $\gamma - \infty$ and encircling all poles of

$\Gamma(1 - a_k + \alpha_k \xi); k = 1, 2, \ldots$, one in the positive direction, but none of the poles of

$\Gamma(b_j - \beta_j \xi); j = 1, 2, \ldots, m$. The integral converges if $p_i \geq 1$ and either $p_i > q_i$ or $p_i = q_i$ and $|z| > 1$ for all $i = 1, 2, \ldots, r$.

**Special Cases**

(i) If we take $r = 1$ it reduces to Fox’s H-Function [100] $\ldots(1.4.3)$

$$
I_{p,q}^{m,n} \left\{ \left( \begin{array}{c}
(a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1,p_i} \\
(b_j, \beta_j)_{1,m}; (b_{ji}, \beta_{ji})_{m+1,q_i}
\end{array} \right) \right\} = H_{p,q}^{m,n} \left\{ (a_j, \alpha_j)_{1,n}; (b_j, \beta_j)_{1,m} \right\}
$$

(ii) If we take $\alpha_j = \beta_j = \alpha_{ji} = \beta_{ji} = 1$, it reduces to a function $I_G$, defined as

$$
I_G = \frac{1}{2\pi \omega} \int_L \eta(\xi)z^\xi d\xi
$$

\ldots(1.4.4)

where

$$
\eta(\xi) = \frac{\prod_{j=1}^m \Gamma(b_j - \xi) \prod_{j=1}^n \Gamma(1 - a_j + \xi)}{\sum_{i=1}^r \prod_{j=m+1}^n \Gamma(1 - b_{ji} + \xi) \prod_{j=n+1}^p \Gamma(a_{ji} - \xi)}
$$

\ldots(1.4.5)

(iii) If we take $r = 1, \alpha_j = \beta_j = \alpha_{ji} = \beta_{ji} = \alpha$, it reduces to a Meijer’s G-function

$$
I_{p,q}^{m,n} \left\{ \left( \begin{array}{c}
(a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1,p_i} \\
(b_j, \beta_j)_{1,m}; (b_{ji}, \alpha)_{m+1,q_i}
\end{array} \right) \right\} = \frac{1}{\alpha} G_{p,q}^{m,n} \left\{ (a_j, 1)_{1,n}; (b_j, 1)_{1,m} \right\}
$$

\ldots(1.4.6)

(iv) If we take $r = 1, m = 1, n = p_i = p, q_i = q + 1, b_1 = 0, \beta_1 = 1, a_j = 1 - a_j, b_{ji} = 1 - b_j, \beta_{ji} = \beta_j$, it reduces to Wright’s generalized hypergeometric function...
\[ I_{p,q+1:1}^{1,p} \left\{ \left( 1 - a_j, \alpha_j \right)_{1,p} \left( 0,1; \left( 1 - b_j, \beta_j \right)_{1,q} \right) \right\} = p \Psi_q \left\{ -Z \left( \left( a_j, \alpha_j \right)_{1,p} \right) \left( b_j, \beta_j \right)_{1,q} \right\} \] \quad \ldots \quad (1.4.7)

where

\[ p \Psi_q \left\{ Z \left( \left( a_j, \alpha_j \right)_{1,p} \right) \left( b_j, \beta_j \right)_{1,q} \right\} = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma \left( a_j + \alpha_j r \right) Z^r}{\prod_{j=1}^{q} \Gamma \left( b_j + \beta_j r \right) r!} \] \quad \ldots \quad (1.4.8)

### 1.5 The Aleph (\( \aleph \))-Function

The Aleph (\( \aleph \))-function occurs naturally in certain problems of fractional driftless Fokker-Planck equations. It also provides a generalization of the I-function. Further, on account of the importance and considerable popularity achieved due to its applications in various fields of science and engineering, such as fluid flow, Rheology, diffusive transport akin to diffusion, electric networks and probability, the Aleph (\( \aleph \))-Function has become a topic of interest in recent days.

The Aleph function, introduced by Südland et al. ([174]; see also [214]), is defined in terms of Mellin-Barnes type integrals as:

\[ \aleph[z] = \aleph_{p,q + 1:1}^{m,n} \left\{ \left( a_j, A_j \right)_{1,n}; \left( \tau_i \left( a_{ji}, A_{ji} \right) \right)_{n+1,p;i;r} \left( b_j, B_j \right)_{1,m}; \left( \tau_i \left( b_{ji}, B_{ji} \right) \right)_{m+1,q;i;r} \right\} = \frac{1}{2\pi i} \int_L \Omega_{p,q + 1:1}^{m,n} (s) z^{-s} ds \] \quad \ldots \quad (1.5.1)

where \( \omega = \sqrt{-1} \)

and

\[ \Omega_{p,q + 1:1}^{m,n} (s) = \frac{\prod_{j=1}^{p} \Gamma \left( b_j + B_j s \right) \prod_{j=1}^{q} \Gamma \left( 1 - a_j - A_j s \right)}{\sum_{i=1}^{\infty} \tau_i \prod_{j=m+1}^{q} \Gamma \left( 1 - b_{ji} - B_{ji} s \right) \prod_{j=n+1}^{p} \Gamma \left( a_{ji} + A_{ji} s \right)} \] \quad \ldots \quad (1.5.2)

The integration path \( L = L_{\omega \gamma} \), \( \gamma \in \Re \) extends from \( \gamma - \omega \infty \) to \( \gamma + \omega \infty \), and is such that the poles of \( \Gamma \left( 1 - a_j - A_j s \right), j = 1, n \) do not coincide with the poles of \( \Gamma \left( b_j + B_j s \right), j = 1, m \).

The parameters \( p_i \) and \( q_i \) are non-negative integers satisfying the condition \( 0 \leq n \leq p_i, 0 \leq m \leq q_i, \tau_i > 0 \) for \( i = 1, r \). The parameters \( A_j, B_j, A_{ji}, B_{ji} > 0 \) and \( a_j, b_j, a_{ji}, b_{ji} \in \mathbb{C} \). An empty product in (1.5.2) is interpreted as unity. The existence conditions for (1.5.1) are:

\[ \varphi_i > 0, |\arg z| < \frac{\pi}{2} \varphi_i \quad ; \quad i = 1, r \] \quad \ldots \quad (1.5.3)
\( \varphi_i \geq 0, |\arg z| < \frac{\pi}{2} \varphi_i \) and \( \Re \{ \zeta_i \} + 1 < 0 \) ...(1.5.4)

where

\[
\varphi_i = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \tau_i \left( \sum_{j=n+1}^{p_i} A_{ji} + \sum_{j=m+1}^{q_i} B_{ji} \right) \]

...(1.5.5)

\[
\zeta_i = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_i \left( \sum_{j=m+1}^{q_i} b_{ji} - \sum_{j=n+1}^{p_i} a_{ji} \right) + \frac{1}{2} (p_i - q_i) ; \ i = 1, r \]

...(1.5.6)

If we take \( \tau_i = 1 \), it reduces to I-Function [268]

\[
\aleph_{p_i,q_i}^{m,n} \left\{ \begin{array}{c}
(a_j, A_j)_{1,n}; (a_{ji}, A_{ji})_{n+1,p_i;r} \\
(b_j, B_j)_{1,m}; (b_{ji}, B_{ji})_{m+1,q_i;r}
\end{array} \right\} = I_{p_i,q_i}^{m,n} \left\{ \begin{array}{c}
(a_j, A_j)_{1,n}; (a_{ji}, A_{ji})_{n+1,p_i;r} \\
(b_j, B_j)_{1,m}; (b_{ji}, B_{ji})_{m+1,q_i;r}
\end{array} \right\}
\]

...(1.5.7)

Thus the Aleph (8)- function yields a large number of simpler special functions as its particular cases, on suitably specifying the parameters involved herein.

### 1.6 Multivariable Generalized Hypergeometric Functions

The great success and fruitful nature of the results concerning the special functions of one variable stimulated the study and development of a corresponding theory of special functions involving two or more variables.

In 1880, P. Appell [184] introduced and studied systematically the four functions \( F_1 \), \( F_2 \), \( F_3 \) and \( F_4 \) which are generalizations of the Gaussian hypergeometric function in two variables. These functions are known as Appell functions, and the set was completed by Horn [115] who introduced the remaining ten series (\( G_1 \), \( G_2 \), \( G_3 \), \( H_1 \), \( \ldots \), \( H_7 \)). A list of all these functions is given in the work by Erdélyi et al [8]. The confluent forms \( F_1 \), \( F_2 \), \( F_3 \) and \( F_4 \) were
studied by Humbert [186]. The functions $F_1$ to $F_4$ and their confluent forms were further
generalized by Kampé de Fériet who introduced the function defined by the following series:

$$F_{n; p; u}^{m; q; v} \begin{pmatrix} (a_m); (c_p); (e_u); \xi, \eta \end{pmatrix} = \sum_{r, s=0}^{\infty} \prod_{j=1}^{m} (a_i + q s) \prod_{j=1}^{u} (e_j + p r) \prod_{j=1}^{n} (c_j + 1) \prod_{j=1}^{n} (d_j + q s) \frac{x^r y^s}{r! s!}$$

...(1.6.1)

The notation used on the left of (1.6.1), due essentially to Burchnell and Chaundy [121],
is more compact than the one used originally by Kampé de Fériet [185, p.150].

The above double series is absolutely convergent for all values of $x$ and $y$, if $m + p < n + q + 1$ and $m + u < n + v + 1$. Also if $n + p = n + q + 1$ and $m + u = n + v + 1$, we must have any one of the following sets of conditions:

(i) $m \leq n$, for $\max(|x|, |y|) < 1$; \hspace{1cm} ...(1.6.2)

(ii) $m > n$, for $|x|^{\frac{1}{m-n}} + |y|^{\frac{1}{m-n}} < 1$ \hspace{1cm} ...(1.6.3)

The details regarding the convergence of the double series (1.6.1) can be found in the
book by Exton [67, p.25].

Thereafter many eminent persons notably W.N. Bailey, T.W. Chaundy, J. L. Burchnell,
etc. have studied these aforementioned functions. The hypergeometric functions of one and
more variables have been discussed in detail by Exton ([66], [67]) and Srivastava and Karlsson
[92].

The Kampé de Fériet function has further been generalized by Srivastava and Daoust
[89]. Their general function is defined and represented as follows:

$$S_{n; q; v}^{m; p; u} \begin{pmatrix} (a_j; \alpha_j, A_j)_{1,m}; (c_j, \gamma_j)_{1,p}; (e_j, E_j)_{1,u}; \xi, \eta \end{pmatrix} = S[\xi, \eta] = S \begin{pmatrix} x \end{pmatrix}$$

$$= \sum_{r, s=0}^{\infty} \prod_{j=1}^{m} \Gamma(a_j + \alpha_j r + A_j s) \prod_{j=1}^{p} \Gamma(c_j + \gamma_j r) \prod_{j=1}^{u} \Gamma(e_j + E_j s) \frac{x^r y^s}{r! s!}$$

...(1.6.4)

Where $(a_j; \alpha_j, A_j)_{1,m}$ abbreviates the array of $m$ parameters $(\alpha_1, \alpha_1, A_1) \ldots (\alpha_m, \alpha_m, A_m)$ and
so on. The series given by (1.6.4) converges absolutely, if

$$1 + \sum_{j=1}^{n} \beta_j + \sum_{j=1}^{q} \delta_j - \sum_{j=1}^{m} \alpha_j - \sum_{j=1}^{p} \gamma_j \geq 0$$

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and

\[ 1 + \sum_{j=1}^{n} B_j + \sum_{j=1}^{v} F_j - \sum_{j=1}^{m} A_j - \sum_{j=1}^{u} E_j \geq 0 \]

where each of the equalities hold when the variables are suitably constrained (see for details, Srivastava and Daoust [90]).

The study of the functions of two variables has greatly been increased after the independent introduction of G-function of two variables by Agarwal [219] and Sharma [33].

Encouraged by the popularity of the aforementioned functions of two variables, Pathak [220], Chaturvedi and Goyal [146], Munot and Kalla [188], Bora and Kalla [232], Saxena [205], Verma [221], Shah ([161],[162]) and several others have studied functions of two variables which are slightly more general than the G-function of two variables studied by R. P. Agarwal and B. L. Sharma.

During the period 1970-1990, several researchers developed and studied a double Mellin-Barnes type of contour integral popularly known as the H-function of two variables. This function is quite general in nature and includes as its special cases, all the functions of one and two variables cited in the previous paragraphs together with Fox’s H-function. A rather complete and systematic study of this function can be found in the research monograph by Srivastava, Gupta and Goyal [83]. Buschman [197, 198], Prasad and Prasad [271], Prasad and Mishra [272] and others have introduced and studied special functions which are more general than the H-function of two variables.

Due to the useful nature of the theory of hypergeometric functions of one and two variables, a corresponding theory of hypergeometric functions involving more than two complex variables has been developed by a number of mathematicians. Lauricella [59] introduced fourteen complete hypergeometric series in three variables of second order. Since then, a few additional triple hypergeometric series have been introduced. In their book, Srivastava and Karlsson [92, pp.74-87] constructed the entire set of distinct triple hypergeometric series. A unification of all these triple series was introduced by Srivastava [77, p.428].

In 1893, Lauricella [59] further generalized the four Appell functions \( F_1, F_2, F_3 \) and \( F_4 \) to functions of several variables and defined his functions as follows (see, e.g. [92, p.33]):
\[ F_A^{(n)}(a, b_1, ..., b_n; c_1, ..., c_n; x_1, ..., x_n) = \sum_{m_1, ..., m_n=0}^{\infty} \frac{(a)_{m_1+...+m_n} (b_1)_{m_1} ... (b_n)_{m_n} x_1^{m_1} ... x_n^{m_n}}{(c_1)_{m_1} ... (c_n)_{m_n} m_1! ... m_n!} \] ...(1.6.5)

Where \(|x_1|+...+|x_n| < 1\);

\[ F_B^{(n)}(a_1, ..., a_n, b_1, ..., b_n; c; x_1, ..., x_n) = \sum_{m_1, ..., m_n=0}^{\infty} \frac{(a_1 m_1 ... (a_n)_{m_n} (b_1)_{m_1} ... (b_n)_{m_n} x_1^{m_1} ... x_n^{m_n}}{(c)_{m_1+...+m_n} m_1! ... m_n!} \] ...(1.6.6)

where max\(|x_1|, ..., |x_n|\) < 1

\[ F_C^{(n)}(a, b; c_1, ..., c_n; x_1, ..., x_n) = \sum_{m_1, ..., m_n=0}^{\infty} \frac{(a)_{m_1+...+m_n} (b)_{m_1+...+m_n} x_1^{m_1} ... x_n^{m_n}}{(c_1)_{m_1} ... (c_n)_{m_n} m_1! ... m_n!} \] ...(1.6.7)

where \(\sqrt{|x_1|+...+\sqrt{|x_n|}} < 1\);

\[ F_D^{(n)}(a, b_1, ..., b_n; c; x_1, ..., x_n) = \sum_{m_1, ..., m_n=0}^{\infty} \frac{(a)_{m_1+...+m_n} (b_1)_{m_1} ... (b_n)_{m_n} x_1^{m_1} ... x_n^{m_n}}{(c)_{m_1+...+m_n} m_1! ... m_n!} \] ...(1.6.8)

where max\(|x_1|, ..., |x_n|\) < 1

A summary of Lauricella’s work is given by Appell and Kampé de Fériet [185]. A number of limiting forms of Lauricella’s functions exist and have been discussed by Humbert [186], Erdélyi [3], Exton [66], Srivastava and Exton [80] etc. Also Erdélyi [5] defined a general series which unifies Lauricella’s multiple series \(F_A^{(n)}, F_B^{(n)}\) and Horn’s double series \(H_2\).

In order to further generalize the Lauricella functions and some of their confluent forms, Srivastava and Daoust ([89], [90]) introduced and studied the following generalized Lauricella function:

\[ F_{\alpha; \beta; \gamma; \delta}^{(n)}(a_1, ..., a_n; b_1, ..., b_n; c_1, ..., c_n; x_1, ..., x_n) \]

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\[
\begin{align*}
\sum_{m_1, \ldots, m_n=0}^{\infty} \Omega(m_1, \ldots, m_n) \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \\
&= F_{p: p_1; \ldots; p_n}^{p_1; \ldots; p_n} \left[ \frac{\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_n}{\gamma_1, \ldots, \gamma_n} \right]_x (x_1, \ldots, x_n)
\end{align*}
\]

where for convenience

\[
\Omega(m_1, \ldots, m_n) = \frac{\prod_{j=1}^{p} (a_j)_{m_1 \beta_j, \ldots, +m_n \beta_j} \prod_{j=1}^{q} (b_j)_{m_1 \beta_j, \ldots, +m_n \beta_j} \prod_{j=1}^{r} (c_j)_{m_1 \gamma_j, \ldots, +m_n \gamma_j} \prod_{j=1}^{q} (d_j)_{m_1 \delta_j, \ldots, +m_n \delta_j}}{\prod_{j=1}^{p} (c_j)_{m_1 \gamma_j, \ldots, +m_n \gamma_j} \prod_{j=1}^{q} (d_j)_{m_1 \delta_j, \ldots, +m_n \delta_j}}
\]

The coefficients \( \alpha_k, j = 1, \ldots, p; \gamma_k, j = 1, \ldots, p \); \( \beta_k, j = 1, \ldots, q; \delta_k, j = 1, \ldots, p \)

are real and positive. The multiple series (1.6.9) converges absolutely (see [77]) for all

\( x_1, \ldots, x_k \), where \( Q_i > 0 \) or for \( Q_i = 0 \) and \( |x_i| < \rho_i, (i = 1, 2, \ldots, n) \) where \( \rho_i \) is defined by equation (5.3) in [85, p.157] and

\[
Q_i = 1 + \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{q} \delta_j^{(i)} - \sum_{j=1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{p} \gamma_j^{(i)}, i = 1, 2, \ldots, n
\]

If each of the positive numbers listed in (1.6.11) is equated to 1, the generalized Lauricella series (1.6.9) reduces to a multivariable extension of Kampé de Fériet series in several variables in the following form:

\[
\begin{align*}
\sum_{m_1, \ldots, m_n=0}^{\infty} \Lambda(m_1, \ldots, m_n) \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \\
&= F_{q_1: q_1; \ldots; q_n}^{q_1; q_1; \ldots; q_n} \left[ \frac{\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_n}{\gamma_1, \ldots, \gamma_n} \right]_x (x_1, \ldots, x_n)
\end{align*}
\]

where
\[ \Lambda(m_1, \ldots, m_n) = \frac{\prod_{j=1}^{p} (a_j)_{m_1+\ldots+m_n}}{\prod_{j=1}^{q} (b_j)_{m_1+\ldots+m_n}} \prod_{j=1}^{p_1} (c_j)_{m_1} \cdots \prod_{j=1}^{p_n} (c_j^{(n)})_{m_n} \]

and for convergence of the multiple hypergeometric series (1.6.13), we have

\[ 1 + q + q_k - p - p_k \geq 0; \quad k = 1, 2, \ldots, n \]

\[ \ldots \quad (1.6.15) \]

A detailed account of multiple hypergeometric functions can be found in the books by Exton [66] and Srivastava and Karlsson [92], wherein a near complete bibliography has also been given. Pathan [169] has also defined a generalized Gaussian series of (n+1) variables.

Again Khadia and Goyal [242] introduced and studied the G-function of several variables which is an obvious generalization of the G-function of two variables studied by R.P. Agarwal and B.L. Sharma. In 1974 and 1977, Saxena ([206], [207]) introduced and studied another generalized function of several variables.

In 1976, Srivastava and Panda introduced a general function of several complex variables which is well known in the literature as the H-function of several complex variables. They have studied many properties associated with this function and have placed this function on a firm footing through a series of research papers ([93], through [94] see also [192] to [194], [83], [84], [96], [99], [100] and [196]). The definition, some simple properties and special cases of the multivariable H-function are given in Appendix I at the end of this thesis.

The importance of the results for the multivariable H-function lies in the fact that they include a large number of special functions of one and more variables mentioned in previous paragraphs as particular cases.

Buschman [197] has studied a multivariable function which is even more general than this H-function of several variables. We shall, however, not study Buschman’s function in the present work.

1.7 Fractional Calculus

Fractional calculus has its origin in the question of extension of meaning. For example, extension of real numbers to complex numbers, factorials of natural numbers to generalized factorials or gamma functions and many such others. The original question that led to the name
of fractional calculus was: Can the meaning of derivative of integer order \( \frac{d^n y}{dx^n} \) be extended to have a meaning when \( n \) is a fraction? Later the question became: Can \( n \) be any number, fractional, irrational or complex? Because this question was answered affirmatively, the name fractional calculus has become a misnomer and might better be called integration and differentiation to an arbitrary order.

The earliest systematic studies seem to have been made in the beginning and middle of the 19th century by Liouville [117], Riemann [30] and Holmgren [72]. The list of mathematicians who provided important contributions up to the middle of 20th century, includes Abel ([175],[176]), Fourier [113], Weyl [70], Davis ([102],[103],[104]), Zygmund [12], Erdélyi ([4],[6],[7]), Kober [68], Hadamard [114], Hardy and Littlewood [61], Grunwald [21], Letnikov ([28],[29]), Riesz [156] and several others.

In 1974 the first international conference on fractional calculus was held at the University of New Haven, Connecticut, U.S.A. The proceedings of the conference were published by Springer-Verlag [195]. Again in 1984 and 1989, the second and third international conferences were held at University of Starthclyde, Glasgow, Scotland [20] and at Nihon University, Tokyo, Japan [125] respectively. Many distinguished mathematicians attended these conferences. These luminaries included R. Askey, M. Mikolas, M. Al-Bassam, P. Heywood, W. Lamb, R. Bagley, Y.A. Brychkov, R. Gorenflo, S.L. Kalla, E.R. Love, K. Nishimoto, S. Owa, A.P. Prudnikov, B. Ross, S. Samko, H.M. Srivastava, J.M.C. Joshi and many others. The papers on the fractional calculus and generalized functions, inequalities obtained by use of the fractional calculus and applications of the fractional calculus to probability theory presented in the conference were quite electric.

A systematic (and historical) account of investigations carried out by various authors in the field of fractional calculus and its applications can be found in the paper by Srivastava and Saxena [97] wherein extensive bibliography on the subject has been given. One can also refer to the research papers by Pandey and Srivastava [218], Duren, Kalla and Srivastava [118], Galue, Kiryakova and Kalla [148], Nishimoto and Srivastava [127], Srivastava, Saigo and Owa [85], Saigo, Raina and Kilbas [164], Srivastava and Owa [98], Manocha [74], Saigo [160], Srivastava and Goyal [99], Saigo and Raina [163] and several others. The excellent research monographs by Oldham and Spanier [130], Nishimoto [126], Miller and Ross [147] and Samko, Kilbas and Marichev [231] contain extensive and useful literature concerning the fractional calculus.
In 1819, Lacroix [230] developed a formula of n-th derivative of \( y = x^m \):

\[
\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!} x^{m-n} \quad , (m \geq n, n \in N, m \in N_0)
\]

...(1.7.1)

and using Gamma function he wrote

\[
\frac{d^n y}{dx^n} = \frac{\Gamma(m + 1)}{\Gamma(m - n + 1)} x^{m-n}
\]

...(1.7.2)

and then set \( n = 1/2 \) to find the derivative of order 1/2.

The history of fractional calculus starts with the work by Abel ([175], [176]). He solved the integral equation

\[
\int_{a}^{z} \frac{f(t)}{(z-t)^\mu} dt = \phi(z), z > a, 0 < \mu < 1
\]

...(1.7.3)

in connection with tautochrone problem and the solution was given for \( \mu = 1/2 \). In 1832, Liouville [116] suggested a definition based on the differentiation of an exponential function and is applicable to \( f(x) \) which may be expanded as the series

\[
f(x) = \sum_{i=0}^{\infty} c_i e^{a_i x}
\]

The definition is

\[
D^n f(x) = \sum_{i=0}^{\infty} c_i a_i^n e^{a_i x}
\]

...(1.7.4)

for any complex \( n \). In the same paper he derived a formula

\[
D^{-n} f(x) = \frac{1}{(-1)^n \Gamma(n)} \int_{0}^{\infty} f(x + t)t^{n-1} dt \quad ; -\infty < t < \infty, Re(n) > 0
\]

...(1.7.5)

In 1847, Riemann [30] had arrived at the expression for fractional integration as

\[
\frac{1}{\Gamma(\alpha)} \int_{0}^{x} \frac{f(t)}{(x-1)^{1-\alpha}} dt \quad , x > 0
\]

...(1.7.6)
Holmgren [72] suggested that fractional differentiation can be considered as an inverse to fractional integration.


\[ D_z^{-\mu}[f(z)] = \frac{1}{\Gamma(\mu)} \int_{a}^{z} (z - t)^{\mu-1} f(t) \, dt, \quad \text{Re}(\mu) > 0 \]

...(1.7.7)

for integration to an arbitrary order. It is also known as Riemann version of fractional integral. When \( \alpha = -\infty \), then (1.7.7) becomes

\[ -D_z^{-\mu}[f(z)] = \frac{1}{\Gamma(\mu)} \int_{-\infty}^{z} (z - t)^{\mu-1} f(t) \, dt, \quad \text{Re}(\mu) > 0 \]

...(1.7.8)

It is also known as Liouville version of fractional integral.

The most used version occurs when \( \alpha = 0 \) in (1.7.7), we get

\[ D_z^{-\mu}[f(z)] = \frac{1}{\Gamma(\mu)} \int_{0}^{z} (z - t)^{\mu-1} f(t) \, dt, \quad \text{Re}(\mu) > 0 \]

...(1.7.9)

which is known as Riemann-Liouville fractional integral of order \( \mu \).

When \( z \to \infty \), equation (1.7.7) may be identified with definition of the familiar Weyl fractional operator of order \( \mu \),

\[ W_z^{-\mu}[f(z)] = \frac{1}{\Gamma(\mu)} \int_{z}^{\infty} (t - z)^{\mu-1} f(t) \, dt, \quad \text{Re}(\mu) > 0 \]

...(1.7.10)

Due to important role played by Riemann-Liouville and Weyl integral operators in different branches of science, engineering and mathematical analysis, a number of generalizations of fractional integral operators are proposed from time to time by many authors namely Sneddon ([108],[109],[110]), Saxena ([203],[204]), Kalla and Saxena [234], Kalla [233], Koul [37], Raina and Kiryakova [200], Srivastava and Goyal [99], Goyal and Jain [239], Saxena and Kumbhat ([211],[212]). Definition (1.7.7) is for integration of arbitrary order. For
differentiation of arbitrary order it cannot be used directly. However, by means of a simple trick it can be done. Suppose, for \( Re(v) > 0 \), we want to find \( a^D_z^\mu f(z) \) i.e., the fractional derivative of \( f(z) \) of order \( v \). So, let \( n \) be the smallest positive integer greater than \( v \). Putting \( \mu = n - v \), obviously \( 0 < Re(\mu) \leq 1 \) and the fractional derivative of order \( v \) can be defined as

\[
a^D_z^\nu f(z) = a^D_z^{n-\mu} f(z) = a^D_z^n [ a^D_z^\mu f(z) ] = \frac{d^n}{dz^n} \frac{1}{\Gamma(\mu)} \int_a^z (z-t)^{\mu-1} f(t) \, dt
\]

...\( (1.7.11) \)

for \( z > 0 \), provided that it exists.

In particular, if \( \alpha = 0 \) and \( f(z) = z^\gamma \), \( Re(\gamma) > -1 \), then

\[
o^D_z^\nu [z^\gamma] = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma - v + 1)} z^{\gamma-v}
\]

...\( (1.7.12) \)

where \( Re(\nu) > 0, Re(\gamma) > -1, z > 0 \).

Pair of operators generalizing the operators \( (1.7.9) \) and \( (1.7.10) \) are known as Erdélyi-Kober operators given by

\[
E^{\alpha,\eta}_{0,z} [f(x)] = \frac{z^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} t^\eta f(t) \, dt
\]

...\( (1.7.13) \)

\[
K^{\alpha,\eta}_{z,\infty} [f(x)] = \frac{z^\eta}{\Gamma(\alpha)} \int_z^\infty (t-z)^{\alpha} t^{-\alpha-\eta} f(t) \, dt
\]

...\( (1.7.14) \)

where \( Re(\alpha) > 0 \).

In 1966, Saxena [203] introduced a generalization of the Erdélyi-Kober operators in the following form

\[
I[f(z)] = I[\alpha, \beta, \gamma, m; f(z)] = \frac{z^{-\mu-1}}{\Gamma(1-\alpha)} \int_0^z 2F_1 \left( \alpha, \beta + m; \beta; \frac{t}{z} \right) t^\mu f(t) \, dt
\]

...\( (1.7.15) \)

and
\[ J[f(z)] = J[\alpha, \beta, \delta, m; f(z)] = \frac{z^{\delta}}{\Gamma(1 - \alpha)} \int_{z}^{\infty} 2F_1 \left( \alpha, \beta + m; \beta; \frac{z^\delta}{t} \right) t^{-\delta-1} f(t) dt \]

...(1.7.16)

where \( \alpha, \beta, \gamma \) are complex numbers, \( \text{Re}(1 - \alpha) > m \).

Saxena and Kumbhat ([211], [212]) defined and studied another pair of fractional integral operators associated with Gauss’s hypergeometric function as follows

\[ R \left[ \alpha, \beta, \mu; f(z) \right] = \frac{z^{-\sigma - \rho}}{\Gamma(\rho)} \int_{0}^{z} t^{\sigma} (z - t)^{\rho - 1} 2F_1 \left( \alpha, \beta; \mu; a \left( 1 - \frac{z}{t} \right) \right) f(t) dt \]

...(1.7.17)

\[ K \left[ \delta, \rho, \alpha; f(z) \right] = \frac{z^{\delta}}{\Gamma(\rho)} \int_{z}^{\infty} t^{-\delta - \rho} (t - z)^{\rho - 1} 2F_1 \left( \alpha, \beta; \mu; a \left( 1 - \frac{z}{t} \right) \right) f(t) dt \]

...(1.7.18)

Saigo [157] gave another modification of classical fractional integration and differentiation operators and introduced two kinds of fractional integrals as follows

\[ I_{0,z}^{\alpha,\beta,\eta} [f(z)] = \frac{z^{-\alpha - \beta}}{\Gamma(\alpha)} \int_{0}^{z} (z - t)^{\alpha - 1} 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{z} \right) f(t) dt \]

\[ = \frac{d^{r}}{dz^{r}} I_{0,z}^{\alpha+r,\beta-r,\eta-r} f(z), 0 < \text{Re}(\alpha) + r \leq 1, r = 1,2, ... \]

...(1.7.19)

and

\[ J_{z,\infty}^{\alpha,\beta,\eta} [f(z)] = \frac{1}{\Gamma(\alpha)} \int_{z}^{\infty} (t - z)^{\alpha - 1} t^{-\alpha - \beta} 2F_1 \left( \alpha + \beta, \eta; \alpha; 1 - \frac{z}{t} \right) f(t) dt \]

\[ = (-1)^{r} \frac{d^{r}}{dz^{r}} J_{z,\infty}^{\alpha+r,\beta-r,\eta-r} f(z), 0 < \text{Re}(\alpha) + r \leq 1, r = 1,2, ... \]

...(1.7.20)

A number of persons have also studied and applied the theory of fractional calculus operators in obtaining certain fruitful results. The research papers contributed by Saigo and Saxena [165], Samko, Kilbas and Marichev [231], Saigo, Raina and Kilbas [164], Raina and Koul [201], Manocha and Sharma [75], Kilbas and Saigo ([16],[17]), Gupta and Soni [137], Banerji and Choudhari [190], Srivastava and Owa [98], Srivastava [76], Gupta, Goyal and Tariq [144], Gupta, Goyal and Garg [141], Chaurasia and Godika ([254],[255]), Chaurasia and Singhal...
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Chaurasia and Gupta [261], Chaurasia and Pandey [262], Chaurasia and Parihar [259], Chaurasia and Srivastava ([256], [257]) are worth mentioning.

The NASA STI (National Aeronautics and Space Administration Scientific and Technical Information) USA, report series TP (Technical Publication) and TM (Technical Memorandum) contain a huge amount of useful research material on practical applications of fractional calculus ([247], [248], [249], [35], [36]).

The fractional calculus finds use in many fields of science and engineering including the fluid flow, rheology, quantitative biology, electro-chemistry, scattering theory, diffusion transport theory, chemical physics and statistical probability theory, potential theory and many branches of mathematical analysis, like integral and differential equations, operational calculus and univalent function theory. Holmgren [72] made use of the fractional integral to the solution of ordinary differential equation. Bagley [216], Bagley and Torvik [217] found the application of fractional calculus in visco-elasticity and electrochemistry of corrosion. Oldham and Spanier ([131], [132]) explained its applications in electrochemistry and general transport problem. Virtually no area of classical analysis is left untouched by fractional calculus.

In Chapter 3 of this thesis, we study a pair of unified and extended fractional integral operators involving the multivariable H-function, I-function and general class of polynomials.

Also in Chapter 4, we have used the generalized fractional integral operators (The Saigo operators [157]) involving Gaussian hypergeometric function defined by:

\[
\left( J_{0+}^{\alpha,\beta,n} f \right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right) f(t) dt ; \quad \text{Re}(\alpha) > 0
\]

\[\text{...(1.7.21)}\]

and

\[
\left( J^{\alpha,\beta,n} f \right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} 2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{X}{t} \right) f(t) dt ; \quad \text{Re}(\alpha) > 0
\]

\[\text{...(1.7.22)}\]

In chapter 5, we have used N-fractional calculus defined by Nishimoto [126] to establish the N-fractional calculus of product of a general class of functions and I function & \( \bar{H} \)-function.
1.8 Chapterwise Summary

The Second chapter is dedicated to the review of literature done. The motive of this exhaustive review is to study the research papers of the field of study deeply and to find the contribution of the papers and the gaps in those to find our problem of further research work. The summary of each paper reviewed is presented, Problem Statement and its objectives are also given in the chapter.

Third chapter is devoted to the study of a pair of unified and extended fractional integral operators involving the multivariable H-Function, I-Function and general class of polynomials. During the course of study, we establish five theorems pertaining to Mellin transforms of these operators. Further, some properties of these operators have also been investigated.

In the Fourth chapter, we study and develop the generalized fractional integral operators given by Saigo [157, 158, 159]. During the course of study, we establish two theorems that give the images of the product of I-function and general class of polynomials in Saigo operators. On account of the general nature of the Saigo operators, I-function and a general class of polynomials, a large number of new and known images involving Riemann-Liouville and Erdélyi-Kober fractional integral operators and several special functions follow as special cases of our main findings.

The chapter 5 is divided into two sections – A and B. In section A, we establish two theorems pertaining to N-fractional calculus of product of the general class of functions due to Kumar [251] and I-function. Next, in section B, we establish two theorems pertaining to N-fractional calculus of product of Kumar’s general class of functions [251] and $\bar{H}$ - Function. Due to the general nature of the functions involved herein, the main results provide useful extension and unification of a number of results obtained earlier in the literature.

In the Sixth chapter, we establish seven integral formulas involving product of the I-function [268] and Fox - Wright’s Generalized Hypergeometric Function [46]. Being unified and general in nature, these integrals yield a number of known and new results as particular cases.

Seventh chapter has been divided into two sections- A and B. In the section A, we establish three integrals involving Srivastava’s Polynomials [78], Aleph ($\aleph$)- Function [174] and Jacobi polynomials [43]. Next, in section B, we establish three theorems involving Srivastava’s Polynomials, Aleph ($\aleph$)- Function and Gauss hypergeometric function [133, 150].
On account of general nature of the functions and polynomials involved therein, our results yield a large number of new and known results involving simpler special functions and polynomials, which may find useful applications in the field of Science and Engineering.

In the Eighth chapter, we evaluate a general class of multiple Eulerian integral with integrands involving a product of general class of polynomials, a general sequence of functions and the multivariable H-function with general arguments. On account of most general nature of the functions and polynomials involved in the integral, our result provide interesting unifications and generalizations of a large number of new and known results.

In the Ninth chapter, we establish four theorems exhibiting the comparative study of Sumudu and Laplace transforms of $\tilde{H}$-function. On account of general nature of the $\tilde{H}$-function, the results obtained here are capable of yielding a very large number of results (new and known) in terms of simpler functions.

In the tenth chapter, we present a Mathematical Model involving the I-Function to study the effect of environmental pollution on the growth and existence of Biological Populations. The results established in this chapter, being general in nature, yield a numerous interesting cases in terms of simpler functions on suitable specifications of parameters involved therein.

The results obtained during the course of study, are discussed in chapter eleven and the summary of the work is presented in the chapter twelve.

With the hope that the subject matter presented in this thesis may be new, interesting and useful, it is being submitted for the award of Ph.D. degree.