Chapter 1

Preliminaries

This chapter is essentially a collection of the basic definitions, notations and terminology which are repeatedly used in the later chapters. Most of the materials that are presented here are well-known. They are included here only for completeness and for quick reference.

In Section 1.1 we consider the notations and terminology relating to the general theory of semigroups. We have considered Green's relations, regular semigroups, rectangular bands and biordered sets. Section 1.2 is devoted to a discussion of the semigroup of linear endomorphisms of a finite dimensional vector space. Section 1.3 deals with the definition of manifolds. We have included here a brief treatment of the Grassmann manifolds. Section 1.4 contains the definition of Lie groups. Fibre bundles and vector bundles are treated in Section 1.5.

1.1 Semigroups

With regard to the notations and terminology relating to semigroups, we have followed [CP61]. In particular, we shall always take composition of functions in the diagram order.

**Definition 1.1.1.** A set $S$ together with an associative binary operation in $S$ is called a semigroup.

**Definition 1.1.2.** An element $e$ in a semigroup $S$ is called an idempotent if $e^2 = e$. 
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1.1 Semigroups

If X is any subset of a semigroup S, then the set of idempotents in X is denoted by \( E(X) \).

1.1.1 Green's Relations

Let S be a semigroup. We define \( S^1 \) to be S if S has an identity element. Otherwise we define \( S^1 \) to be S with an identity element 1 adjoined.

Definition 1.1.3. Let S be a semigroup. The relations \( \mathcal{L} \), \( \mathcal{R} \), \( \mathcal{J} \), \( \mathcal{D} \) and \( \mathcal{H} \) in S defined as below are called Green's relations in S. For \( a, b \in S \):

1. \( a \mathcal{L} b \iff aS^1 = bS^1 \)
2. \( a \mathcal{R} b \iff S^1a = S^1b \)
3. \( a \mathcal{J} b \iff S^1aS^1 = S^1bS^1 \)
4. \( \mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L} \)
5. \( \mathcal{H} = \mathcal{L} \cap \mathcal{R} \)

Green's relations in a semigroup S are equivalence relations in S. If \( a \in S \), the set of all elements in S which are \( \mathcal{L} \)-equivalent to \( a \) is denoted by \( L_a \). \( L_a \) is the \( \mathcal{L} \)-class containing \( a \). The notations \( R_a, J_a, D_a, H_a \) have similar meanings. These classes are referred to as the Green classes in S.

The following result (see Lemma in 2.2 [CP61]), known as Green's lemma, is occasionally invoked later.

Lemma 1.1.4. Let S be a semigroup. Let \( a, b \in S \) be such that \( a \mathcal{R} b \). There exists elements \( s, s' \in S^1 \) such that \( as = b \) and \( bs' = a \). Then the mappings \( x \mapsto xs \) (\( x \in L_a \)) and \( y \mapsto ys' \) (\( y \in L_b \)) are mutually inverse, \( \mathcal{R} \)-class preserving, one-to-one mappings of \( L_a \) onto \( L_b \) and of \( L_b \) onto \( L_a \) respectively.

1.1.2 Regular Semigroups

Definition 1.1.5. An element \( a \) in a semigroup S is called a regular element if \( a = axa \) for some element \( x \) in S. If every element in S is a regular element then S is called a regular semigroup.
Definition 1.1.6. Two elements $a, a'$ in a semigroup $S$ are called inverse elements if $aa' = a$, $a'a = a'$.

A few of the important properties of regular semigroups which are required later are given below (for details, see Section 2.3 in [CP61]).

Theorem 1.1.7. Let $S$ be a regular semigroup and $a, b \in S$. Then $a \mathcal{L} b$ if and only if $S^{-1}a = S^{-1}b$, and $a \mathcal{R} b$ if and only if $aS = bS$.

Theorem 1.1.8. Let $S$ be a regular semigroup. Then every $\mathcal{L}$-class and every $\mathcal{R}$-class contained in $S$ contains an idempotent.

Theorem 1.1.9. If $a, b$ and $ab$ all belong to the same $\mathcal{H}$-class $H$ of a semigroup $S$, then $H$ is a subgroup of $S$.

Theorem 1.1.10. Let $a, b$ be elements of a semigroup $S$. If $R_b \cap L_a$ contains an idempotent then

$$aH_b = H_a b = H_a H_b = H_{ab} = R_a \cap L_b.$$ 

Let $U$ be a subsemigroup of a semigroup $S$. Let $a, b \in U$. We write $a \mathcal{L}^U b$ to mean $U^{-1}a = U^{-1}b$, and $a \mathcal{L}^S b$ to mean $S^{-1}a = S^{-1}b$. In a similar way we assign meanings to $\mathcal{R}^U$ and $\mathcal{R}^S$. We have the following result (see Proposition II.4.5 in [How76]) involving these relations.

Proposition 1.1.11. If $U$ is a regular subsemigroup of a semigroup $S$, then:

1. $\mathcal{L}^U = \mathcal{L}^S \cap (U \times U)$.
2. $\mathcal{R}^U = \mathcal{R}^S \cap (U \times U)$.
3. $\mathcal{H}^U = \mathcal{H}^S \cap (U \times U)$.

1.1.3 Unit Regular Semigroups

An element $u$ in a monoid $S$ is said to be a unit if there exists an element $u'$ in $S$ such that $uu' = u'u = 1$ (see [CP61]). An element $x$ of $S$ is said to be unit regular if there is a unit $u$ in $S$ such that $xux = x$. If all elements of $S$ are unit regular, then $S$ is said to be a unit regular semigroup (see [Vee86]).
Let $S$ be a unit regular semigroup with group of units $G$. Two elements $x, y$ of $S$ are said to be conjugate relative to $G$ if there is an element $u$ in $G$ such that $y = u x u^{-1}$. If any two $\mathcal{D}$-related idempotents in $S$ are conjugate relative to $G$ then $S$ is called a strongly unit regular semigroup (see [Vee86]).

### 1.1.4 Rectangular Bands

In view of Proposition IV.3.2 in [How76], a rectangular band can be defined as follows.

**Definition 1.1.12.** A semigroup $S$ is a rectangular band if $E(S) = S$ and, for all $a, b, c \in S$ we have $abc = ac$.

The following elementary property of rectangular bands is required.

**Proposition 1.1.13.** Let $S$ be a rectangular band and $a \in S$.

1. The map $(x, y) \mapsto xy$ is a bijection from $L_a \times R_a$ onto $S$, and so $L_a R_a = S$.

2. $R_a L_a = \{a\}$.

**Proof.** Let $z \in S$. Since every $\mathcal{H}$-class in a rectangular band contains a unique element, the $\mathcal{H}$-classes $L_a \cap R_z$ and $L_z \cap R_a$ contain unique elements $x$ and $y$. Now, since $x \in L_a \cap R_z$ we have $zx = x$. Similarly we also have $yz = y$. Therefore $xy = (zx)(yz) = z(xy)z = zz = z$. Thus, to each element $z \in S$, there is an element $(x, y) \in L_a \times R_a$ such that $xy = z$.

To show that the pair $(x, y)$ is unique, let $(x', y') \in L_a \times R_a$ be such that $x'y' = z$. Then $x'z = x'(x'y') = x'y' = z$ and $zx' = (x'y')x' = x'x' = x'$. These imply that $x' \in R_z$ and hence $x' \in L_a \cap R_z$. Since $L_a \cap R_z$ contains a unique element $x$ we must have $x' = x$. Similarly we have $y' = y$. Hence the map specified in the statement of the proposition is a bijection.

If $y \in R_a$ and $x \in L_a$ then

$$yx = (ay)(xa) = a(yx)a = aa = a.$$  

This shows that $R_a L_a = \{a\}$. \qed
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1.1.5 Biordered Sets

For the sake of later reference, we give here the complete set of axioms for a biordered set (see Definition 1.1 in [Nam79]).

Let $E$ denote a partial algebra and let $D_E$ denote the domain of the partial binary operation on $E$; that is, let

$$D_E = \{(e, f) \in E \times E : ef \text{ exists in } E\}.$$ 

On $E$ we define the following relations, called biorder relations:

- $\omega^l = \{(e, f) \in E \times E : ef = e\}$,
- $\omega^r = \{(e, f) \in E \times E : ef = f\}$,
- $\omega = \omega^l \cap \omega^r$,
- $L = \omega^l \cap (\omega^l)^{-1}$,
- $R = \omega^r \cap (\omega^r)^{-1}$.

If $\tau$ be one of these biorder relations and $e$ an element in $E$, then the set $\{f \in E : (f, e) \in \tau\}$ is denoted by $\tau(e)$. These sets are called the biorder ideals in $E$.

Recall that a quasiorder on a set $X$ is a reflexive and transitive relation on $X$.

**Definition 1.1.14.** A biordered set $E$ is a partial algebra satisfying the following axioms; here $e, f, g, h$ etc. be arbitrary elements in $E$:

1. (B1) $\omega^l$ and $\omega^r$ are quasiorders on $E$, and $D_E = (\omega^l \cup \omega^r) \cup (\omega^l \cup \omega^r)^{-1}$.

2. (B2) $g \omega^l f \omega^l e \Rightarrow eg, ef \in \omega(e)$ and $fg = f(eg)$;
   $g \omega^r f \omega^r e \Rightarrow ge, fe \in \omega(e)$ and $gf = (ge)f$.

3. (B3) $g \omega^l f$ and $g, f \in \omega^r(e)$ \Rightarrow $ge \omega^l fe$ and $(fg)e = (fe)(ge)$;
   $g \omega^r f$ and $g, f \in \omega^l(e)$ \Rightarrow $eg \omega^r ef$ and $e(gf) = (eg)(ef)$.

4. (B4) If $g, h \in \omega^l(e)$ and $eg \omega^r eh$ then there exists $g_1 \in \omega^l(e)$ such that $g_1 \omega^r h$ and $eg_1 = eg$.
   If $g, h \in \omega^r(e)$ and $eg \omega^l he$ then there exists $g_1 \in \omega^r(e)$ such that $g_1 \omega^l h$ and $g_1 e = ge$. 

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Remark 1.1.1. The set of axioms given above are slightly different from those given in [Nam79]. The axiom (B2) above combines axioms (B21) and (B31) of [Nam79]. The present (B3) combines (B22) and (B32) of [Nam79]. If \( fe \in e \) and \( f = ff = (fe)f \). Then \( fe \in e \), and so again by (B2), \( f(fe) = (fe)(fe) = fe \) which implies that \( f \in e \). Thus (B21) holds. Also the present axiom (B4) is the same as (B4') of [Nam79], which by Proposition 2.4 in [Nam79], is equivalent to (B4) of [Nam79]. (See also [Nam93].)

Let \( E \) be a biordered set and \( e, f \in E \). We write

\[
M(e, f) = \omega^l(e) \cap \omega^r(f)
\]

and define a relation \( \prec \) in \( M(e, f) \) as follows.

\[
g \prec h \iff df w f \quad \text{and} \quad eg w f \quad \text{eh}.
\]

This relation is a quasiorder on \( M(e, f) \). The sandwich set \( S(e, f) \) is defined as follows:

\[
S(e, f) = \{ h \in M(e, f) : g \prec h \quad \text{for all} \quad g \in M(e, f) \}.
\]

Definition 1.1.15. Let \( E \) be a biordered set. \( E \) is said to be regular if \( S(e, f) \neq \emptyset \) for all \( e, f \in E \).

We have the following basic result (see Theorem 1.1 in [Nam79]).

Theorem 1.1.16. Let \( S \) be a regular semigroup. Then \( E(S) \) is a regular biordered set. Also, for any \( e, f \in E(S) \) we have

\[
S(e, f) = \{ h \in M(e, f) : ef = ef \}.
\]

It may be noted here that, for any semigroup \( S \), \( E(S) \) is a biordered set and further the biorder relations \( \mathcal{L} \) and \( \mathcal{R} \) in \( E(S) \) are the restrictions to \( E(S) \) of the Green's relations \( \mathcal{L} \) and \( \mathcal{R} \) in \( S \).
Chains and cycles

Let $E$ be a biordered set. By an $E$-sequence in $E$ we mean a finite sequence $s = s(e_0, e_1, \ldots, e_r)$ of elements in $E$ such that, for $i = 1, \ldots, r$, we have

$$e_{i-1} \mathcal{L} e_i \quad \text{or} \quad e_{i-1} \mathcal{R} e_i.$$ 

The elements $e_0, e_1, \ldots, e_r$ are called the vertices of the sequence $s$. The vertex $e_0$ is called the origin and the vertex $e_r$ the extremity of the sequence $s$. If $s = (e_0, e_1, \ldots, e_r)$ and $s' = s'(f_0, f_1, \ldots, f_t)$ are $E$-sequences, and if $e_r = f_0$, we denote by $ss'$ the $E$-sequence $(e_0, e_1, \ldots, e_r, f_1, \ldots, f_t)$. Note that $ss'$ is undefined if $e_r \neq f_0$. Further, if $s = s(e_0, e_1, \ldots, e_r)$ is an $E$-sequence, then the sequence $(e_r, e_{r-1}, \ldots, e_1, e_0)$ is also an $E$-sequence. This sequence is denoted by $s^{-1}$.

We say that an $E$-sequence $s = s(e_0, e_1, \ldots, e_r)$ is reduced if there exists no $i$ with $1 \leq i \leq r - 1$ such that $e_{i-1} \mathcal{L} e_i \mathcal{L} e_{i+1}$ or $e_{i-1} \mathcal{R} e_i \mathcal{R} e_{i+1}$. Given any $E$-sequence $s$, by combining consecutive similar edges, one can obtain a unique reduced sequence whose origin and extremity are the same as those of $s$. Such a reduced sequence is called an $E$-chain (or, simply, a chain). Chains are denoted as $c(e_0, e_1, \ldots, e_r)$. An $E$-chain $c = c(e_0, e_1, \ldots, e_r)$ is called a $E$-cycle (or, simply, a cycle) based at $e$ if $e_0 = e_r = e$. An $E$-cycle having four edges is called an $E$-square. The $E$-square $c = c(e, f, g, h, e)$ is denoted as a matrix $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$.

Let $S$ be a regular semigroup and let $c(e_0, e_1, \ldots, e_{r-1}, e_0)$ be a cycle in the biordered set $E(S)$. $c$ is said to be commutative if $e_0 e_1 \cdots e_{r-1} e_0 = e_0$.

1.2 Linear Endomorphisms

As standard references to linear algebra we have followed [Hal58, Her88].

It will be assumed throughout that the letter $\mathbb{K}$ denotes either the field $\mathbb{R}$ of all real numbers, or the field $\mathbb{C}$ of all complex numbers. Unless otherwise specified, it will also be assumed that $V$ denotes an $n$-dimensional vector space over $\mathbb{K}$, where $n$ is a positive integer.

The set of all linear endomorphisms of $V$ is denoted by $\text{End}(V)$. The set of all singular endomorphisms of $V$ is denoted by $S_n$. The set of all nonsingular endomorphisms of $V$, that is, the set of all invertible elements in $\text{End}(V)$, is denoted by
GL(V) or GL(n). Choosing an ordered basis for V we can represent the elements of End(V) by square matrices of order n. Thus, End(V) can be identified with the set $M_n(\mathbb{K})$ of all square matrices of order n over $\mathbb{K}$.

If we consider the elements of $M_n(\mathbb{K})$ as endomorphisms of V, then the operation of composition of mappings is a binary operation in $M_n(\mathbb{K})$. $M_n(\mathbb{K})$ is a semigroup under this binary operation. If we consider elements of $M_n(\mathbb{K})$ as square matrices of order n, the operation of multiplication of matrices is a binary operation in $M_n(\mathbb{K})$. $M_n(\mathbb{K})$ is also a semigroup under this operation. This is the usual binary operation in $M_n(\mathbb{K})$.

1.2.1 Green's Relations in $M_n(\mathbb{K})$

To each element $a$ in $M_n(\mathbb{K})$ we associate two subspaces of V, namely, the range $R(a)$ and the null space $N(a)$ of $a$. These are given by

$$R(a) = \{ x \in V : x = ya \text{ for some } y \in V \},$$
$$N(a) = \{ x \in V : xa = 0 \}.$$ 

Green's relations in $M_n(\mathbb{K})$ can be characterized in terms of these subspaces (see p.57 in [CP61]).

**Proposition 1.2.1.** Let $a, b \in M_n(\mathbb{K})$. Then:

1. $a \mathcal{L} b \iff R(a) = R(b)$.
2. $a \mathcal{R} b \iff N(a) = N(b)$.
3. $a \mathcal{D} b \iff \text{Rank}(a) = \text{Rank}(b)$.
4. $\mathcal{J} = \mathcal{D}$.

We give another characterization of Green's relations in $M_n(\mathbb{K})$.

**Proposition 1.2.2.** Let $a, b \in M_n(\mathbb{K})$. Then:

1. $a \mathcal{L} b \iff b = ua$ for some $u \in \text{GL}(n)$.
2. $a \mathcal{R} b \iff b = au$ for some $u \in \text{GL}(n)$.
3. \( a \not\sim b \iff b = uav \) for some \( u, v \in \text{GL}(n) \).

**Proof.** If \( b = ua \), obviously, we have \( a \not\sim b \).

Conversely, let \( a \not\sim b \). By Proposition 1.2.1, we have \( R(a) = R(b) \). Let \( W, W' \) be subspaces of \( V \) such that

\[ N(a) \oplus W = N(b) \oplus W' = V. \]

Then the restrictions \( a|W : W \to R(a) \) and \( b|W' : W' \to R(b) \) are linear isomorphisms. Let \( u_1 : N(b) \to N(a) \) be any linear isomorphism and define \( u_2 : W' \to W \) by \( u_2 = (b|W') \circ (a|W)^{-1} \). If we now set \( u = u_1 \oplus u_2 \), then we have \( b = ua \).

The other characterizations can be proved in a similar way.

The next proposition gives a representation of the \( \mathcal{L} \) and \( \mathcal{R} \)-classes in \( M_n(\mathbb{K}) \).

**Proposition 1.2.3.** Let \( a \in S_n \), \( R(a) = W \) and \( N(a) = N \).

1. \( L_a \) is the set of all surjective mappings in the linear space \( L(V, W) \) of all linear mappings of \( V \) into \( W \).

2. Let \( L^0(V, N) \) be the set of all linear mappings \( x \) of \( V \) into \( V \) such that \( N \subseteq N(x) \). Let \( W' \) be a subspace of \( V \) such that \( W' \oplus N = V \). Then the map \( \rho : x \mapsto x|W' \) is a linear isomorphism of \( L^0(V, N) \) onto \( L(W', V) \). Moreover, \( \rho(R_a) \) is the set of all injective elements in the linear space \( L(W', V) \).

3. If \( W \oplus N = V \), then \( H_a \) is a group and it is isomorphic to the group \( \text{GL}(W) \).

**Proof.**

1. This immediately follows from the fact that \( x \in L_a \) if and only if \( R(x) = R(a) = W \) (see Proposition 1.2.7).

2. Let \( x, y \in L^0(V, N) \) be such that \( x|W' = y|W' \). Since \( N \subseteq N(x) \) we have \( x|N = 0 \). Similarly we have \( y|N = 0 \). Further we have \( W' \oplus N = V \). From all these we see that \( x = y \). Therefore the mapping \( \rho_{W'} : x \mapsto x|W' \) is injective. Now let \( y' \in L(W', V) \). Define \( y \) by

\[ xy = \begin{cases} xy' & \text{if } x \in W' \\ 0 & \text{if } x \in N. \end{cases} \]
Now, extending \( y \) to the whole of \( V \) linearly in an obvious way, we have \( y \in \text{End}(V) \). Also we have \( y|W' = y' \). Thus the map \( \rho \) is surjective also. It can be easily seen that this map is indeed a linear isomorphism.

Let \( y \in R_a \) so that \( N(y) = N \). Let \( y' = y|W' \). Let \( x \in W' \) be such that \( xy' = 0 \). Then we also have \( xy = 0 \). This means that \( x \in N \). Thus \( x \in W' \cap N \). Since \( W' \oplus N = V \), this implies that \( x = 0 \). Therefore \( y' \) is injective. Conversely, if \( y' = y|W' \) is injective, then, for any \( x \in W' \), \( xy' = 0 \) if and only if \( x = 0 \). So we must have \( N(y) = N \) implying that \( y \in R_a \).

3. See p.57 in [CP61].

Let \( W \subseteq V \) \( [N \subseteq V] \) be a subspace of \( V \). Then the \( \mathcal{L} \)-\([\mathcal{R}]\)-class determined by \( W \) \([N]\) is denoted by \( L_W \) \([R_N]\). These are

\[
\begin{align*}
L_W &= \{ x \in M_n(\mathbb{K}) : R(x) = W \} & (1.2.1) \\
R_N &= \{ x \in M_n(\mathbb{K}) : N(x) = N \}. & (1.2.2)
\end{align*}
\]

We have the following interesting result.

**Lemma 1.2.4.** Let \( a, b \in M_n(\mathbb{K}) \) be such that \( b = uav^{-1} \) for some \( u, v \in \text{GL}(n) \). Then

1. \( L_b = uL_av^{-1} = L_{av^{-1}} \).
2. \( R_b = uR_av^{-1} = R_{av} \).

**Proof.** Let \( x \in L_a \) so that \( y = uxv^{-1} \in uL_av^{-1} \). Now,

\[
\begin{align*}
M_n(\mathbb{K})y &= M_n(\mathbb{K})uxv^{-1} \\
&= M_n(\mathbb{K})xv^{-1} & \text{since } u \in \text{GL}(n) \\
&= M_n(\mathbb{K})av^{-1} & \text{by Theorem 1.1.7, since } x \in L_a \\
&= M_n(\mathbb{K})uav^{-1} & \text{since } u \in \text{GL}(n) \\
&= M_n(\mathbb{K})b.
\end{align*}
\]
Therefore, by Theorem 1.1.7, we have $y \in L_b$. Therefore $uL_av^{-1} \subseteq L_b$. The proof of the reverse inclusion is also similar. Thus we have $uL_av^{-1} = L_b$.

If $y \in uL_av^{-1}$ then we have already shown that $M_n(\mathbb{K})y = M_n(\mathbb{K})av^{-1}$. As above, this implies that $uL_av^{-1} = L_av^{-1}$. Hence $L_b = L_av^{-1}$.

The proof of the result concerning $\mathcal{R}$-classes is similar.

Let $k$ be an integer such that $0 \leq k \leq n$. We write

$$D_k = \{ a \in M_n(\mathbb{K}) : \text{Rank}(a) = k \}. \quad (1.2.3)$$

By Proposition 1.2.1, $D_k$ is a $\mathcal{D}$-class of $M_n(\mathbb{K})$.

### 1.2.2 Idempotents and Biorder Relations

**Proposition 1.2.5.** Let $e$ be an idempotent in $S_n$. Then, since $e$ is singular, the dimension of $N(e)$ is at least equal to 1. Also we have $R(e) \oplus N(e) = V$. Conversely, if $W, N$ are subspaces of $V$ such that

$$\dim(N) \geq 1 \text{ and } W \oplus N = V$$

then there is a unique idempotent $e \in S_n$ such that $R(e) = W$ and $N(e) = N$. (This unique idempotent will be denoted by $(W : N)$.)

The set of all idempotents in $S_n$ is denoted by $E_n$. The set of all idempotents in the $\mathcal{D}$-class $D_k$ is denoted by $E(k)$. Thus $E(k)$ is the set of all idempotents of rank $k$. Note that if $e = (W : N) \in E(k)$ then $1 - e = (N : W) \in E(n - k)$. The map

$$\iota : E(k) \to E(n - k), \quad e \mapsto 1 - e \quad (1.2.4)$$

is an isometry.

The biorder relations in $S_n$ are characterized as follows:

**Proposition 1.2.6.** Let $e, f \in E_n$. Then:

1. $e \omega f \iff R(e) \subseteq R(f)$.
2. $e \omega^* f \iff N(f) \subseteq N(e)$. 


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3. \( e \omega f \Leftrightarrow R(e) \subseteq R(f) \) and \( N(f) \subseteq N(e) \).

4. \( e \mathcal{L} f \Leftrightarrow R(e) = R(f) \).

5. \( e \mathcal{R} f \Leftrightarrow N(e) = N(f) \).

1.2.3 Algebraic Structure of \( M_n(\mathbb{K}) \)

Here we consider some of the most important algebraic properties of \( M_n(\mathbb{K}) \) which are significant in our subsequent work.

The following result is well-known (see, for example, p.57 in [CP61]).

**Proposition 1.2.7.** \( S_n \) and \( M_n(\mathbb{K}) \) are regular semigroups under the usual binary operations in them.

As a consequence of Theorem 1.1.16 we have the following corollary to the above result.

**Proposition 1.2.8.** \( E_n \), the set of idempotents in \( S_n \), is a regular biordered set.

From the results in [Erd67], we have the following theorem.

**Theorem 1.2.9.** \( S_n \) is an idempotent generated regular semigroup.

The next proposition gives an interesting property of the semigroup \( S_n \) which has a bearing on the topological properties of \( S_n \) (see [Vee86]).

**Proposition 1.2.10.** \( M_n(\mathbb{K}) \) is a strongly unit regular semigroup.

**Proof.** We begin by noting that the group of units in \( M_n(\mathbb{K}) \) is \( GL(n) \). We first prove that \( M_n(\mathbb{K}) \) is unit regular.

Let \( x \in M_n(\mathbb{K}) \). If \( x \) is invertible, then \( x \in GL(n) \) and so \( x^{-1} \) exists in \( GL(n) \). Therefore, if we take \( u = x^{-1} \) then \( u \in GL(n) \) and \( xux = x \).

Now, let \( x \) be singular. By Theorem 1.1.8, there is \( e \in E_n \) such that \( x \mathcal{L} e \). By Proposition 1.2.2, there is \( v \in GL(n) \) such that \( x = ve \). Now \( xv^{-1}x = xe = x \). Hence, for every \( x \in M_n(\mathbb{K}) \), there exists \( u \in GL(n) \) such that \( xux = x \). We conclude that \( M_n(\mathbb{K}) \) is unit regular.
Finally we show that $M_n(\mathbb{K})$ is strongly unit regular. Let $e, f$ be two $\mathcal{D}$-related idempotents in $M_n(\mathbb{K})$. Then, by Proposition 1.2.1, we must have $\text{Rank}(e) = \text{Rank}(f)$. Therefore

$$\dim(R(e)) = \dim(R(f)), \quad \dim(N(e)) = \dim(N(f)).$$

Since $R(e) \oplus N(e) = V$ and $R(f) \oplus N(f) = V$, we can find $u \in \text{GL}(n)$ such that it maps $R(f)$ onto $R(e)$ and $N(f)$ onto $N(e)$. This means that $R(f)u = R(e)$ and $N(f)u = N(e)$. Hence we have

$$R(f) = R(e)u^{-1} = Veu^{-1} = Vueu^{-1} = R(ueu^{-1}).$$

If $x \in N(f)$ then $xu \in N(e)$. This means that $xue = 0$ and so $xueu^{-1} = 0$ which in turn implies that $x \in N(ueu^{-1})$. The converse implication is also true, that is, if $x \in N(ueu^{-1})$ then $x \in N(f)$. Thus we have $N(f) = N(ueu^{-1})$. Since $f = \langle R(f) : N(f) \rangle = \langle R(ueu^{-1}) : N(ueu^{-1}) \rangle$, we must have $f = ueu^{-1}$. Therefore any two $\mathcal{D}$-related idempotents in $M_n(\mathbb{K})$ are conjugates. We conclude that $M_n(\mathbb{K})$ is a strongly unit regular semigroup.

1.2.4 Perpendicular Projections

Let us now assume that an inner product $\langle \cdot , \cdot \rangle$ has been defined in $V$ so that $V$ becomes an inner product space. If $a \in \text{End}(V)$, then the (Hilbert space) adjoint of $a$ is the element $a^*$ in $\text{End}(V)$ such that $\langle xa, y \rangle = \langle x, ya^* \rangle$ for all $x, y \in V$. From the properties of the adjoint (see Theorem 25.7 in [Lim81]) we can easily see that the map $a \mapsto a^*$ is a homeomorphism of $\text{End}(V)$ onto $\text{End}(V)$.

Let $a$ be represented by the matrix $A$ with respect to a certain ordered orthonormal basis of $V$. Then the conjugate transpose of $A$ represents the adjoint of $a$ with respect to the same ordered orthonormal basis of $V$.

If $W$ is a subspace of $V$, we denote by $W^\perp$ the orthogonal complement of $W$. The next lemma follows from Theorem 27.2 in [Lim81].

**Lemma 1.2.11.** For any $a \in M_n(\mathbb{K})$, we have:

1. $R(a^*) = N(a)^\perp$
2. \( N(a^*) = R(a)^\perp \).

The next result connecting Green's relations and adjoints is implied by the above lemma.

**Proposition 1.2.12.** Let \( a, b \in S_n \). Then

1. \( a \mathcal{L} b \Rightarrow a^* \mathcal{R} b^* \)
2. \( a \mathcal{R} b \Rightarrow a^* \mathcal{L} b^* \)

**Definition 1.2.13.** An element \( e \in \text{End}(V) \) is called a perpendicular projection (or orthogonal projection), if \( e = e^2 = e^* \).

The set of all perpendicular projections of rank \( k \) is denoted by \( P(k) \). The set of all perpendicular projections in \( E_n \) is denoted by \( P \).

Let \( e \) be a perpendicular projection. Then \( R(e)^\perp = N(e) \) and \( N(e)^\perp = R(e) \).

Conversely, let \( W \) be a fixed subspace of \( V \) such that \( 0 < \dim(W) < n \). Then there is a unique perpendicular projection \( e \) such that \( R(e) = W \) and \( N(e) = W^\perp \). This projection is denoted by \( p_W \).

### 1.2.5 The Trace Function

The trace function \( \text{tr} \) is defined as follows:

\[
\text{tr}(a) = \text{The sum of the characteristic values of } a
\]

\[
= \text{The sum of the diagonal elements in the matrix representation of } a
\]

**Lemma 1.2.14.** If \( e \in E(k) \), then \( \text{tr}(e) = k \).

**Proof.** Choosing a basis for \( V \) properly, \( e \) can be represented by a matrix of the form \( \begin{bmatrix} I & O \\ O & O \end{bmatrix} \), where \( I \) is the unit matrix of order \( k \) and \( O \)’s are zero matrices. Hence the characteristic values of \( e \) are 1 (repeated \( k \) times) and 0 (repeated \( n - k \) times). Therefore, the sum of the characteristic values of \( e \) is \( k \)

For future reference, we list the main properties of the trace function (see p.314 [Her88]). The last property given below can be proved by selecting \( P \) to be a matrix in which exactly one of the elements is 1 and all other elements are zero.
Proposition 1.2.15. Let \( a, b \in M_n(\mathbb{K}) \) and \( \alpha \) a scalar.

1. \( \text{tr}(a + b) = \text{tr}(a) + \text{tr}(b) \)

2. \( \text{tr}(ab) = \text{tr}(ba) \)

3. \( \text{tr}(\alpha a) = \alpha \text{tr}(a) \)

4. \( \text{tr}(a^*) = \overline{\text{tr}(a)} \)

5. Let \( A \) be a \( p \times q \) matrix. If for any \( q \times p \) matrix \( P \) we have \( \text{tr}(AP) = 0 \) then \( A = 0 \). Also, if for any \( q \times p \) matrix \( P \) we have \( \text{tr}(PA) = 0 \) then \( A = 0 \).

1.3 Manifolds

1.3.1 Definitions

Since the concept of a manifold appears repeatedly at several places in the sequel, we present here a precise definition of the concept (see § 16.1 in [Die72] for more details).

Let \( X \) be a topological space. A chart of \( X \) is a triplet \( c = (U, \phi, n) \), where \( U \) is an open set in \( X \), \( n \) is an integer \( \geq 0 \), and \( \phi \) is a homeomorphism of \( U \) onto an open set in \( \mathbb{R}^n \). The number \( n \) is the dimension of the chart \( c \). Two charts \( c = (U, \phi, n) \) and \( c' = (U', \phi', n') \), with the same domain of definition \( U \), are said to be compatible if the homeomorphisms

\[
\phi^{-1} \circ \phi' : \phi(U) \to \phi'(U) \quad \text{and} \quad \phi'^{-1} \circ \phi : \phi'(U) \to \phi(U)
\]

(called transition homeomorphisms) are indefinitely differentiable (that is, these functions are \( r \) times differentiable for any positive integer \( r \)). In particular, this implies that \( n = n' \). Two arbitrary charts \( c = (U, \phi, n) \) and \( c' = (U', \phi', n') \) are said to be compatible if either \( U \cap U' = \emptyset \) or the restrictions \( (U \cap U', \phi|(U \cap U'), n) \) and \( (U \cap U', \phi'|(U \cap U'), n') \) of \( c \) and \( c' \) to \( U \cap U' \) are compatible. An atlas of \( X \) is a set \( A \) of charts, each pair of which are compatible and whose domains of definition cover \( X \). Two atlases \( A, B \) of \( X \) are said to be compatible if each chart in \( A \) is
compatible with every chart in \( B \). It can be proved that on the set of atlases of \( X \) the relation "\( A \) and \( B \) are compatible" is an equivalence relation.

**Definition 1.3.1.** A differential manifold is a separable metrizable topological space on which is given an equivalence class of atlases under the equivalence relation "\( A \) and \( B \) are compatible".

If \( X \) is a differential manifold, an atlas of \( X \) is any atlas in the equivalence class of atlases defining \( X \), and any chart of \( X \) is any chart belonging to one of these atlases. An atlas \( A \) on a separable metrizable space \( X \) defines a structure of a differential manifold on \( X \). This structure is the one defined by the equivalence class of atlases to which \( A \) belongs.

Let \( X \) be a differential manifold and \( x \) a point of \( X \). If \( c = (U, \phi, n) \) is a chart of \( X \) such that \( x \in U \) then \( n \) is called the dimension of \( X \) at \( x \) and is denoted by \( \dim_x(X) \). If the dimension of \( X \) is the same constant at every point \( x \in X \), then \( X \) is said to be a pure manifold. If \( X \) is pure and nonempty, the common value of the numbers \( \dim_x(X) \) is called the dimension of \( X \).

A differential manifold as defined above is also called a \( C^{\infty} \)-manifold. If, in the above definitions, we replace "indefinitely differentiable" by "analytic" we get a real analytic manifold. If we now replace \( \mathbb{R}^n \) by \( \mathbb{C}^n \) in all the above definitions we get the definition of a complex analytic manifold.

Here is a useful construction for introducing manifold structures on arbitrary topological spaces (see (16.2.6) in [Die72]). Let \( X \) be a manifold (differential or analytic), \( X' \) a topological space and \( u : X \to X' \) a homeomorphism of \( X \) onto \( X' \). For each chart \( c = (U, \phi, n) \) of \( X \), the triplet \( (u(U), u^{-1} \circ \phi, n) \) is a chart on \( X' \) which we denote by \( u(c) \). As the chart \( c \) runs through an atlas on \( X \), the charts of the form \( u(c) \) form an atlas on \( X' \). This atlas defines the structure of a manifold on \( X' \) which is said to be obtained by transporting the manifold structure on \( X \) by means of the homeomorphism \( u \).

We record here a very elementary property of differential manifolds for future reference (see (16.1.4) in [Die72]).

**Proposition 1.3.2.** Let \( X \) be a manifold (differential or analytic). Then \( X \) is a locally compact and locally connected space. Every point in \( X \) has a neighbourhood
homeomorphic to a complete metric space. The set of open connected components of \( X \) is at most denumerable.

We next consider mappings of differential manifolds.

**Definition 1.3.3.** Let \( X, Y \) be differential (or, analytic) manifolds. A mapping \( u : X \to Y \) is a diffeomorphism (or, an isomorphism of manifolds) if \( u \) is a homeomorphism and if the structure of manifold on \( Y \) is the same as that obtained by transporting the manifold structure on \( X \) by means of the map \( u \).

**Definition 1.3.4.** Let \( X, Y \) be differential manifolds. A mapping \( f : X \to Y \) is said to be indefinitely differentiable (or, simply, differentiable) [analytic] if \( f \) is continuous on \( X \) and satisfies the following condition: For each pair of charts \((U, \phi, n)\) and \((U', \phi', n')\) of \( X \) and \( Y \), respectively, such that \( f(U) \subseteq U' \), the mapping

\[
\phi^{-1} \circ (f|U) \circ \phi' : \phi(U) \to \phi'(U')
\]

(which is called the local expression of \( f \)) is indefinitely differentiable [analytic].

We now consider some special mappings. If for a mapping \( f : X \to Y \), the local expression of \( f \) is differentiable we say that \( f \) is a mapping of class \( C^1 \).

Let \( X \) be any manifold and \( x \) any point in \( X \). Let \( f_1, f_2 \) be two functions of class \( C^1 \) defined on an open neighbourhood of \( 0 \) in \( K \) such that \( f_1(0) = f_2(0) = x \). We say that "\( f_1 \) and \( f_2 \) are tangent at \( 0 \)" if the local expressions of \( f_1 \) and \( f_2 \) have the same derivative at \( 0 \). The relation "\( f_1 \) and \( f_2 \) are tangent at \( 0 \)" is an equivalence relation. The family of equivalence classes under this relation, denoted by \( T_x(X) \), has the structure of a vector space. The space \( T_x(X) \) with this vector space structure is called the tangent space to \( X \) at \( x \).

Now let \( X \) and \( Y \) be two manifolds, \( f : X \to Y \) be a mapping of class \( C^1 \), \( x \in X \) and \( y = f(x) \). \( f \) induces a linear mapping of \( T_x(X) \) into \( T_y(Y) \) which we denote by \( T_x(f) \). The rank of this linear mapping is denoted by \( \text{rk}_x(f) \). The mapping \( f \) is called a subimmersion if the map \( x \mapsto \text{rk}_x(f) \) is constant on \( X \). The mapping \( f \) is called an immersion [respectively, submersion] if the linear map \( T_x(f) \) is injective [respectively, surjective] at every \( x \in X \).

We now define the notion of a submanifold.
Definition 1.3.5. Let $Y$ be a manifold and $X$ a subspace of $Y$. Let $f : X \to Y$ be the canonical injection. Suppose that to each $x \in X$ there exists an open neighbourhood $U$ of $x$ in $X$ and a chart $(U', \phi', n')$ of $Y$ such that $f(U) \subseteq U'$ and such that $(f|_U) \circ \phi'$ is a homeomorphism of $U$ onto the intersection of $\phi'(U')$ with a linear subvariety of $\mathbb{R}^d$. Then the space $X$ endowed with the structure of a manifold which is the inverse image under $f$ of that of $Y$, is said to be a submanifold of $Y$.

We have an interesting result on submanifolds (see (16.8.6.1) in [Die72]).

Proposition 1.3.6. Let $Z$ be a manifold, $Y$ a submanifold of $Z$ and $X$ a subspace of $Y$. Then $X$ is a submanifold of $Z$ if and only if $X$ is a submanifold of $Y$.

1.3.2 Matrix Manifolds

Let $m, n$ be positive integers. Let $M(m, n)$ be the set of all $m \times n$ matrices with elements from $\mathbb{K}$. By listing out the elements of a matrix in $M(m, n)$ row-wise we get a vector of the linear space $\mathbb{K}^{mn}$. Thus we have a bijection $$\kappa : M(m, n) \to \mathbb{K}^{mn}. \quad (1.3.1)$$

Using the usual inner product in the linear space $\mathbb{K}^{mn}$, we can define a norm and a metric in $\mathbb{K}^{mn}$. This norm is the Euclidean norm in $\mathbb{K}^{mn}$ (see p.38 in [Lim81]). This metric can be transported to $M(m, n)$ via the map $\kappa$. Using the metric on $M(m, n)$ so defined, we can define a topology on $M(m, n)$. This is the usual topology on $M(m, n)$. It is also referred to as the Euclidean topology on $M(m, n)$.

However, when $m = n$ the set $M(m, n)$ becomes identical with $M_n(\mathbb{K})$, and $M_n(\mathbb{K})$ is identical with $\text{End}(V)$. $\text{End}(V)$ consists of linear endomorphisms of $V$ into itself. Hence each element in $\text{End}(V)$ is a linear operator on $V$. The elements of $\text{End}(V)$ can be assigned the operator norm (see p.176 in [Hal58]). The elements of $M_n(\mathbb{K})$ can also be assigned the operator norm. For computational convenience, whenever we consider elements of $M_n(\mathbb{K})$, unless otherwise specified, we always assign them the operator norm.

Thus, the set $M_n(\mathbb{K})$ can be assigned a topology in two different ways. One of them is defined via the norm induced by the map $\kappa$ (see Eq.(1.3.1)). The other
topology is defined by the operator norm. Since $M_n(\mathbb{K})$ is a finite-dimensional linear space the two topologies are identical (see, for example, Corollary 6.7 in [Lim81]).

Obviously, $\mathbb{K}^{mn}$ is an analytic manifold of dimension $mn$. The bijection $\kappa$ defined above can now be used to transport the structure of analytic manifold in $\mathbb{K}^{mn}$ to the space $M(m, n)$. Thus $M(m, n)$ becomes an analytic manifold.

We have the following more general result (see pp. 168–169 in [B+85]).

**Proposition 1.3.7.** If $M(m, n; k)$ is the set of all matrices of rank $k$ in $M(m, n)$ then, with the subspace topology inherited from $M(m, n)$, the space $M(m, n; k)$ is an analytic manifold of dimension $k(m - n - k)$. Moreover, it is a submanifold of $M(m, n)$.

### 1.3.3 Grassmann Manifolds

Let $V$ a fixed vector space of dimension $n$ over $\mathbb{K}$. Let $k$ be a fixed integer such that $0 < k < n$. We denote by $G_k$ the set of all $k$-dimensional subspaces of $V$. We shall show below that a manifold structure can be defined on $G_k$. The resulting manifold is referred to as the Grassmann manifold.

Here we describe one of the most elementary ways for defining a manifold structure on $G_k$ (see pp. 169–170 in [B+85]). For other equivalent ways for defining manifold structures on $G_k$, see [Die72].

First, we select an ordered basis for $V$. Relative to this basis, every element in $V$ can be expressed as a row vector in $\mathbb{K}^n$. If $A \in M(k, n; k)$ then, since the rank of $A$ is $k$, the $k$ rows of $A$ are linearly independent. So the $k$ rows of $A$ span a $k$-dimensional subspace of $V$. Hence we can define a map

$$ q : M(k, n; k) \to G_k $$

which associates each matrix $A$ in the set $M(m, n; k)$ to the subspace of $V$ spanned by the rows of $A$. We assign to $G_k$ the quotient topology induced by the map $q$.

We next construct charts on $G_k$. Let $W \in G_k$. Choose $A \in M(k, n; k)$ such that $q(A) = W$. Since the rank of $A$ is $k$, some submatrix of $A$ of order $k$ is nonsingular. We denote one such submatrix by $A_J$. Let this submatrix be formed
by the \((j_1, \ldots, j_k)\)-th columns of \(A\) (where we assume that \(j_1 < \cdots < j_k\)). If \(X\) is any element of \(M(k, n; k)\), we denote by \(X_J\) the submatrix of \(X\) formed by the \((j_1, \ldots, j_k)\)-th columns of \(X\) and by \(X_{J'}\) the submatrix formed by the remaining columns of \(X\). We write

\[
U_W = \{q(X) : X \in M(k, n; k) \text{ with } X_J = I_k \text{ the unit matrix of order } k\}.
\]

It can be shown that \(U_W\) is an open set in \(G_k\) (see [B+85]). Note that \(X_{J'} \in M(k, n - k)\). Now have the map

\[
\phi_W : U_W \rightarrow \mathbb{R}^{k(n-k)}, \quad \phi_W(q(X)) = \kappa(X_{J'}),
\]

where \(\kappa : M(k, n - k) \rightarrow \mathbb{R}^{k(n-k)}\) is as defined in Eq.(1.3.1). This map is a homeomorphism of \(U_W\) onto \(\mathbb{R}^{k(n-k)}\). Thus the triplet \((U_W, \phi_W, k(n - k))\) is a chart on \(G_k\).

It can also be shown that any two charts as defined above are compatible. Further the open sets \(U_W\) cover \(G_k\). Hence the collection of all charts of the form \((U_W, \phi_W, k(n - k))\) forms an atlas on \(G_k\). This atlas defines the structure of a manifold on \(G_k\).

\(G_k\), with the manifold structure on it constructed as above, is called the Grassmann manifold of \(k\)-dimensional subspaces of \(V\). Note that the method of construction of the manifold structure on \(G_k\) as described above is applicable in both real and complex cases. Since the transition homeomorphisms are rational functions, they are analytic. Therefore the Grassmann manifolds are analytic manifolds.

The following result establishes a simple relation between \(G_k\) and \(G_{n-k}\).

**Proposition 1.3.8.** The map \(\text{Perp} : G_k \rightarrow G_{n-k}\) defined by \(W \mapsto W^\perp\) is a homeomorphism.

**Proof.** Let \(J = (j_1, \ldots, j_k)\) be an ordered set of \(k\) integers such that \(1 \leq j_1 < j_2 < \cdots < j_k \leq n\) and let \(J'\) be the ordered (in the ascending order) set of the remaining \(n - k\) integers in \(\{1, \ldots, n\}\). Let \(X\) be any matrix having \(n\) columns. Let \(X_J\) denote the submatrix of \(X\) formed by the \((j_1, \ldots, j_k)\)-th columns of \(X\) and \(X_{J'}\) the submatrix formed by the remaining columns of \(X\). We denote by \(q(X)\), irrespective of the dimension of \(X\), the subspace of \(V\) spanned by the rows of \(X\).
We also write

\[ A_J = \{ X \in M(k, n; k) : X_J = I_k \}, \]
\[ B_J = \{ Y \in M(n-k, n, n-k) : Y_{J'} = I_{n-k} \}, \]
\[ U_J = q(A_J) \subseteq G_k, \]
\[ V_J = q(B_J) \subseteq G_{n-k}. \]

From the definition of the topology of the Grassmann manifolds given above we see that \( U_J \) is an open set in \( G_k \) and the map \( q|A_J : A_J \rightarrow U_J \) is a homeomorphism. Further, the map \( q|B_J : B_J \rightarrow V_J \) is also a homeomorphism.

We now define a map \( p : A_J \rightarrow B_J \) by setting

\[ (p(X))_{J'} = -X_{J'}^*, \quad (p(X))_{J''} = I_{n-k} \]

for any \( X \in A_J \) where \( X_{J'}^* \) is the conjugate transpose of \( X_{J'} \). For \( X \in A_J \), the maps \( X \rightarrow X_{J'} \) and \( X_{J'} \rightarrow -X_{J'}^* \) are obviously homeomorphisms. Hence the map \( p \) is also a homeomorphism. It is easy to show that every row of \( X \) is orthogonal to each row of \( p(X) \) under the usual inner product. It follows that, for any \( X \in A_J \), we have \( p(q(X)) = q(X)^{-1} \). Therefore, we also have

\[ \text{Perp } |U_J = (q|U_J)^{-1} \circ p \circ (q|V_J). \]

This shows that \( \text{Perp } |U_J \) is continuous.

The family of \( U_J \)'s, obtained by considering different ordered \( k \)-tuples of the form \( J \), is an open covering of \( G_k \). Hence (see Theorem III.9.4 in [Dug66]) the map \( \text{Perp} \) is continuous.

The map \( \text{Perp} \) is clearly a bijection. As above we can prove that the map \( \text{Perp}^{-1} \) is also continuous. Hence \( \text{Perp} \) is a homeomorphism.

**1.4 Lie Groups and Their Actions**

**1.4.1 Actions of Groups**

We follow the terminology in [Die70].
Let $G$ be a group and $X$ a set. A **left action** of $G$ on $X$ is a mapping

$$G \times X \to X, \quad (s, x) \mapsto s \cdot x$$

satisfying the following conditions:

1. If $e$ is the identity element of $G$, then $e \cdot x = x$ for all $x \in X$.

2. For all $s, t \in G$ and for all $x \in X$ we have $s \cdot (t \cdot x) = (st) \cdot x$.

For each $x \in X$, the set $G \cdot x = \{ s \cdot x : s \in G \}$ is called the **orbit** of $x$ in $X$ and the set $S_x = \{ s \in G : s \cdot x = x \}$ is called the **stabilizer** of $x$. $G/S_x$ is the set of left cosets $sS_x$ ($s \in G$) of $S_x$ in $G$. The group is said to act **faithfully** if the intersection of the stabilizers $S_x$, as $x$ runs through $X$, consists only of the identity element of $G$. $G$ is said to act **transitively** if, given any two elements $x, y \in X$, there is an element $s \in G$ such that $y = s \cdot x$.

Let $G$ be a topological group acting on a topological space $X$. We say that the action of $G$ on $X$ is continuous if the mapping $G \times X \to X$ defined by $(s, x) \mapsto s \cdot x$ is continuous. If the action of $G$ on $X$ is continuous, then for any fixed $s \in G$, the map $x \mapsto s \cdot x$ is a homeomorphism of $X$ onto $X$.

Let $G$ be a topological group acting continuously on a topological space $X$. Let $X/G$ be the set of orbits. Let $\pi : X \to X/G$ be the mapping $x \mapsto G \cdot x$. Let $T$ be the set of subsets $U$ of $X/G$ such that $\pi^{-1}(U)$ is open in $X$. Then $T$ is a topology on $X/G$. The set $X/G$ with this topology is called the **orbit space** of the action of $G$ on $X$. With this topology on $X/G$, the mapping $\pi : X \to X/G$ as defined above is continuous.

We have the following general result on actions of groups on topological spaces (see [Hoh87]).

**Theorem 1.4.1.** Let $G$ be a Hausdorff, locally compact, $\sigma$-compact group and let $G$ act (on the left) on a Hausdorff, locally compact space $X$. Assume further that $X$ is an orbit under the action of $G$ on $X$ so that $X = G \cdot x$ for some $x \in X$. Then the surjection $G \to X$ defined by $s \mapsto s \cdot x$ is continuous and open.

**Remark 1.4.1.** Analogous to the above definition of a left action of a group $G$ on a topological space $X$, we can define a right action $f G$ on $X$. However, to fix notations and terminology, we have used only left actions in the present work.
1 Preliminaries

1.4 Lie Groups

Here, we follow the terminology in [Die72].

**Definition 1.4.2.** Let $G$ be a set endowed with a group structure and a structure of a differential manifold. These structures are said to be compatible if the mappings

\[(x, y) \mapsto xy \quad \text{of} \quad G \times G \quad \text{into} \quad G\]

\[x \mapsto x^{-1} \quad \text{of} \quad G \quad \text{into} \quad G\]

are differentiable. A group endowed with the structure of a differential manifold, which is compatible with its group structure, is called a **Lie group**.

Let a set $G$ be endowed with the structure of a group and the structure of an **analytic** manifold. These structures are compatible if the mappings specified in the above definition are analytic. A group endowed with the structure of an analytic manifold, which is compatible with its group structure, is called an **analytic group**.

Let $G$ be a Lie [analytic] group and a $H$ a subgroup of $G$ which is also a submanifold of $G$. The set $H$, endowed with its structures of group and manifold, is called a **Lie [analytic] subgroup** of $G$.

1.4.3 Actions of Lie Groups

Let $G$ be a Lie [analytic] group and $X$ a differential [analytic] manifold. We say $G$ acts differentiably [analytically] on $X$ if we are given a left action $(s, x) \mapsto s \cdot x$ of $G$ on $X$ which is a differentiable [analytic] mapping of $G \times X$ into $X$ (see (16.10) in [Die72]).

Suppose that, on the orbit space $X/G$, there exists the structure of a manifold for which the underlying topological space is the orbit space $X/G$ and for which the mapping

\[\pi : X \to X/G, \quad x \mapsto G \cdot x\]

is a submersion. Then the space $X/G$ endowed with this structure of a manifold is called the **orbit manifold** of the action of $G$ on $X$ (see (16.10.3) in [Die72]).

We now state a few results on actions of Lie groups on manifolds. These help us answer questions on the nature of the orbit of an element in the manifold, the
properties of the stabilizer of an element in the manifold, etc. (For more details, see (16.10) in [Die72]).

**Proposition 1.4.3.** Let a Lie [analytic] group $G$ act differentiably [analytically] on a differential [analytic] manifold $X$. For each $x \in X$ the mapping $s \mapsto s \cdot x$ of $G$ into $X$ is a subimmersion. The stabilizer $S_x$ of $x$ is a Lie [analytic] subgroup of $G$.

**Proposition 1.4.4.** Let $H$ be a Lie [analytic] subgroup of a Lie [analytic] group $G$. Then the orbit manifold $G/H$ exists, and

$$\dim(G/H) = \dim(G) - \dim(H).$$

Recall [Die70] that a subset $A$ of a topological space $X$ is said to be locally closed if $A$ is the intersection of an open subset and a closed subset of $X$.

**Proposition 1.4.5.** Let $G$ be a Lie [analytic] group acting differentiably [analytically] on a differential [analytic] manifold $X$. If a point $x \in X$ is such that the orbit $G \cdot x$ is a locally closed subspace of $X$, then $G \cdot x$ is a submanifold of $X$ and the mapping

$$f_x : G/S_x \to G \cdot x, \quad sS_x \mapsto s \cdot x$$

is an isomorphism of manifolds.

**Proposition 1.4.6.** Let $G$ be a Lie [analytic] group which acts differentiably [analytically] and transitively on a differential [analytic] manifold $X$, then for each $x \in X$ the mapping $f_x : G/S_x \to X$ defined by $sS_x \mapsto s \cdot x$ is an isomorphism of manifolds.

### 1.5 Bundle Structures

In the study of the topology of $S^n$, bundle structures arise naturally. In this section we give the necessary definitions (see [Ste51]).

#### 1.5.1 Coordinate Bundles

An effective topological transformation group of a topological space $X$ is a topological group $G$ acting continuously and faithfully on $X$. 
Definition 1.5.1. A coordinate bundle is a quintuple

\[ B = (X, B, F, G, p), \]

where \( X, B, F \) are topological spaces, \( G \) an effective topological transformation group of \( F \) and \( p : X \to B \) a continuous surjection, together with a family \( \{U_j\} \) of open sets covering \( B \) indexed by a set \( J \) and, for each \( j \in J \), a homeomorphism

\[ \phi_j : U_j \times F \to p^{-1}(U_j), \]

all satisfying the following conditions:

1. For all \( x \in U_j \) and \( y \in F \), we have \( p(\phi_j(x, y)) = x \); that is, the following diagram where \( \text{pr}_1 \) represents projection onto the first coordinate space is commutative:

\[
\begin{array}{ccc}
U_j \times F & \xrightarrow{\phi_j} & p^{-1}(U_j) \\
\downarrow \text{pr}_1 & & \downarrow p \\
U_j & & 
\end{array}
\]

2. For \( x \in U_j \), define the map

\[ \phi_{j,x} : F \to p^{-1}(x), \quad y \mapsto \phi_j(x, y). \]

Then, for each pair \( i, j \) in \( J \) and each \( x \in U_i \cap U_j \), the homeomorphism

\[ \phi_{i,x} \circ \phi_{j,x}^{-1} : F \to F \]

coincides with the action of an element of \( G \).

3. For each pair \( i, j \) in \( J \), the map

\[ g_{ij} : U_i \cap U_j \to G, \quad x \mapsto \phi_{i,x} \circ \phi_{j,x}^{-1} \]
1 Preliminaries

The space $X$ in the above definition is called the total space, $B$ the base space, $F$ the fibre space (or, simply the fibre), $G$ the group of the bundle (or, the structure group) and $p$ the projection map. The open sets $U_j$ are the coordinate neighbourhoods and the functions $\phi_j$ are the coordinate functions. For each $b \in B$ the inverse image $p^{-1}(b)$ is homeomorphic to the fibre space $F$. It is called the fibre of the bundle over $b$.

If $X$, $B$ are differentiable [analytic] manifolds, we usually assume that the projection map $p$ is differentiable [analytic]. If $X$, $B$ are manifolds, the fibre $p^{-1}(b)$ over $b \in B$ is a closed submanifold of $X$.

1.5.2 Fibre Bundles

Two coordinate bundles $B$, $B$ are equivalent in the strict sense if they have the same total space, base space, fibre, group, and projection and their coordinate functions $\{\phi_j\}$ and $\{\phi'_k\}$ satisfy the condition that the map

$$g'_{kj}(x) = \phi_{j,x} \circ \phi'^{-1}_{k,x} \quad \text{for} \quad x \in U_j \cap U'_k$$

coincides with the operation of an element of $G$, and the map $g'_{kj} : U_j \cap U'_k \rightarrow G$ so obtained is continuous.

Definition 1.5.2. A fibre bundle is an equivalence class of coordinate bundles under the above equivalence relation.

Every coordinate bundle defines a unique fibre bundle. So, a coordinate bundle itself is sometimes referred to as a fibre bundle.

Definition 1.5.3. A coordinate bundle is called a product bundle if there is just one coordinate neighbourhood $U = B$ and the group $G$ consists of the identity alone.

We now define a vector bundle.

Definition 1.5.4. A vector bundle is a fibre bundle $B = (X, B, F, G, p)$ in which the fibre space $F$ is a finite dimensional vector space and the group $G$ of the bundle is a subgroup of the group of all invertible linear endomorphisms of $F$. 
As an application of the bundle structure theorem (see § 7.4 in [Ste51]) we have the following result (see also (16.14.9) in [Die72]).

**Theorem 1.5.5.** Let $B$ be a Lie [analytic] group, $G$ a closed subgroup of $B$ and $H$ a closed subgroup of $G$. Then the quintuple

$$B = (B/H, B/G, G/H, G/H_0, p),$$

where $p : B/H \to B/G$ is the map induced by inclusion of cosets and $H_0$ is the largest subgroup of $H$ invariant in $G$, is a fibre bundle. The group of the bundle acts on $G/H$ by left translations.

A local cross-section of $G$ in $B$ is a function $f$ mapping a neighbourhood $U$ of $G \in B/G$ continuously into $B$ such that $p'(f(x)) = x$, where $p' : B \to B/G$ is the natural map $b \mapsto bG$, for all $x \in V$. To construct coordinate functions, we choose some local cross-section $f : U \to B$ of $G$ in $B$. We choose $B$ itself as the indexing set. For each $b \in B$, we define the coordinate neighbourhood $U_b$ in $B/G$ by $U_b = bU$. Define $f_b : U_b \to B$ by $f_b(x) = bf(b^{-1}x)$. The coordinate function $\phi_b$ for the bundle in Theorem 1.5.5 is now defined by

$$\phi_b(x, y) = f_b(x)y, \quad \text{for } x \in U_b, y \in G/H$$

(see p.33 in [Ste51]). Since any two local cross-sections lead to strictly equivalent bundles, the bundle structure is independent of the choice of the local cross-sections.

### 1.5.3 Equivalent Bundles

We now consider the concept of a map from one bundle into another bundle. The concept of such a map as given below is slightly more general than that given in [Ste51]. In [Ste51], one considers only situations where the the bundles under consideration have the same fibre space and the same group. Here we consider cases where the fibre spaces are only homeomorphic, and the bundle groups are isomorphic (of course, topologically).

Given the fibre bundles $B = (X, B, F, G, p)$ and $B' = (X', B', F', G', p')$, a
map $\Psi : B \rightarrow B'$ is a quadruple

$$\Psi = (\Psi_X, \Psi_B, \Psi_F, \Psi_G)$$

(1.5.1)

consisting of the following:

- A continuous map $\Psi_X : X \rightarrow X'$
- A continuous map $\Psi_B : B \rightarrow B'$
- A homeomorphism $\Psi_F : F \rightarrow F'$
- An isomorphism $\Psi_G : G \rightarrow G'$

These must satisfy the following conditions.

1. The following diagram is commutative:

2. Let $b \in B$ and $b' = \Psi_B(b) \in B'$. Then $\Psi_X|p^{-1}(b)$ maps $p^{-1}(b)$ homeomorphically onto $p'^{-1}(b')$.

3. The group action is preserved under the map, that is, for all $y \in F$ and $g \in G$ we have

$$\Psi_F(g \cdot y) = \Psi_G(g) \cdot \Psi_F(y).$$

4. Let $U_j$ be a coordinate neighbourhood and $\phi_j$ the corresponding coordinate function in $B$. Let $U'_k$ be a coordinate neighbourhood and $\phi'_k$ the corresponding coordinate function in $B'$. Let $x \in U_j \cap \Psi_B^{-1}(U'_k)$ and $x' = \Psi_B(x) \in U'_k$. Then the map (see diagram below)

$$\bar{\theta}_{jk}(x) = \phi_{j,x} \circ (\Psi_X|p^{-1}(x)) \circ \phi'^{-1}_{k,x'} \circ \Psi_F^{-1}.$$
of $F$ into $F$ coincides with the action of an element of $G$.

$$\phi_{j,x} : p^{-1}(x) \to \Psi_X^{-1}(x')$$

$$\tilde{g}_{jk}(x)$$

$$F \leftarrow F'$$

$$\Psi_F^{-1}$$

5. The map

$$\tilde{g}_{jk} : U_j \cap \Psi_B^{-1}(U_k') \to G, \quad x \mapsto \tilde{g}_{jk}(x)$$

is continuous.

Two fibre bundles $B$ and $B'$ are equivalent if there is a map $\Psi : B \to B'$ such that the map $\Psi_B : B \to B'$ is a homeomorphism.

### 1.5.4 Transporting a Fibre Bundle Structure

The method of transporting the bundle structure of a given space onto another given space is helpful in establishing the bundle structure of of certain spaces. The basic idea is a generalisation of the idea of transporting the manifold structure of a space onto another homeomorphic space as indicated in (16.26) of [Die72].

Let $B = (X, B, F, G, p)$ be a fibre bundle. Let $X'', B'', F''$ be topological spaces and the maps

$$\Psi'_X : X \to X''$$

$$\Psi'_B : B \to B''$$

$$\Psi'_F : F \to F''$$

be surjective homeomorphisms. Let $p'' : X'' \to B''$ be a continuous surjection such that $\Psi'_X \circ p'' = p \circ \Psi'_B$. We define an action of $G$ on $F''$ as follows:

$$g \cdot y'' = \Psi'_F(g \cdot (\Psi'_F)^{-1}(y'')), \quad g \in G, y'' \in F''.$$
With this action, $G$ becomes an effective topological transformation group of $F''$. Let $\{U_j\}$ be a family of coordinate neighbourhoods and $\{\phi_j\}$ the corresponding set of coordinate functions in a coordinate bundle representation of $B$. We now write $U''_j = \Psi_X(U_j)$ and define the map

$$\phi''_j : U''_j \times F'' \to \varphi''^{-1}(U''_j)$$

$$(x'', y'') \mapsto \Psi'_X \left( \phi_j \left( \Psi'_X^{-1}(x''), \, \Psi'_F^{-1}(y'') \right) \right).$$

The family of sets $\{U''_j\}$ is a family of open sets covering $X''$. It can be easily verified that this family of sets together with the family of maps $\{\phi''_j\}$ defines a coordinate bundle structure on the quintuple $(X'', B'', F'', G, p'')$. Let $B''$ be the resulting fibre bundle. It is also easy to prove that the fibre bundles $B$ and $B''$ are equivalent under the map $\Psi' = (\Psi'_X, \Psi'_B, \Psi'_F, \text{Id}_G)$ where $\text{Id}_G$ is the identity automorphism of the group $G$. In such a situation we say that the fibre bundle $B''$ is obtained by transporting the fibre bundle structure in $B$ by means of the map $\Psi'$. 