Chapter – 11

Results and Discussions

11.1 Introduction

This chapter presents a brief note on the results established during the course of our study along with the relevant description.

11.2 Results and Discussion

This section presents point wise discussion on the results established in the chapters of this thesis. The notations used for various functions occurring in the following discussions have already been made clear in the relevant chapters.

- Development of new operators
  Fractional integral operators involving various special functions, have found significant importance and applications in various sub-field of applicable mathematical analysis. A lot of research work has recently come up on the study and development of fractional calculus operators to obtain image formulas in terms of several special functions like Meijer’s G-Function, Fox’s H-Function and the $\tilde{H}$-Function that is a generalization of Fox’s H-Function. Motivated by these avenues of applications, in the chapter 3, we study a pair of unified and extended fractional integral operators involving the multivariable H-Function, I-Function and general class of polynomials. During the course of our study, we establish five theorems pertaining to Mellin transforms of these operators. Further, some properties of these operators have also been investigated. On account of the general nature of the functions involved herein, a large number of (known and new) fractional integral operators
involving simpler functions can be obtained. Also, in the chapter 4, we study and develop the generalized fractional integral operators given by Saigo. Our main findings are contained in two theorems which give the images of the product of I-function and a general class of polynomials in Saigo operators, thereby generalizing several important results obtained earlier in the literature.

We have developed the following four operators in these two chapters -

**Operator – 1**

\[
Q_{\gamma_n}^{\alpha,\beta} [f(x)] = tx^{-\alpha-t\beta-1} \int_0^x y^\alpha (x^t - y^t) \times H \left[ \begin{array}{c} y^t_1 \nu \\ \cdots \\ y^t_n \nu \end{array} \right] \\
\times \prod_{j=1}^k \prod_{p_{i,j},q_{i,j},x^t} \left\{ z_j \left( \frac{y^t_i}{x^t} \right)^{a_j} \left( 1 - \frac{y^t_i}{x^t} \right)^{b_j} \left( \frac{e_{i,j}^t, E_{j,i}^t}{1, n_j} \right) \left( \frac{e_{i',j'}^t, E_{j',i'}^t}{n_j+1, p_{i,j}} \right) \right\} \\
\times \prod_{i=1}^r S_{\gamma_i}^{u_i} \left[ z_i \left( \frac{y^t_i}{x^t} \right)^{g_i} \left( 1 - \frac{y^t_i}{x^t} \right)^{h_i} \right] \psi \left( \frac{y^t_i}{x^t} \right) f(y) dy
\]

...(11.2.1)

provided the conditions are satisfied those are mentioned with (3.2.22).

**Operator – 2**

\[
R_{\gamma_n}^{\rho,\beta} [f(x)] = tx^p \int_0^x y^{-\rho-t\beta-1}(y^t - x^t)^\beta \times H \left[ \begin{array}{c} y^t_1 \mu \\ \cdots \\ y^t_n \mu \end{array} \right] \\
\times \prod_{j=1}^k \prod_{p_{i,j},q_{i,j},x^t} \left\{ z_j \left( \frac{x^t}{y^t} \right)^{a_j} \left( 1 - \frac{x^t}{y^t} \right)^{b_j} \left( \frac{e_{i,j}^t, E_{j,i}^t}{1, n_j} \right) \left( \frac{e_{i',j'}^t, E_{j',i'}^t}{n_j+1, p_{i,j}} \right) \right\} \\
\times \prod_{i=1}^r S_{\gamma_i}^{u_i} \left[ z_i \left( \frac{x^t}{y^t} \right)^{g_i} \left( 1 - \frac{x^t}{y^t} \right)^{h_i} \right] \psi \left( \frac{x^t}{y^t} \right) f(y) dy
\]

...(11.2.2)

provided the conditions are satisfied those are mentioned with (3.2.23).
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Operator – 3

\[
\left[ j^{\alpha, \beta, \eta} \right] \left[ \sum_{j=1}^{k} S_{j}^{m_{j}} \left[ c_{j} t^{\lambda_{j}} \right] \right]_{P_{1}, Q_{1}: R}^{M,N} \left[ \left( a_{j', \alpha_{j'}}, \alpha_{j'}, \alpha_{j'} \right)_{1,N'}^{(a_{j', \alpha_{j'}}, \alpha_{j'} \alpha_{j'})_{N+1,P_{1}}} \right] (x)
\]

\[
= x^{\mu-\beta-1} \sum_{l_{1}=0}^{[n_{1}/m_{1}]} \sum_{l_{2}=0}^{[n_{2}/m_{2}]} \sum_{l_{k}=0}^{[n_{k}/m_{k}]} \left\{ \frac{(-n_{1})_{m_{1}l_{1}} \ldots (-n_{h})_{m_{h}l_{h}}}{l_{1}! \ldots l_{h}!} \right\}
\]

\[
A_{n_{1}, l_{1}}^{(k)} a_{n_{h}, l_{h}} c_{1} \ldots c_{h} x^{\sum_{j=1}^{k} \lambda_{j} l_{j}}
\]

\[
x^{\mu-\beta-1} \left[ l_{P_{1}^{M+2}} ^{Q_{1}^{N+2}} \left( 1 - \mu - \sum_{j=1}^{k} \lambda_{j} l_{j}, v; 1 \right) \right] (b_{j', \beta_{j'}}, \beta_{j'} \beta_{j'} \gamma_{M+1,Q_{1}}^{1})
\]

\[
\left( 1 - \mu - \eta + \beta - \sum_{j=1}^{k} \lambda_{j} l_{j}, v; 1 \right), \left( a_{j', \alpha_{j'}}, \alpha_{j'} \alpha_{j'} \gamma_{N+1,P_{1}}^{1} \right)
\]

\[
\left( 1 - \mu + \beta - \sum_{j=1}^{k} \lambda_{j} l_{j}, v; 1 \right), \left( 1 - \mu - \alpha - \eta - \sum_{j=1}^{k} \lambda_{j} l_{j}, v; 1 \right)
\]

\[
\ldots (11.2.3)
\]

provided the conditions are satisfied those are mentioned with (4.4.1).

Operator – 4

\[
\left[ j^{\alpha, \beta, \eta} \right] \left[ \sum_{j=1}^{k} S_{j}^{m_{j}} \left[ c_{j} t^{\lambda_{j}} \right] \right]_{P_{1}, Q_{1}: R}^{M,N} \left[ \left( a_{j', \alpha_{j'}}, \alpha_{j'}, \alpha_{j'} \right)_{1,N'}^{(a_{j', \alpha_{j'}}, \alpha_{j'} \alpha_{j'})_{N+1,P_{1}}} \right] (x)
\]

\[
= x^{\mu-\beta-1} \sum_{l_{1}=0}^{[n_{1}/m_{1}]} \sum_{l_{2}=0}^{[n_{2}/m_{2}]} \sum_{l_{k}=0}^{[n_{k}/m_{k}]} \left\{ \frac{(-n_{1})_{m_{1}l_{1}} \ldots (-n_{h})_{m_{h}l_{h}}}{l_{1}! \ldots l_{h}!} \right\}
\]

\[
A_{n_{1}, l_{1}}^{(k)} a_{n_{h}, l_{h}} c_{1} \ldots c_{h} x^{\sum_{j=1}^{k} \lambda_{j} l_{j}}
\]
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\[ \times I_{P_i+2, Q_i+2: R}^{M,N+2} \left\{ \left( \mu - \beta + \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right), \left( b_j', \beta_j' \right)_{1, N}; \left( b_j', \beta_j' \right)_{M+1, Q_i} \right\} \]

\[ \times X^\nu \left\{ \left( \mu - \eta + \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right), \left( a_j', \alpha_j' \right)_{1, N}; \left( a_j', \alpha_j' \right)_{N+1, P_i} \right\} \]

\[ \times \left\{ \left( \mu + \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right), \left( \mu - \alpha - \beta - \eta - \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right) \right\} \]

\[ \ldots (11.2.4) \]

provided the conditions are satisfied those are mentioned with (4.4.2).

- **Establishment of certain properties in operational calculus**

  The Fractional calculus has significant role in studies of viscoelastic materials, as well as in many fields of science and engineering including fluid flow, rheology, diffusive transport, electrical networks, electromagnetic theory and probability. In the chapter 5 we establish two theorems pertaining to N-fractional calculus of product of a general class of functions and \( I \)-function and two theorems pertaining to N-fractional calculus of product of a general class of functions and \( \tilde{H} \)-function. Due to the general nature of the functions involved herein, the main results established in this chapter provide useful extension and unification of a number of (known or new) results lying hitherto in the literature.

The results obtained are -

**Result - 1**

\[ z^\nu V_n^{h_m,d,g_j} \left[ p, \tau, k, w, q, k_m, a_j, b_r, \alpha, \beta, \delta; \zeta z^\mu \right] \]

\[ \times I_{P_i,Q_i: R}^{M,N} \left\{ a z^\sigma \left( a_j', \alpha_j' \right)_{1, N}; \left( a_j', \alpha_j' \right)_{N+1, P_i} \right\} v \]

\[ = (-1)^v \sum_{n=0}^{\infty} \frac{(-p)^n}{\prod_{j=1}^{\tau} (g_j, n + a_j)} \frac{\prod_{m=1}^{\nu} \left( (h_m, n) + k_m \right)}{\prod_{r=1}^{\mu} \left( (d, an\delta + b_r) \right)^{n_k + dw + q}} \]
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\[ x \times z^{\rho + \mu \eta + \mu d w + \mu q - v} \]

\[ \times I_{P_l+1,Q_l+1;R}^{M+1,N} \left\{ a z^\sigma \left[ (a_j', \alpha_j')_{1,N}; (a_j', \alpha_j')_{N+1,P_l} \right] \begin{array}{l} \left( v - \rho - \mu \eta - \mu d w - \mu q, \sigma \right) \end{array} \right\} \left( b_j', \beta_j' \right)_{1,M}; \left( b_j', \beta_j' \right)_{M+1,Q_l} \right\} \]

provided the conditions are satisfied those are mentioned with (5.3.1).

**Result - 2**

\[ \left[ z^\rho V^{h_m,d,j}_n \left[ p, k, w, q, k_m, a_j, b_r, \alpha, \beta, \delta; \zeta z^\mu \right] \right] \]

\[ \times I_{P_l,Q_l;R}^{M,N} \left\{ a z^\sigma \left[ (a_j', \alpha_j')_{1,N}; (a_j', \alpha_j')_{N+1,P_l} \right] \begin{array}{l} \left( v - \rho - \mu \eta - \mu d w - \mu q, \sigma \right) \end{array} \right\} \left( b_j', \beta_j' \right)_{1,M}; \left( b_j', \beta_j' \right)_{M+1,Q_l} \right\} \]

\[ = \left[ z^\rho V^{h_m,d,j}_n \left[ p, k, w, q, k_m, a_j, b_r, \alpha, \beta, \delta; \zeta z^\mu \right] \right] \]

\[ \times I_{P_l,Q_l;R}^{M,N} \left\{ a z^\sigma \left[ (a_j', \alpha_j')_{1,N}; (a_j', \alpha_j')_{N+1,P_l} \right] \begin{array}{l} \left( v - \rho - \mu \eta - \mu d w - \mu q, \sigma \right) \end{array} \right\} \left( b_j', \beta_j' \right)_{1,M}; \left( b_j', \beta_j' \right)_{M+1,Q_l} \right\} \]

\[ = \left[ z^\rho V^{h_m,d,j}_n \left[ p, k, w, q, k_m, a_j, b_r, \alpha, \beta, \delta; \zeta z^\mu \right] \right] \]

\[ \times I_{P_l,Q_l;R}^{M,N} \left\{ a z^\sigma \left[ (a_j', \alpha_j')_{1,N}; (a_j', \alpha_j')_{N+1,P_l} \right] \begin{array}{l} \left( v - \rho - \mu \eta - \mu d w - \mu q, \sigma \right) \end{array} \right\} \left( b_j', \beta_j' \right)_{1,M}; \left( b_j', \beta_j' \right)_{M+1,Q_l} \right\} \]

provided the conditions are satisfied those are mentioned with (5.3.3).
Result - 3

\[
\begin{align*}
&z^\rho V_n^{h.m.d.gj}[p, \tau, k, w, q, k_m, a_j, b_r, \alpha, \beta, \delta; \zeta z^\mu] \\
&\times \hat{H}_{P,Q}^{M',N'} \left\{ az^\sigma \left( \begin{array}{c}
(a_j, \alpha_j; A_j)_{1,N'}; (a_j, \alpha_j)_{N'+1,P} \\
(b_j, \beta_j; B_j)_{1,M'}; (b_j, \beta_j; B_j)_{M'+1,Q}
\end{array} \right) \right\}_{v'}
\end{align*}
\]

\[= (-1)^{\nu} \lambda \sum_{n=0}^{\infty} (-p)^n \prod_{m=1}^{t} [(h_m)_n + k_m] (d + an + \beta)^{-\tau} \left( \frac{\zeta}{\nu} \right)^{n(k+dw+q)}
\]

\[\times z^{\rho+\mu n k+\mu dw+\mu q-v} \times \hat{H}_{P+1,Q+1}^{M'+1,N'} \left\{ az^\sigma \left( \begin{array}{c}
(a_j, \alpha_j; A_j)_{1,N'}; (a_j, \alpha_j)_{N'+1,P'} \\
(v - \rho - \mu n k - \mu dw - \mu q, \alpha) \\
(b_j, \beta_j; B_j)_{1,M'}; (b_j, \beta_j; B_j)_{M'+1,Q}
\end{array} \right) \right\}_{v'}
\]

provided the conditions are satisfied those are mentioned with (5.7.1).

Result - 4

\[
\begin{align*}
&z^\rho V_n^{h.m.d.gj}[p, \tau, k, w, q, k_m, a_j, b_r, \alpha, \beta, \delta; \zeta z^\mu] \\
&\times \hat{H}_{P,Q}^{M,N} \left\{ az^\sigma \left( \begin{array}{c}
(a_j, \alpha_j; A_j)_{1,N'}; (a_j, \alpha_j)_{N+1,P} \\
(b_j, \beta_j; B_j)_{1,M'}; (b_j, \beta_j; B_j)_{M+1,Q}
\end{array} \right) \right\}_{v'}
\end{align*}
\]

\[= \left[ z^\rho V_n^{h.m.d.gj}[p, \tau, k, w, q, k_m, a_j, b_r, \alpha, \beta, \delta; \zeta z^\mu] \\
\times \hat{H}_{P,Q}^{M,N} \left\{ az^\sigma \left( \begin{array}{c}
(a_j, \alpha_j; A_j)_{1,N'}; (a_j, \alpha_j)_{N+1,P} \\
(b_j, \beta_j; B_j)_{1,M'}; (b_j, \beta_j; B_j)_{M+1,Q}
\end{array} \right) \right\}_{v'} \right]
\]
provided the conditions are satisfied those are mentioned with (5.7.2).

- **Integral formulas involving special functions**

  The I-Function is the generalized form of the hypergeometric function in one argument and it contains an important class of symmetric Fourier kernel of a very general nature and a vast number of well-known analytic functions as special cases like Fox’s H-Function and Meijer’s G-Function which are also generalization of many higher transcendental functions. Therefore the results obtained in the chapter 6 are useful in obtaining many new results involving the products of simple commonly used functions appearing in mathematical analysis both pure and applied.

  Aleph ($\aleph$)- Function occurs naturally in certain problems of fractional driftless Fokker-Planck equations. Further, on account of the importance and considerable popularity achieved due to its applications in various fields of science and engineering, such as fluid flow, Rheology, diffusive transport akin to diffusion, electric networks and probability, the Aleph ($\aleph$)- Function has become a topic of interest in recent days. In the chapter 7, we establish certain integrals and theorems involving Srivastava’s Polynomials and Aleph ($\aleph$)- Function. The importance of our main results of this chapter lies in their many fold generality. On account of general nature of the functions and polynomials involved in the results, our results provide interesting unifications and generalizations of a large number of new and known results, which may find useful applications in the field of Science and Engineering.

  The following integral formulas in terms of I function and Aleph function have been established in the chapters 6 and 7 -
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**Result - 1**

\[
\int_0^t x^{\beta-1}(t-x)^{\sigma-1}e^{-xz} p_\psi q \left[ \frac{(e_j, y_j)_{1,p}}{(f_j, \delta_j)_{1,q}} ; ax^\xi(t-x)^\eta \right]
\]

\[
\times I_{p_\psi q;r}^{m,n} \left\{ yx^\mu(t-x)^\nu \left\{ (a_j, \alpha_j)_{1,n} ; (a_{ji}, \alpha_{ji})_{n+1,p_i} \right\} dx \right. \\
= e^{-zt}t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{u} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\xi+\eta-1)k+u} \\
\times I_{p_\psi q;r+1}^{m,n+2} \left\{ yt^{\mu+v} \left\{ (a_j, \alpha_j)_{1,n} ; (a_{ji}, \alpha_{ji})_{n+1,p_i} \right\} \right. \\
\left. \left( 1 - \rho - \sigma - (\xi + \eta - 1)k - u, \mu + \nu \right) \right\} \\
\]

provided the conditions are satisfied those are mentioned with (6.3.1).

**Result - 2**

\[
\int_0^t x^{\beta-1}(t-x)^{\sigma-1}e^{-xz} p_\psi q \left[ \frac{(e_j, y_j)_{1,p}}{(f_j, \delta_j)_{1,q}} ; ax^\xi(t-x)^\eta \right]
\]

\[
\times I_{p_\psi q;r}^{m,n} \left\{ yx^{-\mu}(t-x)^{\nu} \left\{ (a_j, \alpha_j)_{1,n} ; (a_{ji}, \alpha_{ji})_{n+1,p_i} \right\} dx \right. \\
= e^{-zt}t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{u} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\xi+\eta-1)k+u} \\
\times I_{p_\psi q;r+1}^{m,n+2} \left\{ yt^{-\mu-v} \left\{ (a_j, \alpha_j)_{1,n} ; (a_{ji}, \alpha_{ji})_{n+1,p_i} \right\} \right. \\
\left. \left( \rho + \xi k, \mu \right) \left( \sigma + (\eta - 1)k + u, \nu \right) \right\} \\
\]

\[
\left( \rho + \sigma + (\xi + n - 1)k + u, \mu + \nu \right) \\
\left( b_j, \beta_j \right)_{1,n} ; \left( b_{ji}, \beta_{ji} \right)_{n+1,q_i} \\
\]

provided the conditions are satisfied those are mentioned with (6.3.3).
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Result – 3

\[ \int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{-xz} p\psi_q \left[ \frac{(e_j, \gamma_j)_{1,p}}{(f_j, \delta_j)_{1,q}}; ax^\gamma(t-x)^\eta \right] \]

\times I_{p_i,q_i:r}^{m,n} \left\{ yx^\mu(t-x)^\nu \frac{(a_j, \alpha_j)_{1,n}}{(b_j, \beta_j)_{1,m}} \right\} dx 

= e^{-zt}t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{u} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta+\eta-1)k+u} 

\times I_{p_i+2,q_i+1:r}^{m+1,n+1} \left\{ yt^{\mu-\nu} \frac{(1-\rho-\zeta k, \mu, (a_j, \alpha_j)_{1,n}, (a_{ji}, \alpha_{ji})_{n+1,p_i})}{(\sigma + (\eta - 1)k + u, \nu)} \right\} 

\left( \rho + \sigma + (\zeta + \eta - 1)k + u, \nu - \mu \right) 

\left( b_j, \beta_j \right)_{1,m} 

\left( b_{ji}, \beta_{ji} \right)_{m+1,q_i} 

...(11.2.11)

provided the conditions are satisfied those are mentioned with (6.3.4).

Result – 4

\[ \int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{-xz} p\psi_q \left[ \frac{(e_j, \gamma_j)_{1,p}}{(f_j, \delta_j)_{1,q}}; ax^\gamma(t-x)^\eta \right] \]

\times I_{p_i,q_i:r}^{m,n} \left\{ yx^\mu(t-x)^\nu \frac{(a_j, \alpha_j)_{1,n}}{(b_j, \beta_j)_{1,m}} \right\} dx 

= e^{-zt}t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{u} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta+\eta-1)k+u} 

\times I_{p_i+2,q_i+1:r}^{m+1,n+1} \left\{ yt^{\mu-\nu} \frac{(1-\sigma - (\eta - 1)k - u, \nu, (a_j, \alpha_j)_{1,n})}{(\rho + \zeta k, \mu, (b_j, \beta_j)_{1,m})} \right\} 

\left( a_{ji}, \alpha_{ji} \right)_{n+1,p_i} \left( \rho + \sigma + (\zeta + \eta - 1)k + u, \mu - \nu \right) 

\left( b_{ji}, \beta_{ji} \right)_{m+1,q_i} 

...(11.2.12)

provided the conditions are satisfied those are mentioned with (6.3.5).
Result – 5

\[
\int_0^t x^{\rho-1}(t-x)^{\sigma-1} e^{-xz} p\psi_q \left[ \frac{(e_j, y_j)_{1,p}}{(f_j, \delta_j)_{1,q}} ; ax^\xi (t-x)^\eta \right] dx
\]

\[
\times I^{m,n}_{p_i,q_i;r} \left\{ yx^{-\mu}(t-x)^{-\nu} \left[ \frac{(a_j, \alpha_j)_{1,n}}{(b_j, \beta_j)_{1,m}} ; (a_{ji}, \alpha_{ji})_{n+1,p_i} \right] \right\} dx
\]

\[
= e^{-zt} t^{\rho + \sigma - 1} \sum_{u=0}^{\infty} \sum_{k=0}^{u} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta + \eta - 1)k + u} \times
\]

\[
\times I^{m+1,n+1}_{p_i+2,q_i+1;r} \left\{ yt^{-\mu} \left[ 1 - \rho - \zeta k, \mu), (a_j, \alpha_j)_{1,n} ; (a_{ji}, \alpha_{ji})_{n+1,p_i} \right] \right\}
\]

\[
(\sigma + (\eta - 1)k + u, \nu),
\]

\[
(\rho + \sigma + (\zeta + \eta - 1)k + u, \nu - \mu)
\]

\[
(b_j, \beta_j)_{1,m} ; (b_{ji}, \beta_{ji})_{m+1,q_i}
\]

provided the conditions are satisfied those are mentioned with (6.3.6).

Result – 6

\[
\int_0^t x^{\rho-1}(t-x)^{\sigma-1} e^{-xz} p\psi_q \left[ \frac{(e_j, y_j)_{1,p}}{(f_j, \delta_j)_{1,q}} ; ax^\xi (t-x)^\eta \right] dx
\]

\[
\times I^{m,n}_{p_i,q_i;r} \left\{ yx^{-\mu}(t-x)^{-\nu} \left[ \frac{(a_j, \alpha_j)_{1,n}}{(b_j, \beta_j)_{1,m}} ; (a_{ji}, \alpha_{ji})_{n+1,p_i} \right] \right\} dx
\]

\[
= e^{-zt} t^{\rho + \sigma - 1} \sum_{u=0}^{\infty} \sum_{k=0}^{u} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta + \eta - 1)k + u} \times
\]

\[
\times I^{m+1,n+1}_{p_i+1,q_i+1;r} \left\{ yt^{-\mu} \left[ (1 - \sigma - (\eta - 1)k - u, \nu),
\right.
\]

\[
(\rho + \zeta k, \mu), (b_j, \beta_j)_{1,m} ; (b_{ji}, \beta_{ji})_{m+1,q_i}
\]

\[
(1 - \rho - \sigma - (\zeta + \eta - 1)k - u, \nu - \mu)
\]

\[
(1 - \rho, \alpha_j)_{1,n} ; (a_{ji}, \alpha_{ji})_{n+1,p_i}
\]

\[
(1 - \rho - \sigma - (\zeta + \eta - 1)k - u, \nu - \mu)
\]

\[
... (11.2.14)
\]
provided the conditions are satisfied those are mentioned with (6.3.7).

**Result – 7**

\[
\int_0^\infty \chi^{\eta-1}e^{ax} P_{\rho,\sigma} \left[ (e_j, \gamma_j)_1,p; (f_j, \delta_j)_1,q; ax^p \right] \times \chi^{h_\sigma} \left\{ (a_j, \sigma_j)_1,m; (a_{j+1} \sigma_{j+1})_m+1,p \right\} \\
\times H_{p,q}^m \left\{ \omega x \left[ (c_j, \zeta_j)_1,n; (c_j, \zeta_j)_n+1,p \right] \right\} dx \\
= \omega^{-\eta} \sum_{u=0}^\infty \sum_{k=0}^u f(k) \frac{a^{u-k}}{(u-k)!} \omega^{-(\rho-1)k-u} \\
\times \chi^{m+n+1+m} \left\{ (a_j, \sigma_j)_1,m; (1 - d_j - (\eta + (\rho - 1)k + u)\tau_j, \sigma_j)_m+1,q \right\} \\
(\chi^{0} \chi^{m+n+1+m} \left\{ (a_{j+1}, \sigma_{j+1})_m+1,p; (b_{j+1}, \beta_{j+1})_m+1,q \right\} \\
\chi^{(11.2.15)} \right) \]

provided the conditions are satisfied those are mentioned with (6.3.8).

**Result – 8**

\[
\int_0^t (1 - x)^{\rho-1} (1 + x)^{\sigma-1} \chi^{m,n} \left[ w(1 - x)^u (1 + x)^v \right] \\
\times \chi^{n+1,n+1} \left\{ (a_{j+1}, \sigma_{j+1})_m+1,p; (b_{j+1}, \beta_{j+1})_m+1,q \right\} dx \\
= 2^{\rho+\sigma-1} \sum_{l_1=0}^{[n_1/m_1]} \sum_{l_2=0}^{[n_2/m_2]} \ldots \sum_{l_\lambda=0}^{[n_\lambda/m_\lambda]} \prod_{g=1}^{[n_g]} \frac{(-n_g)}{l_g!} A_{n_g,l_g} w^{l_g} 2^{(u+v)l_g} \\
\times \chi^{m,n+2} \left\{ (1 - \rho - ul_g, h), (1 - \sigma - vl_g, k), \\
(b_{j+1}, \beta_{j+1})_m+1,q \right\} \\
\chi^{(11.2.15)} \right) \]
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\[
(a_j, A_j)_{1,n} \left[ \tau_i (a_{ji}, A_{ji}) \right]_{n+1, p_i; r} \\
(1 - \rho - \sigma - (u + v)l, h + k)
\]

porvided the conditions are satisfied those are mentioned with (7.3.1).

**Result – 9**

\[
\int_{-1}^{1} (1 - x)^{\rho-1} (1 + x)^{\sigma-1} \ p_{\mu}^{(\alpha, \beta)} (t + y) \ p_{\mu}^{(\alpha, \beta)} (t - y) \\
\times \mathcal{K}_{p_i, q_i; \tau_i; r} \left\{ z(1 - x)^h (1 + x)^k \right\} \left( a_j, A_j \right)_{1,n} \left[ \tau_i (a_{ji}, A_{ji}) \right]_{n+1, p_i; r} \ dx
\]

\[
= 2^{\rho+\sigma-1} (-1)^\mu \Gamma (1 + \alpha + \mu) \Gamma (1 + \beta + \mu) / (\mu!)^2
\]

\[
\times \sum_{\eta=0}^{\mu} \sum_{l=0}^{\eta} \left( -\mu \right)_\eta (1 + \alpha + \beta + \mu)_\eta (-\eta)_l (\eta + \alpha) (\alpha + \beta + \eta + 1)_l / (\alpha + 1)_l
\]

\[
\times \mathcal{K}_{p_i+2, q_i+1, \tau_i; r} \left\{ z \ 2^{(h+k)} \right\} \left( 1 - \rho - l, h \right), (1 - \sigma, k), \\
\left( a_j, A_j \right)_{1,n} \left[ \tau_i (a_{ji}, A_{ji}) \right]_{n+1, p_i; r} \left( 1 - \rho - \sigma - l, h + k \right)
\]

provided the conditions are satisfied those are mentioned with (7.3.3).

**Result – 10**

\[
\int_{-1}^{1} (1 - x)^{\rho-1} (1 + x)^{\sigma-1} \ y (1 - t + y)^{-\alpha} (1 + t + y)^{-\beta} \\
\times \mathcal{K}_{p_i, q_i; \tau_i; r} \left\{ z(1 - x)^h (1 + x)^k \right\} \left( a_j, A_j \right)_{1,n} \left[ \tau_i (a_{ji}, A_{ji}) \right]_{n+1, p_i; r} \ dx
\]

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\[ 2^{-\alpha - \beta + \rho + \sigma - 1} \sum_{\eta=0}^{\mu} \sum_{l=0}^{\eta} \frac{(-\eta)_l}{l!} \sqrt{\eta + \alpha} \left( \frac{\alpha + \beta + \eta + 1}{\alpha + 1} \right)_l \]
\[ \times \mathcal{K}_{m,n+2}^{m,n} \left\{ \begin{array}{c} z^2 (h+k) \\ (b_j, B_j)_{1,m}; [\tau_i(b_{ji}, B_{ji})]_{m+1,q;r} \\
(a_j, A_j)_{1,n}; [\tau_i(a_{ji}, A_{ji})]_{n+1,p;r} \end{array} \right\} \\
(1 - \rho - \sigma - l, h + k) \]  

provided the conditions are satisfied those are mentioned with (7.3.4).

**Result – 11**

\[ \int_0^\infty z^{-\sigma - 1} F_1 \left( \alpha, \beta; \gamma + \frac{1}{2}; z \right) F_1 \left( \gamma - \alpha, \gamma - \beta; \gamma + \frac{1}{2}; z \right) \]
\[ \times \mathcal{S}_{m_1,m_2,\ldots,m_k}^{m_1,m_2,\ldots,m_k} \left[ \prod_{g=1}^{\lambda} z \gamma^{-\mu_g} \mathcal{K}_{m_1,m_2,\ldots,m_k}^{m_1,m_2,\ldots,m_k} [\gamma] \right] dx \]
\[ \frac{\sqrt{\pi}}{2a(4ab + c)^{\sigma + \frac{1}{2}}} \sum_{k=0}^{\infty} \sum_{l_1=0}^{n_1/m_1} \sum_{l_2=0}^{n_2/m_2} \ldots \sum_{l_k=0}^{n_k/m_k} \prod_{g=1}^{\lambda} \frac{(-n_g)_m g!}{l_g!} A_{n_g,l_g}(y_g)^l_g \]
\[ \times \frac{\lambda}{(4ab + c)^{-k+\mu_g l_g}} \left( \frac{\gamma}{\gamma + \frac{1}{2}} \right)_k \]
\[ \times \mathcal{K}_{m,n+1}^{m,n+1} \left\{ \begin{array}{c} z \\ (4ab + c)^{\delta} \end{array} \right\} \left( \frac{1}{2} - \sigma + k - \sum_{g=1}^{\lambda} \mu_g l_g, \delta \right); \\
(b_j, B_j)_{1,m}; [\tau_i(b_{ji}, B_{ji})]_{m+1,q;r} \\
(a_j, A_j)_{1,n}; [\tau_i(a_{ji}, A_{ji})]_{n+1,p;r} \right\} \]
\[ (-\sigma + k - \sum_{g=1}^{\lambda} \mu_g l_g, \delta) \]  

provided the conditions are satisfied those are mentioned with (7.5.2).
\begin{equation}
\int_0^\infty \frac{1}{\sqrt{2\pi}} z^{-\sigma-1} 2F_1 \left( \alpha, \beta; \frac{1}{2}; z \right) 2F_1 \left( \gamma - \alpha, \gamma - \beta; \frac{1}{2}; z \right) \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{[n_1/m_1] [n_2/m_2]}{[\lambda/m]} \prod_{g=1}^{\lambda} y_g \sum_{l_1=0}^{l_1} \sum_{l_2=0}^{l_2} \ldots \sum_{l_\lambda=0}^{l_\lambda} \prod_{g=1}^{\lambda} (-n_g)_m l_g ! \times \mathbf{N}_{p_i q_i + 1, r_i \delta} \left[ \frac{1}{2} - \sigma + k - \sum_{g=1}^{\lambda} \mu_g l_g , \delta \right] ;
\end{equation}

\begin{equation}
\times \mathbf{N}_{p_i + 1, q_i + 1, r_i \delta} \left[ \frac{1}{2} - \sigma + k - \sum_{g=1}^{\lambda} \mu_g l_g , \delta \right] ;
\end{equation}

provided the conditions are satisfied those are mentioned with (7.5.3).

\begin{equation}
\text{Result – 13}
\end{equation}

\begin{equation}
\int_0^\infty \left( a + b x^2 \right) z^{-\sigma-1} 2F_1 \left( \alpha, \beta; \frac{1}{2}; z \right) 2F_1 \left( \gamma - \alpha, \gamma - \beta; \frac{1}{2}; z \right) \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{[n_1/m_1] [n_2/m_2]}{[\lambda/m]} \prod_{g=1}^{\lambda} y_g \sum_{l_1=0}^{l_1} \sum_{l_2=0}^{l_2} \ldots \sum_{l_\lambda=0}^{l_\lambda} \prod_{g=1}^{\lambda} (-n_g)_m l_g ! \times \mathbf{N}_{p_i q_i + 1, r_i \delta} \left[ \frac{1}{2} - \sigma + k - \sum_{g=1}^{\lambda} \mu_g l_g , \delta \right] ;
\end{equation}

\begin{equation}
\times \mathbf{N}_{p_i + 1, q_i + 1, r_i \delta} \left[ \frac{1}{2} - \sigma + k - \sum_{g=1}^{\lambda} \mu_g l_g , \delta \right] ;
\end{equation}
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\[ \frac{\sqrt{\pi}}{2a(4ab + c)^{\sigma+\frac{1}{2}}} \sum_{k=0}^{\infty} \sum_{l_1=0}^{n_1/m_1} \sum_{l_2=0}^{n_2/m_2} \cdots \sum_{l_\lambda=0}^{n_\lambda/m_\lambda} \prod_{g=1}^{\lambda} \frac{(-n_g)^{m_g} l_g!}{l_g^\sigma} \] 
\[ \times \frac{(y)_k a_k}{(y + 1/2)_k} \] 
\[ \times \mathbb{K}_{p_i+1,q_i+1,\tau_i;r} \left\{ \frac{z}{(4ab + c)^{\delta}} \left( \frac{1}{2} - \sigma + k - \sum_{g=1}^{\lambda} \mu_g l_g, \delta \right) ; \right. \]
\[ (b_j, B_j)_{1,m}; [\tau_i(b_{ji}, B_{ji})]_{m+1,q_i;r}; \]
\[ (a_j, A_j)_{1,n}; [\tau_i(a_{ji}, A_{ji})]_{n+1, p_i;r}; \]
\[ (-\sigma + k - \sum_{g=1}^{\lambda} \mu_g l_g, \delta) \] 
\[ \left. \right\} \]
\[ \cdots \]

provided the conditions are satisfied those are mentioned with (7.5.4).

- **A general class of multiple Eulerian integral**

In study of the screening properties of a charged impurity located inside and near the surface of a metal subjected to a magnetic field, there arises an interesting class of Eulerian integrals involving the Bessel function \( J_\alpha(z) \) or \( J_0(z) \), which were expressed in closed forms by Glasser. To provide generalizations of these Bessel function integrals of Glasser, a number of Eulerian integrals involving various general class of polynomials, Meijer’s G-function and Fox’s H-function of one and more variables with general arguments have been established hitherto in the literature. In the **chapter 8**, we evaluate a general class of multiple Eulerian integral with integrands involving a product of general class of polynomials, a general sequence of functions and the multivariable H-function with general arguments. On account of most general nature of the functions and polynomials involved in the integral, our results provide interesting unifications and generalizations of a large number of new and known results.

**Result – 1**

\[ \int_{y_1}^{z_1} \cdots \int_{y_\mu}^{z_\mu} \prod_{j=1}^{u} \left\{ \frac{(x_j - y_j)^{\lambda_j} (z_j - x_j)^{\mu_j}}{X_j^{\lambda_j+\mu_j+2}} \right\} S_j^u \left[ a \prod_{j=1}^{u} \frac{(x_j - y_j)^{\delta_j} (z_j - x_j)^{t_j}}{X_j^{\delta_j+t_j}} \right] \]
provided the conditions are satisfied those are mentioned with (8.3.1).

- **Applications of integral transforms**
  In the literature there are numerous integral transforms widely used in physics, astronomy as well as in engineering. In early 90’s Watugala introduced a new integral transform, named the Sumudu transform and further applied it to the solution of ordinary differential equation in control engineering problems. In the chapter 9, we apply both Laplace and Sumudu transforms to $\tilde{H}$-function and present a comparative study. On account of general nature of the $\tilde{H}$-function, the results obtained here are capable of yielding a very large number of results (new and known).
  
  The results obtained are as under:
Result - 1

\[
S \left[ t^{\rho-1} \overline{H}^{M,N}_{P,Q} \left\{ zt^{\sigma} \left[ (a_j, \alpha_j; A_j)_{1,N'} (a_j, \alpha_j)_{N+1,P} \right] \right\} \right] \\
= u^{\rho-1} \overline{H}^{M,N+1}_{P+1,Q} \left\{ zu^{\sigma} \left[ (a_j, \alpha_j; A_j)_{1,N'} (a_j, \alpha_j)_{N+1,P} (1 - \rho, \sigma) \right] \right\}
\]

provided the conditions are satisfied those are mentioned with (9.3.1).

Result - 2

\[
L \left[ t^{\rho-1} \overline{H}^{M,N}_{P,Q} \left\{ zt^{\sigma} \left[ (a_j, \alpha_j; A_j)_{1,N'} (a_j, \alpha_j)_{N+1,P} \right] \right\} \right] \\
= S^{-\rho} \overline{H}^{M,N+1}_{P+1,Q} \left\{ zS^{-\sigma} \left[ (a_j, \alpha_j; A_j)_{1,N'} (a_j, \alpha_j)_{N+1,P} (1 - \rho, \sigma) \right] \right\}
\]

provided the conditions are satisfied those are mentioned with (9.3.5).

Result - 3

\[
S \left[ t^{\rho-1} \overline{H}^{M,N}_{P,Q} \left\{ zt^{-\sigma} \left[ (a_j, \alpha_j; A_j)_{1,N'} (a_j, \alpha_j)_{N+1,P} \right] \right\} \right] \\
= u^{\rho-1} \overline{H}^{M+1,N}_{P,Q+1} \left\{ zu^{-\sigma} \left[ (a_j, \alpha_j; A_j)_{1,N'} (a_j, \alpha_j)_{N+1,P} \right] \right\}
\]

provided the conditions are satisfied those are mentioned with (9.3.9).

Result - 4

\[
L \left[ t^{\rho-1} \overline{H}^{M,N}_{P,Q} \left\{ zt^{-\sigma} \left[ (a_j, \alpha_j; A_j)_{1,N'} (a_j, \alpha_j)_{N+1,P} \right] \right\} \right]
\]
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**Application of special functions**

To study the effect of environmental pollution on the growth and existence of Biological Populations, various investigations have been carried out both experimentally and mathematically. In sequel of such investigations, recently, certain mathematical models have been presented in terms of H-Function and \( \tilde{H} \) - Function. In view of providing an extension to this study, in chapter 10 we have presented a mathematical model involving I-Function, which being more generalized in nature, encompasses numerous results available in the literature.

**The Mathematical model**

The dynamics of the concentration \( C(x, t) \) of the pollutant is considered to be given by the following equation-

\[
\frac{\partial C}{\partial t} = Q_0 - \alpha C + D_c \frac{\partial^2 C}{\partial x^2}
\]

provided the conditions are satisfied those are mentioned with (10.3.2).

**Result**

\[
\frac{1}{t^{p_i+1,q_i+1}} \left\{ \frac{Z x^\sigma t^\mu}{(0, \mu, (a_j, A_j)_{1,n}; (a_j, A_j)_{n+1,p_i}) (b_j, B_j)_{1,n}; (b_j, B_j)_{m+1,q_i}} \right\}
\]

\[
= Q_0 - \alpha I_{p_i+1,q_i+1} \left( \frac{Z x^\sigma t^\mu}{(0, \sigma, (a_j, A_j)_{1,n}; (a_j, A_j)_{n+1,p_i}) (b_j, B_j)_{1,n}; (b_j, B_j)_{m+1,q_i}} \right)
\]

\[
+ D_c \frac{1}{x^2} I_{p_i+1,q_i+1} \left( \frac{Z x^\sigma t^\mu}{(0, \sigma, (a_j, A_j)_{1,n}; (a_j, A_j)_{n+1,p_i}) (b_j, B_j)_{1,n}; (b_j, B_j)_{m+1,q_i}} \right)
\]

provided the conditions are satisfied those are mentioned with (10.4.4).
The above description leads to the conclusion that the work presented in the thesis, besides providing extensions and unifications of the earlier results in the literature, also find useful applications in various fields of pure and applied mathematics.