Preface

The theory of topological semigroups emerged as a distinct branch of the general theory of semigroups during the fifties [Hof95]. Most of the topological theory of semigroups developed during the fifties and sixties was the theory of compact topological semigroups. However, in later periods, the general theory of topological semigroups has undergone considerable development [C+83].

But it appears that there is still no satisfactory theory of topological regular semigroups, in spite of the fact that the algebraic structure of regular semigroups is completely known (see [Hal73, Gri74a, Gri74b, Nam79]). One can see an attempt in this direction in [Raj93]. The definition of a topological regular semigroup as given in [Raj93] includes several classes of semigroups with a natural topology. However, it does not include the best known example of a regular semigroup with a natural topology, namely, the semigroup $M_n(\mathbb{K})$ of linear endomorphisms of a $n$-dimensional real or complex vector space. In the present thesis, we have attempted a detailed investigation of the topological properties of this semigroup, or, more specifically, the subsemigroup $S_n$ formed by the singular endomorphisms in $M_n(\mathbb{K})$. In describing the algebraic structure of a regular semigroup $S$, the biordered set $E$ of idempotents in $S$ plays a crucial role. So, the topological properties of $E$ will strongly influence the topology of $S$. Thus, to study the topology of $M_n(\mathbb{K})$ it is imperative that we study the topology of the space $E_n$ of idempotents in $M_n(\mathbb{K})$. Hence a substantial part of the present research is devoted to describing the topology of $E_n$.

From a geometrical point of view, one of the most interesting findings of the present investigation is that the space $E_n$ can be looked upon as a generalised hyperboloid of one sheet (see Chapter 5). In fact, $E_n$ has two systems of generating
affine spaces and these affine spaces have properties similar to the properties of
generating lines of a hyperboloid of one sheet. Surprisingly, in the special case
when \( n = 2 \), \( E_n \) is actually a hyperboloid of one sheet (see Chapter 2). Moreover,
other geometrical objects like paraboloids, cones and planes appear in the geometrical
study of the semigroup structures in \( M_2(\mathbb{R}) \). For example, it is shown that,
if \( a \in M_2(\mathbb{R}) \) is singular, then the set of all semigroup theoretic inverses of \( a \) is a
paraboloid.

From a topological point of view, there are two structures which appear repeat-
edly in the semigroup structures in \( S_n \). These are the structures of a manifold and
of a fibre bundle (see Chapters 3, 4; see also [KN00]). The Green classes in \( S_n \),
the space of idempotents \( E_n \) and the biorder ideals all have manifold structures in
them. The manifold structures on these spaces can be described in terms of their
semigroup structures. The partition of any \( \mathcal{P} \)-class of \( S_n \) into \( \mathcal{L} \)- and \( \mathcal{R} \)-classes
give rise to natural bundle structures in \( S_n \). Further, the partition of any \( \mathcal{L} \)- or \( \mathcal{R} \-
class into \( \mathcal{K} \)-classes also gives rise to bundle structures. The base spaces for these
bundle structures are the Grassmann manifolds \( G_k \) of \( k \)-dimensional subspaces of
the underlying vector space. The space of idempotents has additional structures.
The space \( E(k) \) of idempotent endomorphisms of rank \( k \) is a vector bundle over \( G_k \)
or \( G_{n-k} \). As a preliminary to these discussions, we have included in Chapter 1 a
brief description the relevant definitions and concepts.

In the study of biordered sets, cycles appear quite naturally. Every cycle in \( E_n \)
defines a polygonal arc in \( E_n \). The homotopy properties of these cycles (see Chapter
6) is another topological characteristic of \( E_n \). Interestingly, when \( \mathbb{K} = \mathbb{R} \) and \( n > 2 \),
the fundamental group of \( E(k) \) is the two-element group \( \mathbb{Z}_2 \). So, every cycle in such
an \( E(k) \) is either null homotopic or its square is null-homotopic. This classification
of cycles based on homotopy properties can be related to a classification of cycles
based on determinants. Another interesting result of this genre is that every path in
\( E_n \) is homotopic to a polygonal path in \( E_n \). Arbitrary paths joining idempotents are
related to path components of \( H_e \).

We have briefly considered in Chapter 7 the topological properties of algebraic
semigroups (see [Put86]) by assigning them the subspace topology inherited from
the Euclidean topology on \( M_n(\mathbb{K}) \). We have considered only regular irreducible
algebraic monoids. Our discussion of these ideas is very preliminary in nature.
Even though $M_n(\mathbb{K})$ is a good example of a regular semigroup with a natural topology, it cannot be considered as a typical example of such a semigroup because it has several special algebraic properties. In fact, $M_n(\mathbb{K})$ is a strongly unit regular semigroup and $S_n$ is an idempotent generated semigroup. Some of the several topological and geometric properties of $M_n(\mathbb{K})$ discovered in the present investigation are consequences of these special algebraic properties.