Chapter – 6

Some Integrals Involving I-Function and Wright’s Generalized Hypergeometric Function

6.1 Introduction

In this chapter, we establish seven integral formulas involving product of the I-function and Fox-Wright’s Generalized Hypergeometric Function. Being unified and general in nature, these integrals yield a number of known and new results as particular cases. For the sake of illustration, as special cases of our results we obtain seven corollaries.

6.2 Prerequisites

We recall here the following definitions required for the present study:

(a) The Wright’s Generalized hypergeometric function \( p\psi_q \) [46] appearing in this chapter is defined by (1.2.8) and using in the form

\[
p\psi_q \left[ \begin{array}{c} (e_1, \gamma_1), \ldots, (e_p, \gamma_p); \\ (f_1, \delta_1), \ldots, (f_p, \delta_p); \end{array} \right] x = p\psi_q \left[ \begin{array}{c} (e_j, \gamma_j)_{1,p}; \\ (f_j, \delta_j)_{1,q}; \end{array} \right] x = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(e_j + \gamma_j k) x^k}{\prod_{j=1}^{q} \Gamma(f_j + \delta_j k) k!}
\]

... (6.2.1)
(b) During the course of present study we require the following result due to Rainville [43]:

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n - k)
\]

\[\ldots (6.2.2)\]

(c) The Mellin Transform of \( H \) – Function [24, 34, 83] is given by [26]:

\[
\int_{0}^{\infty} x^{s-1} H_{p,q}^{m,n} \left\{ ax \left( a_{j}, \alpha_{j} \right)_{1,p}, \left( b_{j}, \beta_{j} \right)_{1,q} \right\} dx = a^{-s} \frac{\Pi_{j=1}^{m} \Gamma(b_{j} + \beta_{j}s) \Pi_{j=1}^{n} \Gamma(1 - a_{j} - \alpha_{j}s)}{\Pi_{j=n+1}^{p} \Gamma(a_{j} + \alpha_{j}s) \Pi_{j=m+1}^{q} \Gamma(1 - b_{j} - \beta_{j}s)}
\]

\[\ldots (6.2.3)\]

where

\[
\sum_{j=1}^{n} \alpha_{j} - \sum_{j=n+1}^{p} \alpha_{j} + \sum_{j=1}^{m} \beta_{j} - \sum_{j=m+1}^{q} \beta_{j} > 0,
\]

\[
|\arg a| < \frac{1}{2} A\pi, \delta = \sum_{j=1}^{q} \beta_{j} - \sum_{j=1}^{p} \alpha_{j} > 0
\]

and

\[
- \min_{1 \leq j \leq m} \left[ Re \left( \frac{b_{j}}{\beta_{j}} \right) \right] < Re(s) < \min_{1 \leq j \leq n} \left[ Re \left( \frac{1 - a_{j}}{\alpha_{j}} \right) \right]
\]

6.3 Main Integrals

**First Integral**

\[
\int_{0}^{t} x^{\rho-1} (t - x)^{\sigma-1} e^{-xz} p_{\psi_{q}} \left[ \left( e_{j}, \gamma_{j} \right)_{1,p}, \left( f_{j}, \delta_{j} \right)_{1,q} \right] \left[ a_{j}, \alpha_{j} \right]_{1,n} \left[ b_{j}, \beta_{j} \right]_{1,m} \left( t - x \right)^{\eta} dx
\]

\[
\times I_{p_{i},q_{i},r}^{m_{i},n_{i}} \left\{ yx^{\mu(t - x)^{\nu}} \left( a_{j}, \alpha_{j} \right)_{1,n} \left( b_{j}, \beta_{j} \right)_{1,m} \left( a_{j}, \alpha_{j} \right)_{n+1,p_{i}} \left( b_{j}, \beta_{j} \right)_{m+1,q_{i}} \right\} dx
\]

\[
= e^{-zt} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{\infty} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta+\eta-1)k+u}
\]
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\[ \times I_{p+2,q+1:r}^{m,n+2} \left\{ yt^{\mu+v} \right\} \left( 1 - \rho - \zeta(k, \mu), (1 - \sigma - (n - 1)k - u, \nu), \right. \\
\left. (b_j, \beta_j)_{1:m}; (b_{ji}, \beta_{ji})_{m+1,q_i}; \right. \\
\left. (a_j, \alpha_j)_{1:n}; (a_{ji}, \alpha_{ji})_{n+1,p_i}; \right. \\
\left. (1 - \rho - \zeta - (\zeta + \eta - 1)k - u, \mu + v) \right\} \]

where\[ f(k) = \frac{\prod_{j=1}^{p} \Gamma(e_j + \gamma_j k) a^k}{\prod_{j=1}^{q} \Gamma(f_j + \delta_j k) k!} \]

provided that-

(i) \( \mu \geq 0, \nu \geq 0 \) (Not both zero simultaneously)

(ii) \( \zeta \) and \( \eta \) are non-negative integers such that \( \zeta + \eta \geq 1 \)

(iii) \( A_i > 0, B_i < 0; |\arg y| < \frac{1}{2} A_i \pi, \forall i \in 1,...,r; \)

where

\[ A_i = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p_i} \alpha_{ji} + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q_i} \beta_{ji} \]

\[ B_i = \frac{1}{2} (p_i - q_i) + \sum_{j=1}^{q_i} b_{ji} - \sum_{j=1}^{p_i} a_{ji} \]

and

(iv) \( \text{Re}(\rho) - \mu \max_{1 \leq j \leq n} \left[ \text{Re} \left( \frac{a_j - 1}{a_j} \right) \right] > 0 \)

(v) \( \text{Re}(\sigma) + \nu \min_{1 \leq j \leq m} \left[ \text{Re} \left( \frac{b_j}{\beta_j} \right) \right] > 0 \)

The I-function occurring in the integral is defined by (1.4.1).

**Second Integral**

\[ \int_{0}^{t} x^{\sigma-1}(t - x)^{\rho-1}e^{-xz} p_{\psi_q} \left[ (e_j, \gamma_j)_{1:p_i}; ax^\zeta(t - x)^{\eta} \right] \]
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\[ \times I_{p_i,q_i:r}^{m,n} \left\{ y x^{-\mu} (t - x)^{-\nu} \left\{ (a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1,p_l} \right\} dx \right. \]

\[ = e^{-z t} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{u} f(k) \frac{z^{u-k}}{(u-k)!} t (\zeta+\eta-1) k+u \]

\[ \times I_{p_i+2,q_{i+1}:r}^{m,n+2} \left\{ y t^{-\mu-\nu} \left\{ (a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1,p_l} \right\} \right. \]

\[ (\rho + \zeta k, \mu), (\sigma + (\eta - 1) k + u, \nu), \]

\[ = e^{-z t} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{u} f(k) \frac{z^{u-k}}{(u-k)!} t (\zeta+\eta-1) k+u \]

provided

\[ Re(\rho) - \mu \max_{1 \leq j \leq n} \left[ Re \left( \frac{a_{j} - 1}{\alpha_{j}} \right) \right] > 0, \]

\[ Re(\sigma) - \nu \max_{1 \leq j \leq n} \left[ Re \left( \frac{a_{j} - 1}{\alpha_{j}} \right) \right] > 0, \]

along with the set of conditions (i) to (iii) given with (6.3.1) and \( f(k) \) is given by (6.3.2)

**Third Integral**

\[ \int_{0}^{t} x^{\rho-1} (t - x)^{\sigma-1} e^{-x z} p \psi_{q} \left\{ (e_j, \gamma_j)_{1,p}; ax^\zeta (t - x)^\eta \right\} \]

\[ \times I_{p_i,q_{i+1}:r}^{m,n} \left\{ y x^{\mu} (t - x)^{-\nu} \left\{ (b_j, \beta_j)_{1,m}; (b_{ji}, \beta_{ji})_{m+1,q_l} \right\} \right. \]

\[ = e^{-z t} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{u} f(k) \frac{z^{u-k}}{(u-k)!} t (\zeta+\eta-1) k+u \]

\[ \times I_{p_{i+1},q_{l+1}:2:r}^{m+1,n+1} \left\{ y t^{\mu-\nu} \left\{ (1 - \rho - \zeta k, \mu); (a_j, \alpha_j)_{1,n}; \right. \right. \]

\[ (\sigma + (\eta - 1) k + u, \nu), (b_j, \beta_j)_{1,m}; \]

\[ \left. \left. \left\{ (a_{ji}, \alpha_{ji})_{n+1,p_l} \right\} \right. \right. \]

\[ (b_{ji}, \beta_{ji})_{m+1,q_l}, (1 - \rho - \sigma - (\zeta + \eta - 1) k - u, \mu - \nu) \}

\[ \ldots (6.3.4) \]
provided

\[
Re(\rho) + \mu \min_{1\leq j \leq m} \left[ Re \left( \frac{b_j}{\beta_j} \right) \right] > 0,
\]

\[
Re(\sigma) - \nu \max_{1\leq j \leq n} \left[ Re \left( \frac{a_j - 1}{\alpha_j} \right) \right] > 0,
\]

along with the set of conditions (i) to (iii) given with (6.3.1) and \( f(k) \) is given by (6.3.2)

**Fourth Integral**

\[
\int_{0}^{t} x^{\rho-1}(t - x)^{\sigma-1}e^{-xz} \quad p\psi_q \left[ (e_j, \gamma_j)_{1.p}; (f_j, \delta_j)_{1,q}; \alpha x^\xi (t - x)^\eta \right] dx
\]

\[
	imes I_{p_i,q_i;r_i}^{m,n} \left\{ yx^{-\mu}(t - x)\nu \left[ (a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1,p_i} \right] \left( b_j, \beta_j \right)_{1,m}; (b_{ji}, \beta_{ji})_{m+1,q_i} \right\}
\]

\[
= e^{-zt} t^{\rho + \sigma - 1} \sum_{u = 0}^{\infty} \sum_{k = 0}^{\infty} f(k) \frac{u^k}{(u - k)!} t^{(\xi + \eta - 1)k + u}
\]

\[
	imes I_{p_i+1,q_i+1;r_i}^{m+1,n+1} \left\{ \left( 1 - \sigma - (\eta - 1)k - u, \nu \right), (a_j, \alpha_j)_{1,n}; (\rho + \zeta k, \mu), (b_j, \beta_j)_{1,m}; (a_{ji}, \alpha_{ji})_{n+1,p_i}, (\rho + \sigma + (\xi + \eta - 1)k + u, \mu - \nu) \right\}
\]

\[
(b_{ji}, \beta_{ji})_{m+1,q_i}
\]

provided

\[
Re(\rho) + \mu \min_{1\leq j \leq m} \left[ Re \left( \frac{b_j}{\beta_j} \right) \right] > 0,
\]

\[
Re(\sigma) - \nu \max_{1\leq j \leq n} \left[ Re \left( \frac{a_j - 1}{\alpha_j} \right) \right] > 0,
\]

along with the set of conditions (i) to (iii) given with (6.3.1) and \( f(k) \) is given by (6.3.2)

**Fifth Integral**

\[
\int_{0}^{t} x^{\rho-1}(t - x)^{\sigma-1}e^{-xz} \quad p\psi_q \left[ (e_j, \gamma_j)_{1.p}; (f_j, \delta_j)_{1,q}; \alpha x^\xi (t - x)^\eta \right] dx
\]
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\[ \times l_{p_i,q_i}^{m,n} \left\{ yx^\mu (t-x)^{-\nu} \left\{ (a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1,p_i} \right\} \right\} dx \]

\[ = e^{-zt} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{u} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta+\eta-1)k+u} \]

\[ \times l_{p_i+2,q_i+1}^{m+1,n+1} \left\{ y^{\mu-\nu} \left( 1 - \rho - \zeta k, \mu, (a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1,p_i} \right) \right\} \]

\[ (\sigma + (\eta - 1)k + u, v), \]

\[ (\rho + \sigma + (\zeta + \eta - 1)k + u, v - \mu) \]

\[ (b_j, \beta_j)_{1,m}; (b_{ji}, \beta_{ji})_{m+1,q_i} \right\} \]

... (6.3.6)

along with the set of conditions (i) to (iv) given with (6.3.1) and \( f(k) \) is given by (6.3.2).

**Sixth Integral**

\[ \int_0^t x^\rho (t-x)^{\sigma-1} e^{-xz} p \psi_q \left\{ (e_j, \gamma_j)_{1,p}; (f_j, \delta_j)_{1,q}; a \right\} x^\xi (t-x)^\eta \]

\[ \times l_{p_i,q_i}^{m,n} \left\{ yx^\mu (t-x)^{-\nu} \left\{ (a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1,p_i} \right\} \right\} dx \]

\[ = e^{-zt} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{u} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta+\eta-1)k+u} \]

\[ \times l_{p_i+1,q_i+2}^{m+1,n+1} \left\{ y^{\mu+\nu} \left( 1 - \sigma - (\eta - 1)k - u, v), (a_j, \alpha_j)_{1,n}; \right) \right\} \]

\[ (\rho + \zeta k, \mu, (b_j, \beta_j)_{1,m}; \]

\[ (a_{ji}, \alpha_{ji})_{n+1,p_i} \]

\[ (b_{ji}, \beta_{ji})_{m+1,q_i}, (1 - \rho - \sigma - (\zeta + \eta - 1)k - u, v - \mu) \right\} \]

... (6.3.7)

provided \( \mu \geq 0, v > 0 \) such that \( \mu - v \geq 0 \).
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\[
\text{Re}(\rho) - \mu \max_{1 \leq j \leq n} \left[ \text{Re}\left\{ \frac{(a_j-1)}{\alpha_j} \right\} \right] > 0, \quad \text{Re}(\sigma) + \nu \min_{1 \leq j \leq m} \left[ \text{Re}\left\{ \frac{b_j}{\beta_j} \right\} \right] > 0,
\]

along with the conditions (i) to (iii) given with (6.3.1).

**Seventh Integral**

\[
I_7 = \int_0^\infty x^{n-1} e^{ax} p \varphi_q \left[ (e_j, \gamma_j)_1, p; (f_j, \delta_j)_1, q; ax\right] \times I_{p_1, q_1}^{m_1, n_1} \left\{ zx^{\sigma} \begin{aligned}
(a_j, \alpha_j)_1, n_1; (a_j, \alpha_j)_{n_1+1, p_1} \\
(b_j, \beta_j)_1, m_1; (b_j, \beta_j)_{m_1+1, q_1}
\end{aligned} \right\} dx
\]

\[= \omega^{-\eta} \sum_{u=0}^{\infty} \sum_{k=0}^{u} f(k) \frac{a^{u-k}}{(u-k)!} \omega^{-(\rho-1)k-u} \times I_{p_1+q_1+p_1}^{m_1+n_1+m} \left\{ z \omega^{-\sigma} \begin{aligned}
(a_j, \alpha_j)_1, n_1; (1 - d_j - (\eta + (\rho-1)k + u)\tau_j, \sigma_j)_1, m_1; \\
(b_j, \beta_j)_1, m_1; (1 - c_j - (\eta + (\rho-1)k + u)\tau_j, \sigma_j)_1, n_1; \\
(a_{j_1}, \alpha_{j_1})_{n_1+1, p_1}; (1 - d_j - (\eta + (\rho-1)k + u)\tau_j, \sigma_j)_m+1, q_1 \\
(b_{j_1}, \beta_{j_1})_{m_1+1, q_1}; (1 - c_j - (\eta + (\rho-1)k + u)\tau_j, \sigma_j)_n+1, p_1
\end{aligned} \right\}
\]

\[\ldots(6.3.8)\]

Provided that

(i) \( \lambda > 0, |\arg z| < \frac{1}{2}\pi\lambda \)

(ii) \( \lambda \geq 0, |\arg z| \leq \frac{1}{2}\pi\lambda, \text{Re}(\theta + 1) < 0 \)

(iii) \( \lambda_1 > 0, |\arg \omega| < \frac{1}{2}\pi\lambda_1 \)

(iv) \( \lambda_1 \geq 0, |\arg \omega| \leq \frac{1}{2}\pi\lambda_1, \text{Re}(\theta_1 + 1) < 0 \)

(v) \( \sigma > 0, -\sigma \min_{1 \leq j \leq m_1} \left[ \text{Re}\left\{ \frac{b_j}{\beta_j} \right\} \right] - \min_{1 \leq j \leq m} \left[ \text{Re}\left\{ \frac{d_j}{\tau_j} \right\} \right] < \text{Re}(\eta) < \sigma < \min_{1 \leq j \leq m_1} \left[ \text{Re}\left\{ \frac{(1-a_j)}{\alpha_j} \right\} \right] + \min_{1 \leq j \leq n} \left[ \text{Re}\left\{ \frac{(1-c_j)}{\zeta_j} \right\} \right] \)
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where

\[
\lambda = \sum_{j=1}^{n_1} \alpha_j + \sum_{j=1}^{m_1} \beta_j - \max_{1 \leq i \leq r} \left[ \sum_{j=n_1+1}^{p_i} \alpha_{ji} + \sum_{j=m_1+1}^{q_i} \beta_{ji} \right]
\]

\[
\theta = \sum_{j=1}^{n_1} b_j - \sum_{j=1}^{m_1} a_j - \min_{1 \leq i \leq r} \left[ \sum_{j=n_1+1}^{p_i} a_{ji} - \sum_{j=m_1+1}^{q_i} b_{ji} + \frac{p_i}{2} - \frac{q_i}{2} \right]
\]

\[
\lambda_1 = \sum_{j=1}^{m} \tau_j + \sum_{j=1}^{n} \zeta_j - \sum_{j=m+1}^{p} \tau_j - \sum_{j=n+1}^{q} \zeta_j
\]

\[
\theta_1 = \frac{1}{2} (p - q) + \sum_{j=1}^{q} d_j - \sum_{j=1}^{p} c_j
\]

**Proofs**

**Proof of the First Integral:** We write the LHS of the result (6.3.1) as \(\mathcal{Z}\) (say)

\[
\mathcal{Z} = e^{-zt} \int_{0}^{t} x^{\rho-1} (t-x)^{\sigma-1} e^{(t-x)z} p_{q} \left[ (e_j, \gamma_j), p ; (f_j, \delta_j), q ; ax^\xi (t-x)^\eta \right] dx
\]

\[
\times I_{p,q}^{m,n} \left\{ yx^\mu (t-x)^\nu \left[ (a_j, \alpha_j), n ; (a_{ji}, \alpha_{ji}), n+1, p_i \right] \right\} dx
\]

Now replacing \(e^{(t-x)z}\) by \(\sum_{u=0}^{\infty} \frac{(t-x)^u}{u!} z^u\) and also using (6.2.1) and (1.5.1), we get

\[
\mathcal{Z} = e^{-zt} \int_{0}^{t} x^{\rho-1} (t-x)^{\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{\infty} \frac{\Pi_{j=1}^{p_i} \Gamma(e_j + \gamma_j k)}{\Pi_{j=1}^{q_i} \Gamma(f_j + \delta_j k)} \frac{z^u}{u!}
\]

\[
\times a^{k} x^{\xi k} (t-x)^{u+\eta k} \frac{1}{k!} \int_{L} \phi(\xi) yx^\mu (t-x)^\nu d\xi
\]

which in view of (6.2.2), becomes

\[
\mathcal{Z} = e^{-zt} \int_{0}^{t} x^{\rho-1} (t-x)^{\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{\infty} \frac{\Pi_{j=1}^{p_i} \Gamma(e_j + \gamma_j k)}{\Pi_{j=1}^{q_i} \Gamma(f_j + \delta_j k)} \frac{z^{u-k}}{(u-k)!}
\]
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\[ a^k x^\xi(t - x)^{u + \eta k - k} \frac{1}{k!} \left( \frac{1}{2\pi\omega} \right) \int_L \phi(\xi) y^\xi x^\mu(t - x)^{v\xi} d\xi \]

Now on interchanging the order of integration and summation, we obtain

\[ \Im = e^{-zt} \sum_{u=0}^{\infty} \sum_{k=0}^{u} f(k) \frac{z^{u-k}}{(u-k)!} \]

\[ \times \frac{1}{2\pi\omega} \int_L \phi(\xi) y^\xi \left\{ \int_0^t x^{\rho + \zeta k + \mu \xi - 1} (t - x)^{\sigma + (\eta - 1)k + u + v\xi - 1} dx \right\} d\xi \]

where \( f(k) \) is given by (6.3.2).

Further, on substituting \( x = ts \) in the inner \( x \)-integral, we get

\[ \Im = e^{-zt} t^{\rho + \sigma - 1} \sum_{u=0}^{\infty} \sum_{k=0}^{u} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\rho + \eta - 1)k + u} \]

\[ \times \frac{1}{2\pi\omega} \int_L \phi(\xi) \frac{\Gamma(\rho + \zeta k + \mu \xi) \Gamma(\sigma + (\eta - 1)k + u + v\xi)}{\Gamma(\rho + \sigma + (\zeta + \eta - 1)k + u + (\mu + v)\xi)} y^\xi t^{(\mu + v)\xi} d\xi \]

Finally by virtue of (1.5.1) we arrive at the desired result.

The rest of the integrals can also be established on similar lines.

6.4 Special Cases

(i) On taking \( \gamma_j = 1 = \delta_j \) in our main results we get the following seven integrals involving generalized hypergeometric function-

\[ I_1 = \int_0^t x^{\rho - 1} (t - x)^{\sigma - 1} e^{-x\zeta} \frac{dF_1}{\rho} \left\{ ax^{\xi} (t - x)^{\eta} \right\} \]

\[ \times \int_{p_i, q_i; \tau}^{m, n} \left\{ y x^{\mu} (t - x)^{v} \left\{ \begin{array}{c} (a_j, a_j)_{1, 1} \tau; (a_j, a_j)_{n+1, p_i} \\ (b_j, b_j)_{1, 1} \tau; (b_j, b_j)_{m+1, q_i} \end{array} \right\} \right\} dx \]

\[ = e^{-zt} t^{\rho + \sigma - 1} \sum_{u=0}^{\infty} \sum_{k=0}^{u} g(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta + \eta - 1)k + u} \]
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\[ \times \int_{p_{l+2,q_{l+1}:r}}^{m,n+2} \left\{ \begin{array}{c}
(1 - \rho - \zeta k, \mu), (1 - \sigma - (n - 1)k - u, \nu), \\
(b_j, \beta_j)_{1:m}; (b_{ji}, \beta_{ji})_{m+1,q_i}, \\
(a_j, \alpha_j)_{1:n}; (a_{ji}, \alpha_{ji})_{n+1,p_i} \\
(1 - \rho - \sigma - (\xi + \eta - 1)k - u, \mu + \nu)
\end{array} \right\} \]

\[ \cdots (6.4.1) \]

where

\[ g(k) = \frac{\prod_{j=1}^n \Gamma(e_j + k) \Gamma(e_j) a^k}{\prod_{j=1}^n \Gamma(f_j + k) \Gamma(f_j) k!} \]

\[ \cdots (6.4.2) \]

provided that the conditions easily obtainable from those mentioned with (6.3.1) are satisfied.

\[ I_2 = \int_0^t x^{\sigma-1}e^{-xz} \sum_{k=0}^{\infty} \sum_{u=0}^{\infty} \frac{g(k) z^{u-k}}{(u-k)!} t^{(\xi+\eta-1)k+u} \]

\[ \times \int_{p_{l+2,q_{l+1}:r}}^{m,n+2} \left\{ yt^{\mu-\nu} \left| (a_j, \alpha_j)_{1:n}; (a_{ji}, \alpha_{ji})_{n+1,p_i} \\
(b_j, \beta_j)_{1:m}; (b_{ji}, \beta_{ji})_{m+1,q_i} \right| \right\} dx \]

\[ \cdots (6.4.3) \]

provided that the conditions easily obtainable from those mentioned with (6.3.3) are satisfied.

\[ I_3 = \int_0^t x^{\sigma-1}e^{-xz} \sum_{k=0}^{\infty} \sum_{u=0}^{\infty} \frac{g(k) z^{u-k}}{(u-k)!} t^{(\xi+\eta-1)k+u} \]

\[ \times \int_{p_{l+2,q_{l+1}:r}}^{m,n+2} \left\{ yt^{\mu-\nu} \left| (a_j, \alpha_j)_{1:n}; (a_{ji}, \alpha_{ji})_{n+1,p_i} \\
(b_j, \beta_j)_{1:m}; (b_{ji}, \beta_{ji})_{m+1,q_i} \right| \right\} dx \]

\[ \cdots (6.4.3) \]
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\[
\times I_{l,p,l,q}^{m,n} \left\{ yx^\mu (t - x)^{-\nu} \left\{ (a_j, \alpha_j)_{1,n}^1 ; (a_j, \alpha_j)_{n+1,p_i}^1 \right\} dx \right. \\
= e^{-zt} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{u} g(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta+\eta-1)k+u} \times I_{l+1,q_i+2:r}^{m+1,n+1} \left\{ yt^{\mu-\nu} \left\{ (1 - \rho - \zeta k, \mu), (\sigma + (\eta - 1)k + u, \nu), (b_j, \beta_j)_{1,m}^1 ; (b_j, \beta_j)_{m+1,q_i}^1 \right\} \\
\left. \left\{ (a_j, \alpha_j)_{1,n}^1 ; (a_j, \alpha_j)_{n+1,p_i}^1 \right\} (1 - \rho - \sigma - (\zeta + \eta - 1)k - u, \mu - \nu) \right\} 
\]
\]

provided that the conditions easily obtainable from those mentioned with (6.3.4) are satisfied.

\[
I_4 = \int_0^t x^{\rho-1} (t - x)^{\sigma-1} e^{-xz} p_F q [ax^\zeta (t - x)^\eta] \\
\times I_{l,p,l,q}^{m,n} \left\{ yx^\mu (t - x)^{-\nu} \left\{ (a_j, \alpha_j)_{1,n}^1 ; (a_j, \alpha_j)_{n+1,p_i}^1 \right\} dx \right. \\
= e^{-zt} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{u} g(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta+\eta-1)k+u} \times I_{l+1,q_i+2:r}^{m+1,n+1} \left\{ yt^{\mu-\nu} \left\{ (1 - \sigma - (\eta - 1)k - u, \nu), (a_j, \alpha_j)_{1,n}^1 ; \\
\left( \rho + \zeta k, \mu \right), (b_j, \beta_j)_{1,m}^1 ; \\
\left( a_j, \alpha_j \right)_{n+1,p_i}^1 \left( \rho + \sigma + (\zeta + \eta - 1)k + u, \mu - \nu \right) \\
\left( b_j, \beta_j \right)_{m+1,q_i}^1 \right\} \\
\right. \\
\left. \right\} 
\]
\]

provided that the conditions easily obtainable from those mentioned with (6.3.5) are satisfied.

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\[ I_5 = \int_0^t x^{\beta-1}(t-x)^{\alpha-1} e^{-xz} p F_q [ a x^\xi (t-x)^\eta ] \]

\[ \times I_{p_1,q_1,r}^{m,n} \left\{ y x^{\mu} (t-x)^{-v} \left[ (a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1,p_i} \right] (b_j, \beta_j)_{1,m}; (b_{ji}, \beta_{ji})_{m+1,q_i} \right\} dx \]

\[ e^{-zt} t^\rho+\sigma-1 \sum_{u=0}^{\infty} \sum_{k=0}^{n} h(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta+\eta-1)k+u} \]

\[ \times I_{p_1+2,q_1+2,r}^{m+1,n+1} \left\{ y t^{-\mu} \left[ (1-\rho-\zeta k, \mu), (a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1,p_i} (\sigma + (\eta - 1)k + u, v), (b_j, \beta_j)_{1,m}; (b_{ji}, \beta_{ji})_{m+1,q_i} \right) \right\} \]

... (6.4.6)

provided that the conditions easily obtainable from those mentioned with (6.3.6) are satisfied.

\[ I_6 = \int_0^t x^{\beta-1}(t-x)^{\alpha-1} e^{-xz} p F_q [ a x^\xi (t-x)^\eta ] \]

\[ \times I_{p_1,q_1,r}^{m,n} \left\{ y x^{\mu} (t-x)^{-v} \left[ (a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1,p_i} \right] (b_j, \beta_j)_{1,m}; (b_{ji}, \beta_{ji})_{m+1,q_i} \right\} dx \]

\[ e^{-zt} t^\rho+\sigma-1 \sum_{u=0}^{\infty} \sum_{k=0}^{n} h(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta+\eta-1)k+u} \]

\[ \times I_{p_1+1,q_1+2,r}^{m+1,n+1} \left\{ y t^{-\mu} \left[ (1-\rho-(\eta - 1)k - u, v), (\rho + \zeta k, \mu), (b_j, \beta_j)_{1,m}; (b_{ji}, \beta_{ji})_{m+1,q_i}, (a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1,p_i} \right) \right\} \]

... (6.4.7)

where
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\[ h(k) = \frac{\prod_{j=1}^{p} \Gamma(e_j + k) a^k}{\prod_{j=1}^{q} \Gamma(f_j + k) k!} \]

...(6.4.8)

provided that the conditions easily obtainable from those mentioned with (6.3.7) are satisfied.

\[ I_7 = \int_0^{\infty} x^{\eta-1} e^{ax} P_{q}^{m_1,n_1} \left( a x^p \right) \times \sum_{u=0}^{\infty} \sum_{k=0}^{u} g(k) \frac{a^{u-k}}{(u-k)!} \omega^{-(\rho+1)k-u} \]

\[ \times \left\{ \begin{array}{ll}
\int_0^\infty \omega x \left[ (a_j, \alpha_j)_{1,n_1+1,p_1} (b_j, \beta_j)_{1,m_1} \right] \times \left[ (c_j, \varsigma_j)_{1,n+1} (d_j, \tau_j)_{m+1,q} \right] \ dx \\
\int_0^\infty \omega x \left[ (a_j, \alpha_j)_{1,n+1} (b_j, \beta_j)_{1,m+1,q} \right] \times \left[ (c_j, \varsigma_j)_{1,n} (d_j, \tau_j)_{m+1,p} \right] \ dx
\end{array} \right. \]

where

\[ g(k) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(e_j + k) a^k x^{\rho k}}{\prod_{j=1}^{q} \Gamma(f_j + k) k!} \]

...(6.4.10)

provided that the conditions easily obtainable from those mentioned with (6.3.8) are satisfied.

(ii) The results due to, Arora and Saha [151] and several others, follow as special cases of our main results under suitable parametric specifications.

On account of unified and general nature of the functions involved in the main integrals, a large number of new and known integral formulas in terms of simpler functions can be obtained as special cases of our main findings.
Conclusion

The integral formulas established in this chapter involve the I function and Wright’s generalized hypergeometric functions which are very general in nature. Hence, these integral formulas provide extension and unification of a large number of results obtained earlier in the literature, thus enhancing the scope of their applications, as is evident from the special cases of our main findings obtained in the present chapter.