Appendix - II

II.1 The $H$-Function

The $H$-function is defined and represented in the following manner [13, 14]):

$$
H_{P,Q}^{M,N} [z] = H_{P,Q}^{M,N} \left\{ \left( \frac{a_j, \alpha_j; A_j}_{1,M} \right) \frac{1}{(b_j, \beta_j; B_j)}_{M+1,Q} \right\} = \frac{1}{2\pi\omega} \int_{-\infty}^{\infty} \psi(\xi) z^{\xi} d\xi
$$

... (II.1.1)

where

$$\omega = \sqrt{-1}$$

and

$$
\psi(\xi) = \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{N} \Gamma(1 - a_j + \alpha_j \xi)}{\prod_{j=M+1}^{Q} \Gamma(1 - b_j - \beta_j \xi) \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j \xi)}
$$

... (II.1.2)

which contains fractional powers of some of the gamma functions.

Here $a_j (j = 1, ..., P)$ and $b_j (j = 1, ..., Q)$ are complex parameters, $\alpha_j \geq 0$ ($j = 1, ..., P$), $\beta_j \geq 0$ ($j = 1, ..., Q$) (not all zero simultaneously) and the exponents $A_j (j = 1, ..., N)$ and $B_j (j = M + 1, ..., Q)$ can take non-integer values.

The contour in (II.1.1) is imaginary axis $Re (\xi) = 0$. It is suitably indented in order to avoid the singularities of the gamma functions and to keep those singularities on appropriate sides.

Again, for $A_j (j = 1, ..., N)$ not an integer, the poles of the gamma function of the numerator in (II.1.2) are converted to branch points. However, as long as there is no coincidence of poles from any $\Gamma(b_j - \beta_j \xi), (j = 1, ..., M)$ and $\Gamma(1 - a_j + \alpha_j \xi), (j = 1, ..., N)$ pair, the branch cuts can be chosen so that the path of integration can be distorted in the usual manner.
The following sufficient conditions for the absolute convergence of the defining integral for the $H$-function given by equation (II.1.1) have been given by Buschman and Srivastava [199].

$$T = \sum_{j=1}^{M} \beta_j + \sum_{j=1}^{N} |A_j\alpha_j| - \sum_{j=M+1}^{Q} |B_j\beta_j| - \sum_{j=N+1}^{P} \alpha_j > 0$$

... (II.1.3)

and

$$|\arg z| < \frac{1}{2} \pi T$$

... (II.1.4)

The behaviour of the $H$-function for small values of $|z|$ follows easily due to Inayat-Hussain ([14], p. 4119-4128), see also [22]:

$$H_{P,Q}^{M,N}(z) = O(|z|^{\alpha}),$$

$$\alpha = \min_{1 \leq j \leq M} \left[ \text{Re} \left( \frac{b_j}{\beta_j} \right) \right], |z| \to 0$$

... (II.1.5)

Again, due to Inayat-Hussain ([14], pp.4119-4128) for large values of $z$, we have

$$H_{P,Q}^{M,N}(z) = O(|z|^{\beta}),$$

$$\beta = \max_{1 \leq j \leq N} \left[ \text{Re} \left( A_j \left( \frac{a_j - 1}{\alpha_j} \right) \right) \right]$$

... (II.1.6)

The series representation of $H$-function ([136], p. 271, Eq. (6)) is as follows:

$$H_{P,Q}^{M,N}(z) \left\{ \begin{array}{l}
(a_j, \alpha_j; A_j)_{1,N'} (a_j, \alpha_j)_{N+1,P} \\
(b_j, \beta_j)_{1,M'} (b_j, \beta_j; B_j)_{M+1,Q}
\end{array} \right\}$$

$$= \sum_{h=1}^{M} \sum_{\gamma=0}^{Q} \prod_{j=M+1}^{P} \Gamma(1 - b_j + \beta_j \xi_{h,r}) \prod_{j=1}^{N} \Gamma(1 - a_j + \alpha_j \xi_{h,r}) \prod_{j=1}^{M} \Gamma(b_j - \beta_j \xi_{h,r}) \prod_{j=1}^{N} \Gamma(1 - a_j + \alpha_j \xi_{h,r}) \prod_{j=1}^{M} \Gamma(1 - b_j + \beta_j \xi_{h,r}) \prod_{j=1}^{N} \Gamma(a_j - \alpha_j \xi_{h,r}) r! \beta_h$$

... (II.1.7)

where
\[ \xi_{h,r} = \frac{b_h + r}{\beta_h} \]

... (II.1.8)

**II.2 Special Cases**

(i) If we take \( A_j = 1 \) \((j = 1, \ldots, N)\) and \( B_j = 1 \) \((j = M + 1, \ldots, Q)\) in (II.1.1.), the \( \tilde{H} \)-function reduces to the well-known Fox’s H-function [34].

(ii) On setting \( A_j = B_j = \alpha_j = \beta_j = 1 \) in (II.1.1), the \( \tilde{H} \)-function reduces to the Meijer’s G-function [39].

It is noteworthy that the following functions which are quite general in nature and of interest in themselves, are special cases of \( \tilde{H} \)-function but not of Fox’s H-function:

(iii) On specifying the parameters in (II.1.6) suitably and then making some obvious changes therein, we get

\[ \tilde{H}^{M,N}_{P,Q} \left\{-z \right\}_{(0,1), (1 - b_j, \beta_j; B_j)_{1,Q}}^{(1 - a_j, \alpha_j; A_j)_{1,P}} = r \tilde{\psi}_Q \left[ (a_j, \alpha_j; A_j)_{1,P}; (b_j, \beta_j; B_j)_{1,Q}; z \right] \]

\[ = \sum_{r=0}^{\infty} \prod_{j=1}^{P} \left\{ \Gamma(a_j + \alpha_j r) \right\}^{A_j} \prod_{j=1}^{Q} \left\{ \Gamma(b_j + \beta_j r) \right\}^{B_j} r! \]

... (II.2.1)

where \( r \tilde{\psi}_Q \) [136] is termed as generalized Wright hypergeometric function because it gives \( r \psi_Q \) ([83], p.19, Eq.(2.6.11)) for \( A_j = 1 \) \((j = 1, \ldots, P)\) and \( B_j = 1 \) \((j = 1, \ldots, Q)\) in it.

(iv) Further, on taking \( \alpha_j = 1 \) \((j = 1, \ldots, P)\) and \( \beta_j = 1 \) \((j = 1, \ldots, Q)\) in (II.2.1), we arrive at

\[ \tilde{H}^{M,N}_{P,Q} \left\{-z \right\}_{(0,1), (1 - b_j, 1; B_j)_{1,Q}}^{(1 - a_j, 1; A_j)_{1,P}} = \prod_{j=1}^{P} \left\{ \Gamma(a_j) \right\}^{A_j} \prod_{j=1}^{Q} \left\{ \Gamma(b_j) \right\}^{B_j} r \tilde{F}_Q \left[ (a_j, 1; A_j)_{1,P}; (b_j, 1; B_j)_{1,Q}; z \right] \]

\[ = \sum_{r=0}^{\infty} \prod_{j=1}^{P} \left\{ \Gamma(a_j + r) \right\}^{A_j} \prod_{j=1}^{Q} \left\{ \Gamma(b_j + r) \right\}^{B_j} r! \]

... (II.2.2)

where the function \( r \tilde{F}_Q \) [136] reduces to well known \( r F_Q \) function for \( A_j = 1 \) \((j = 1, \ldots, P)\) and \( B_j = 1 \) \((j = 1, \ldots, Q)\) in it.
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(v) Now, we mention a function $g_1$ ([139], p.98, Eq. (1.3); [14], p.4125, Eq. (20)) which is connected with a certain class of Feynman integrals and is also a special case of $\rho F_Q$. We have $g_1 = (-1)^p g(\gamma, \eta, \tau, p; z)$

\[
g_1 = \frac{k_{d-1} p \Gamma \left(1 + \frac{\tau}{2} \right) B \left(\frac{1}{2}, \frac{1}{2} + \frac{\tau}{2} \right)}{2^{2-p} \pi \Gamma(\gamma) \Gamma(\gamma - \frac{\tau}{2})} \times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(-\xi) \Gamma(\gamma + \xi) \Gamma(\gamma - \frac{\tau}{2} + \xi) (-z)^\xi}{(\eta + \xi)^{1+p}} \frac{d\xi}{\Gamma(1 + \frac{\tau}{2} + \xi)}
\]

\[
g_1 = \frac{k_{d-1} \Gamma(p+1) \Gamma \left(\frac{1}{2} + \frac{\tau}{2} \right)}{2^{2-p} \pi \Gamma(\gamma) \Gamma(\gamma - \frac{\tau}{2})} \times \mathcal{R}_{1,3}^{1,3} \left\{ \begin{array}{c} (1 - \gamma, 1; 1), (1 - \gamma + \frac{\tau}{2}, 1; 1), (1 - \eta, 1; 1 + p) \\ (0,1), \left(-\frac{\tau}{2}, 1; 1\right), (-\eta, 1; 1 + p) \end{array} \right\}
\]

where

\[
k_d = \frac{2^{1-d} \pi^{d/2}}{\Gamma \left(\frac{d}{2} \right)} \quad ([2], p. 4121, Eq. (5))
\]

(vi) The following function is the exact partition function of the Gaussian model in statistical mechanics [139]:

\[
\beta F(d; \epsilon) = -\frac{1}{4\pi \epsilon (1 + \epsilon)^2} \times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{(-1 + \epsilon)^{-2} \Gamma(-\xi) \Gamma(1 + \xi)}{\Gamma(2 + \xi)} \left[ \frac{\Gamma \left(\frac{3}{2} + \xi\right)}{\Gamma(-\xi)} \right]^d \frac{d\xi}{\Gamma(2 + \xi)}
\]

\[
= -\frac{1}{4\pi \epsilon (1 + \epsilon)^2} \mathcal{H}_{2,2}^{1,2} \left\{ (1 + \epsilon)^{-2} \left| (0,1; 2), \left(-\frac{\tau}{2}, 1; d\right) \right| (0,1), (-1,1; 1 + d) \right\}
\]

\[
= -\frac{1}{4\pi \epsilon (1 + \epsilon)^2} \mathcal{H}_{3,2}^{1,3} \left\{ (1 + \epsilon)^{-2} \left| (0,1; 1), (0,1; 1), \left(-\frac{\tau}{2}, 1; d\right) \right| (0,1), (-1,1; 1 + d) \right\}
\]

\[ \text{... (II.2.5)} \]