Chapter – 4

On Generalized Fractional Integration of I-Function

4.1 Introduction

In the present chapter, we study and develop the generalized fractional integral operators given by Saigo [157, 158, 159]. Here we establish two theorems that give the images of the product of I-function and general class of polynomials in Saigo operators. On account of general nature of the Saigo operators, I-function and general class of polynomials, a large number of new and known images involving Riemann-Liouville and Erdélyi-Kober fractional integral operators and several special functions follow as special cases of our main findings. To illustrate, six corollaries have been recorded here.

4.2 Prerequisites

A useful generalization of the hypergeometric fractional integrals, including the Saigo operators [157, 158, 159], introduced by Marichev [180] (see details in Samko, Kilbas and Marichev [225] and also see Kilbas and Saigo [18, p.258]) is as follows:

Let $\alpha, \beta, \eta$ be complex numbers and $x > 0$, then the generalized fractional integral operators (The Saigo operators [157]) involving Gaussian hypergeometric function are defined by the following equations:
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\[
\left( J_{0+}^{\alpha, \beta, \eta} f \right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \, \text{}_2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right) f(t) \, dt \quad ; \text{Re}(\alpha) > 0
\]

...(4.2.1)

and

\[
\left( J^{\alpha, \beta, \eta} f \right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} \, \text{}_2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{x}{t} \right) f(t) \, dt \quad ; \text{Re}(\alpha) > 0
\]

...(4.2.2)

where \( \text{}_2F_1(\cdot) \) Stands for the well known Gaussian hypergeometric function defined by

\[
\text{}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{x^n}{n!}
\]

...(4.2.3)

When \( \beta = -\alpha \), the above equations (4.2.1) and (4.2.2) reduce to the following classical Riemann-Liouville fractional integral operator (see Samko et al., [225], p.94, Eqns. (5.1), (5.3)):

\[
\left( J_{0+}^{\alpha, -\alpha, \eta} f \right)(x) = \left( J_{0+}^{\alpha} f \right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) \, dt \quad ; x > 0
\]

...(4.2.4)

and

\[
\left( J^{\alpha, -\alpha, \eta} f \right)(x) = \left( J^{\alpha} f \right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t) \, dt \quad ; x > 0
\]

...(4.2.5)

Again, if \( \beta = 0 \), the equations (4.2.1) and (4.2.2) reduce to the following Erdélyi-Kober fractional integral operator (see Samko et al., [225], p.322, Eqns. (18.5), (18.6)):

\[
\left( J_{0+}^{\alpha, 0, \eta} f \right)(x) = \left( J_{0+}^{\alpha} f \right)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\eta} f(t) \, dt \quad ; x > 0
\]

...(4.2.6)

and
\[(J_{\alpha,\eta} f)(x) = (K_{\alpha,0} f)(x) = \frac{x^\eta}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\eta} f(t) \, dt; \quad x > 0\]  

... (4.2.7)

The following two lemmas will be required to establish our main results.

**Lemma 1** (Kilbas and Sebastian [19], p. 871, Eq. (15) to (18)).

Let \(\alpha, \beta, \eta \in C\) be such that \(Re(\alpha) > 0\) and \(Re(\mu) > \max\{0, Re(\beta - \eta)\}\) then the following relation holds:

\[
(J_{0+}^{\alpha,\beta,\eta} t^{\mu-1})(x) = \frac{\Gamma(\mu) \Gamma(\mu + \eta - \beta)}{\Gamma(\mu + \alpha + \eta) \Gamma(\mu - \beta)} x^{\mu-\beta-1}
\]

... (4.2.8)

In particular, if \(\beta = -\alpha\) and \(\beta = 0\) in (4.2.8), we have:

\[
(J_{0+}^\alpha t^{\mu-1})(x) = \frac{\Gamma(\mu)}{\Gamma(\mu + \alpha)} x^{\mu+\alpha-1}, \quad Re(\alpha) > 0 \quad \text{and} \quad Re(\mu) > 0
\]

... (4.2.9)

and

\[
(J_{0+}^\eta t^{\mu-1})(x) = \frac{\Gamma(\mu + \eta)}{\Gamma(\mu + \alpha + \eta)} x^{\mu-1}, \quad Re(\alpha) > 0 \quad \text{and} \quad Re(\mu) > -Re(\eta)
\]

... (4.2.10)

**Lemma 2** (Kilbas and Sebastian [19], p. 872, Eq. (21) to (24)).

Let \(\alpha, \beta, \eta \in C\) be such that \(Re(\alpha) > 0\) and \(Re(\mu) > \max\{Re(\beta), Re(\eta)\}\) then the following relation holds:

\[
(J_{\alpha,\beta,\eta}^\mu t^{\mu-1})(x) = \frac{\Gamma(\beta - \mu + 1) \Gamma(\eta - \mu + 1)}{\Gamma(1 - \mu) \Gamma(\alpha + \beta + \eta - \mu + 1)} x^{\mu-\beta-1}
\]

... (4.2.11)

In particular, if \(\beta = -\alpha\) and \(\beta = 0\) in (4.2.11), we have:

\[
(J_{\alpha}^\mu t^{\mu-1})(x) = \frac{\Gamma(1 - \alpha - \mu)}{\Gamma(1 - \mu)} x^{\mu+\alpha-1}, \quad 1 - Re(\mu) > Re(\alpha) > 0
\]

... (4.2.12)

and
\[(K_{\eta,\alpha}^\mu t^{\mu-1})(x) = \frac{\Gamma(\eta - \mu + 1)}{\Gamma(1 - \mu + \alpha + \eta)} x^{\mu-1}, \quad \text{Re}(\mu) < 1 + \text{Re}(\eta)\]  

\hspace{1cm} \ldots (4.2.13)

### 4.3 Main Theorems

**Theorem 1**

\[
\begin{align*}
\left[ t^{\mu-1} & \sum_{j=0}^{k} \left( \sum_{l_1=0}^{n_1/m_1} \cdots \sum_{l_k=0}^{n_k/m_k} \frac{(-n_1)_{m_1 l_1} \cdots (-n_h)_{m_h l_h}}{l_1! \cdots l_h!} \right) \right] (x) \\
&= x^{\mu-\beta-1} \sum_{l_1=0}^{n_1/m_1} \cdots \sum_{l_k=0}^{n_k/m_k} \left( \frac{1 - \mu}{1 - \mu - \eta + \beta - \sum_{j=1}^{k} \lambda_j l_j; v; 1} \right) \\
&\times I_{P_{l+2}Q_{l+2}:R}^{M+2} \left[ z^v \right] \left( 1 - \mu - \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right) \left( a_j; a_j \right)_{1,N}^{(a_j'; a_j')_{N+1,P_l}} \\
&\left( 1 - \mu + \beta - \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right) \left( 1 - \mu - \alpha - \eta - \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right) \\
&\left( a_j', a_j' \right)_{1,N}^{(a_j'; a_j')_{N+1,P_l}} \\
&\left( b_j', b_j' \right)_{1,N}^{(b_j'; b_j')_{M+1,Q_l}} \left( b_j'; b_j' \right)_{M+1,Q_l} \\
&\ldots (4.3.1)
\end{align*}
\]

The I-function and the general class of polynomials \(S_n^m(x)\) occurring in the above expression are defined by (1.4.1) and (3.2.1) respectively, and the conditions of validity of (4.3.1) are as follows:

(i) \(\alpha, \beta, \eta, a, b, z \in \mathbb{C}\) and \(\lambda_j, v > 0\ \forall \ j \in \{1, k\}\)

(ii) \(|\text{arg} \ z| < \frac{1}{2} \Omega_i \pi; \ \Omega_i > 0\ \text{where}\)
\[
\Omega_l = \sum_{j=1}^{N} \alpha_j - \sum_{j=N+1}^{P_l} \alpha_{ij} + \sum_{j=1}^{M} \beta_j - \sum_{j=M+1}^{Q_l} \beta_{ij} \quad \forall \imath \in \mathbb{I}, R
\]

(iii) \(Re(\alpha) > 0 \) and \( Re(\mu) + \nu \min_{1 \leq j \leq M} Re\left(\frac{b_j}{\beta_j}\right) > \max\{0, Re(\beta - \eta)\}\)

**Proof:** In order to prove (4.3.1), we first express the general class of polynomials occurring in its left-hand side in the series form and also express the I-function in terms of the Mellin–Barnes contour integral by using (3.2.1) and (1.4.1) respectively. Next, on interchanging the order of summations & the integral occurring therein (which is permissible under the conditions stated), the LHS of (4.3.1) takes the following form \(\Delta\) (say):

\[
\Delta = \sum_{l_1=0}^{[n_1/m_1]} \sum_{l_2=0}^{[n_2/m_2]} \ldots \sum_{l_k=0}^{[n_k/m_k]} \left\{ \frac{(-n_1)_{m_1} l_1 \ldots (-n_h)_{m_h l_h}}{l_1! \ldots l_h!} \right\}
\]

\[
\times A'_{n_1,l_1} \ldots A^{(k)}_{n_h,l_h c_1 l_1 \ldots c_h l_h \sum_{j=1}^{k} \lambda_j l_j}
\]

\[
\times \frac{1}{2\pi i} \int L \phi(\xi) z^{\xi} d\xi \left( \int_{0}^{+} \alpha, \beta, \eta t^{\mu + \sum_{j=1}^{k} \lambda_j l_j + \nu \xi - 1}(x) d\xi \right)
\]

Finally, applying Lemma 1 and then re-interpreting the Mellin-Barnes contour integral thus obtained in terms of the I-function, we arrive at the RHS of (4.3.1) after a little simplification.

**Theorem 2**

\[
\left[ \int_{\alpha,\beta,\eta}^{\alpha',\beta',\eta'} \left( t^{\mu-1} \prod_{j=1}^{k} S_{n_j}^{m_j} C_j t^{\lambda_j} \right) \right]_{P_l+Q_l+R}^{M,N} \left\{ z t^{\mu} \left( \begin{array}{c} (a_j', \alpha_j')_{1,N} ; (a_j', \alpha_j')_{N+1,P_l} \\ b_j', \beta_j' ; b_j', \beta_j' \end{array} \right) \right\} (x)
\]

\[
= x^{\mu-\beta-1} \sum_{l_1=0}^{[n_1/m_1]} \sum_{l_2=0}^{[n_2/m_2]} \ldots \sum_{l_k=0}^{[n_k/m_k]} \left\{ \frac{(-n_1)_{m_1 l_1} \ldots (-n_h)_{m_h l_h}}{l_1! \ldots l_h!} \right\}
\]

\[
\times A'_{n_1,l_1} \ldots A^{(k)}_{n_h,l_h c_1 l_1 \ldots c_h l_h \sum_{j=1}^{k} \lambda_j l_j}
\]

\[
\times \int_{P_l+2Q_l+2R}^{M,N+2} \left\{ \begin{array}{c} z x^{\mu} \left( \mu - \beta + \sum_{j=1}^{k} \lambda_j l_j + \nu \right) \right\}
\]

\[
\left( b_j', \beta_j' ; b_j', \beta_j' \right)_{M+1,Q_l'}
\]
\[
\begin{align*}
\left( \mu - \eta + \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right), \left( a_{j', \lambda}, a_{j' \lambda} \right)_{1,N} ; \left( a_{j', \lambda} \right)_{N+1, P_i} \\
\left( \mu + \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right), \left( \mu - \alpha - \beta - \eta - \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right) \end{align*}
\]
\[\ldots(4.3.2)\]

The conditions of validity of (4.3.2) are as follows:

(i) \( \alpha, \beta, \eta, a, b, z \in \mathbb{C} \) and \( \lambda_j, v > 0 \ \forall \ j \in 1, k \)

(ii) \(|\arg z| < \frac{1}{2} \Omega_{l} \pi \); \( \Omega_{l} > 0 \) where

\[
\Omega_{l} = \sum_{j=1}^{N} \alpha_j - \sum_{j=N+1}^{P_i} \alpha_{j'i} + \sum_{j=1}^{M} \beta_j - \sum_{j=M+1}^{Q_i} \beta_{ji} \ \forall \ i \in 1, R
\]

(iii) \( Re(\alpha) > 0 \) and \( Re(\mu) - \nu \ \min_{1 \leq j \leq M} Re \left( \frac{b_j}{\beta_j} \right) < \min \{ Re(\beta), Re(\eta) \} \)

**Proof:** Proceeding on the lines similar to those followed for proving the Theorem 1 and using the Lemma 2, we easily arrive at the desired result.

### 4.4 Special Cases

(i) If we put \( \beta = -\alpha \) in (4.3.1) then in view of (4.3.2), we get the following new and interesting corollary concerning Riemann-Liouville fractional integral operator defined by (4.2.4):

**Corollary 1**

\[
\begin{align*}
&\left[ J_{0+}^{\alpha} \left( t^{\mu - 1} \prod_{j=1}^{k} \sum_{n_j}^{m_j} \left[ c_j t^{\lambda_j} \right] \right) \right]_{P_i, Q_i} \left( \sum_{l=0}^{l+u} \left( \frac{b_j}{\beta_j} \right)_{1,N} ; \left( b_{j'i}, \beta_{j'i} \right)_{M+1, Q_i} \right) \right] (x) \\
= x^{\mu - \beta - 1} \sum_{l_1=0}^{[n_1/m_1]} \sum_{l_2=0}^{[n_2/m_2]} \sum_{l_k=0}^{[n_k/m_k]} \left\{ \frac{(-1)^{m_1 l_1} \ldots (-1)^{m_1 l_1}}{l_1! \ldots l_h!} \right\} \\
& \times A'_{n_1 l_1} \ldots A^{(k)}_{n_h l_h} c_{l_1} \ldots c_{l_h} x^{l_1 a_1 + \ldots + l_h a_h}
\end{align*}
\]
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\[
\times I^{M,N+1}_{P_i+1,Q_i+1;R} \left\{ \begin{array}{c}
Z^{\nu} \left( 1 - \mu - \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right), \\
\left( b_{j'}, \beta_{j'} \right)_{1, N}; \left( b_{j'} \right)'_{M+1, Q_i} \\
(1 - \mu - \alpha - \sum_{j=1}^{k} \lambda_j l_j, v; 1) \end{array} \right\}
\]

which holds under the conditions easily obtainable from those mentioned with (4.3.1).

(ii) On putting \( \beta = 0 \) in (4.3.1), then in view of (4.2.10), we get the following corollary pertaining to Erdélyi-Kober fractional integral operators defined by (4.2.6):

**Corollary 2**

\[
\left[ J^{\mu,\alpha}_{n_1,k} \left( t^{\mu-1} \prod_{j=1}^{k} S^{m_j}_{n_j} \left[ t^{\lambda_j j} \right] I^{M,N}_{P_i,Q_i;R} \left\{ Z^{\nu} \left( a_{j'}, \alpha_{j'} \right)_{1, N}; \left( a_{j'} \right)'_{N+1,P_i} \left( b_{j'} \right)'_{M+1, Q_i}; \left( \alpha_{j'} \right)' \right\} \right) \right] (x) 
\]

\[
= x^{\mu-1} \sum_{l_1=0}^{[n_1/m_1]} \sum_{l_2=0}^{[n_2/m_2]} \ldots \sum_{l_k=0}^{[n_k/m_k]} \left\{ \frac{(-n_1)_{-l_1} \ldots (-n_h)_{-l_h}}{l_1! \ldots l_h!} A'_{n_1,l_1} \ldots A^{(k)}_{n_h,l_h} c_1 l_1 \ldots c_k l_k = \Sigma_{j=1}^{k} \lambda_j j \right\} 
\]

\[
\times I^{M,N+1}_{P_i+1,Q_i+1;R} \left\{ \begin{array}{c}
Z^{\nu} \left( 1 - \mu - \eta - \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right), \\
\left( b_{j'}, \beta_{j'} \right)_{1, N}; \left( b_{j'} \right)'_{M+1, Q_i} \\
(1 - \mu - \alpha - \eta - \sum_{j=1}^{k} \lambda_j l_j, v; 1) \end{array} \right\}
\]

\[
\ldots (4.4.2)
\]

where

\[ Re(\alpha) > 0 \quad \text{and} \quad Re(\mu) + v \min_{1 \leq j \leq M} Re \left( \frac{b_j}{\beta_j} \right) > -Re(\eta) \]
and the conditions (i) and (ii) mentioned with Theorem 1 are also satisfied.

(iii) If we put $\beta = -\alpha$ in (4.3.2), then in view of (4.2.12) we arrive at the following new and interesting corollary concerning Riemann-Liouville fractional integral operator defined by (4.2.4):

**Corollary 3**

\[
J_{\alpha}^{-}(t^{\mu-1} \sum_{j=1}^{k} \int_{\alpha}^{t} \left[ J_{\alpha}^{-}(m_{j} t \lambda_{j}) I_{P_{i},Q_{i};R}^{M,N} \left( x \left( \begin{array}{c} \sum_{j=1}^{m_{j}} c_{j} \left( \begin{array}{c} a_{j}, \alpha_{j} \\ b_{j}, \beta_{j} \end{array} \right)_{1,N}^{1}; \left( \begin{array}{c} a_{j}', \alpha_{j}' \\ b_{j}', \beta_{j}' \end{array} \right)_{N+1,P_{i}} \end{array} \right) \right] \right)(x) \]

\[
= x^{\mu+\alpha-1} \sum_{l_{1}=0}^{[n_{1}/m_{1}]} \sum_{l_{2}=0}^{[n_{2}/m_{2}]} ... \sum_{l_{k}=0}^{[n_{k}/m_{k}]} \left( \frac{(-n_{1})_{l_{1} \ldots (-n_{h})_{m_{h} l_{h}}}{l_{1}! \ldots l_{h}!} \right)
\]

\[A'_{n_{1} l_{1}} \ldots A(k)_{n_{h} l_{h}} c_{1} \ldots c_{h} l_{h} X^{k} f_{j=1}^{k} \lambda_{j} l_{j} \]

\[
\times I_{P_{i},Q_{i}+2;R}^{M,N+2} \left( x \left( \begin{array}{c} \mu + \alpha + \sum_{j=1}^{k} \lambda_{j} l_{j}, v; 1 \\ b_{j}', \beta_{j}' \end{array} \right)_{1,N}^{1}; \left( \begin{array}{c} a_{j}', \alpha_{j}' \\ b_{j}', \beta_{j}' \end{array} \right)_{M+1,Q_{i}'}^{M+1,Q_{i}'} \end{array} \right) \]

\[
(\begin{array}{c} a_{j}', \alpha_{j}' \end{array})_{1,N}^{1}; (\begin{array}{c} a_{j}', \alpha_{j}' \end{array})_{N+1,P_{i}}^{N+1,P_{i}} \end{array} \right) \]

\[
\left( \mu + \sum_{j=1}^{k} \lambda_{j} l_{j}, v; 1 \right) \]

\[\ldots (4.4.3)\]

which holds under the conditions obtainable from those mentioned with (4.3.2).

(iv) On putting $\beta = 0$ in (4.3.2), then in view of (4.2.13) we get the following corollary pertaining to Erdélyi-Kober fractional integral operator defined by (4.2.7):

**Corollary 4**

\[
K_{\eta,a}^{-}(t^{\mu-1} \sum_{j=1}^{k} S_{n_{j}}^{m_{j}} \left[ c_{j} t^{\lambda_{j}} \right] I_{P_{i},Q_{i};R}^{M,N} \left( x \left( \begin{array}{c} \sum_{j=1}^{m_{j}} c_{j} \left( \begin{array}{c} a_{j}, \alpha_{j} \\ b_{j}, \beta_{j} \end{array} \right)_{1,N}^{1}; \left( \begin{array}{c} a_{j}', \alpha_{j}' \\ b_{j}', \beta_{j}' \end{array} \right)_{N+1,P_{i}} \end{array} \right) \right)(x) \]

\[
= x^{\mu-1} \sum_{l_{1}=0}^{[n_{1}/m_{1}]} \sum_{l_{2}=0}^{[n_{2}/m_{2}]} ... \sum_{l_{k}=0}^{[n_{k}/m_{k}]} \left( \frac{(-n_{1})_{l_{1} \ldots (-n_{h})_{m_{h} l_{h}}}{l_{1}! \ldots l_{h}!} \right)
\]

\[A'_{n_{1} l_{1}} \ldots A(k)_{n_{h} l_{h}} c_{1} \ldots c_{h} l_{h} X^{k} f_{j=1}^{k} \lambda_{j} l_{j} \]

\[
\times I_{P_{i},Q_{i}+2;R}^{M,N+2} \left( x \left( \begin{array}{c} \mu + \alpha + \sum_{j=1}^{k} \lambda_{j} l_{j}, v; 1 \\ b_{j}', \beta_{j}' \end{array} \right)_{1,N}^{1}; \left( \begin{array}{c} a_{j}', \alpha_{j}' \end{array} \right)_{M+1,Q_{i}'}^{1}; (\begin{array}{c} a_{j}', \alpha_{j}' \end{array})_{N+1,P_{i}}^{N+1,P_{i}} \end{array} \right) \]

\[
\left( \mu + \sum_{j=1}^{k} \lambda_{j} l_{j}, v; 1 \right) \]

\[\ldots (4.4.3)\]
\begin{equation}
A'_{n_1,l_1} \ldots A^{(k)}_{n_h,l_h} c_1 l_1 \ldots c_h l_h x^{k} \sum_{j=1}^{k} \lambda_j l_j \right) \right) \right) \\
\times i_{P_i+2, Q_i+2; R}^{M,N+2} \left\{ ZX^u \left( \mu - \eta + \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right), (b'_j, \beta'_j)_{1,N}; (b'_j', \beta'_j')_{M+1,Q'_i} \right. \\
\left. \left( a'_j, \alpha'_j \right)_{1,N}; (a'_j', \alpha'_j')_{N+1,P'_i} \right) \right. \\
\left( \mu - \alpha - \eta - \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right) \right) \right) \\
\ldots (4.4.4) \end{equation}

which holds under the conditions easily obtainable from those mentioned with (4.3.2).

(v) If we take $P_i = P, Q_i = Q$ and $R = 1$ in the Theorem 1, we get the following result in terms of Fox’s H-Function [5,7]:

**Corollary 5**

\begin{equation}
\left. \left[ \int_{0^+}^{\alpha, \beta, \eta} \left( t^{\mu-1} \prod_{j=1}^{k} S_{n_j}^{m_j} c_j t^{\lambda_j} \right) H_{P,Q}^{M,N} \left\{ zt^u \left( (a'_j, \alpha'_j)_{1,P} \right) \right) \right] \right) (x) \\
= x^{\mu-\beta-1} \sum_{l_1=0}^{[n_1/m_1]} \sum_{l_2=0}^{[n_2/m_2]} \ldots \sum_{l_k=0}^{[n_k/m_k]} \left\{ \left( \frac{(-1)^{m_1} \ldots (-1)^{m_h}}{l_1! \ldots l_h!} \right) \right. \\
A'_{n_1,l_1} \ldots A^{(k)}_{n_h,l_h} c_1 l_1 \ldots c_h l_h x^{k} \sum_{j=1}^{k} \lambda_j l_j \right) \right) \right) \\
\times H_{P+2, Q+2}^{M,N+2} \left\{ ZX^u \left( 1 - \mu - \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right), (b'_j, \beta'_j)_{1,Q}; (1 - \mu + \beta - \sum_{j=1}^{k} \lambda_j l_j, v; 1) \right) \right) \right) \\
\ldots (4.4.4) \end{equation}
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\[
\left( 1 - \mu - \eta + \beta \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right), (\alpha_j', \alpha_j')_{1,P} \\
\left( 1 - \mu - \alpha - \eta - \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right)
\]

provided that the conditions easily obtainable from those mentioned with Theorem 1 are satisfied.

(vi) On setting \( R = 1, M = 1, N = P_i = P, Q_i = Q + 1, b_i' = 0, \beta_i' = 1, \alpha_i' = 1 - a_i', b_j' = 1 - b_j', \beta_j' = \beta_j' \) in (4.3.1), we arrive at the following result in terms of Wright’s generalized hypergeometric function [15]:

**Corollary 6**

\[
\left[ \frac{\psi_{\alpha, \beta, \eta}}{t^{\mu-1}} \prod_{j=1}^{k} S_{m_j}^{m_j} \left[ c_j t^\lambda_j \right] p \psi_{q} \left( -z t^u \left( (\alpha_j', \alpha_j')_{1,P} \right) \right) \right] (x) \\
= x^{\mu-\beta-1} \sum_{l_1=0}^{[n_1/m_1]} \sum_{l_2=0}^{[n_2/m_2]} \sum_{l_h=0}^{[n_h/m_h]} \left\{ \frac{(-n_1 m_1 l_1 \ldots (-n_h m_h l_h)}{l_1! \ldots l_h!} \right\}

A'_{n_1 l_1} \ldots A^{(k)}_{n_h l_h} c_1^{l_1} \ldots c_h^{l_h x^{\sum_{j=1}^{k} \lambda_j l_j}}

\times \int_{P+2,Q+3:1}^{1,P+2,Q+3:1} \left( 1 - \mu - \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right), (0,1); (1 - b_j', \beta_j')_{1,Q'} \left( 1 - \mu + \beta - \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right),

\left( 1 - \mu - \eta + \beta - \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right), (1 - a_j', \alpha_j')_{1,P}

\left( 1 - \mu - \alpha - \eta - \sum_{j=1}^{k} \lambda_j l_j, v; 1 \right)

\right)
provided that the conditions easily obtainable from those mentioned with Theorem 1 are satisfied.

On suitably specifying the parameters in (4.3.1), we arrive at the results due to Hussain [13], Kilbas [15] and Kilbas and Sebastian [19].

A number of other special cases of Theorem 2 can also be obtained on following the lines similar to those mentioned above for the Theorem 1 but we do not record them here explicitly.

**Conclusion**

In this chapter, the images of the generalized fractional integral operators given by Saigo have been developed in terms of the product of I-function and general class of polynomials. The results obtained here, besides being of very general character, have been put in a compact form, avoiding the occurrence of infinite series and thus making them useful in applications. Also, these results provide unification and extension of the results obtained earlier in the literature.