Chapter 5

Geometry of \( E_n \)

In Chapter 2, we have seen that the space of idempotent endomorphisms of a plane is a hyperboloid of one sheet. In this chapter we examine whether this result can be generalised to the space of idempotent endomorphisms of a higher dimensional vector space.

Obviously, the most important property of a hyperboloid is the existence of the two systems of generating lines on it. The generating lines can be thought of as maximal affine spaces lying in the surface. Hence, we begin our geometrical study of \( E_n \) by considering the maximal affine spaces in \( E(k) \). We have shown that, for \( e \in E(k) \), \( E(L_e) \) and \( E(R_e) \) are maximal affine spaces lying in \( E(k) \). These affine spaces turn out to be the ‘generating affine spaces’ of \( E_n \).

Rectangular bands in \( E_n \), though not directly involved in the geometry of \( E_n \), are important because every sandwich set in a regular semigroup is a rectangular band (see [Nam91]). This consideration is the motivation for Section 5.2. In this section we have obtained the interesting result that the maximal rectangular bands in \( E(k) \) are point-wise products of two affine spaces the sum of whose dimensions is \( 2k(n-k) \).

In Section 5.3 we establish the main result of this chapter which is that each of the components \( E(k) \) of \( E_n \) can be looked upon as a generalized hyperboloid of one sheet. This is done by showing that the space \( E(k) \) has most of the properties of a hyperboloid of one sheet as described in the classical text of R.J.T. Bell (see [Bel60]). This is achieved by considering the affine spaces \( E(L_e) \) and \( E(R_e) \) as the generators of \( E(k) \) through \( e \).
5.1 Maximal Affine Spaces in $E(k)$

Recall the notations of Eq.(3.2.1), namely,

$$S^l_e = (1 - e)S_ne, \quad S^r_e = eS_n(1 - e). \quad (5.1.1)$$

Recall also the characterizations of elements in $S^l_e$ and $S^r_e$ given in Lemma 3.2.1.

We begin our discussion by proving a result which brings out the importance of the sets $S^l_e$ and $S^r_e$ in the study of affine spaces contained in $E_n$.

**Lemma 5.1.1.** Any affine space $A$ contained in $E_n$ and containing $e$ is contained in $e + S^l_e \oplus S^r_e$.

**Proof.** Let $f, g \in E_n$ and $h = (1 - \lambda)f + \lambda g$, where $\lambda$ is scalar, be a point on the line joining $f$ and $g$. By direct computation we see that the condition $h^2 = h$ is equivalent to

$$\lambda(1 - \lambda)(f + g) = \lambda(1 - \lambda)(fg + gf),$$

which is equivalent to $f + g = fg + gf$ when $\lambda \neq 0$. Hence the line joining $f$ and $g$ lies completely in $E_n$ if and only if $f + g = fg + gf$. In this case we also have $fgf = f$ and $gfg = g$.

Now let $\{e, f_1, \ldots, f_r\}$ be an affine basis for $A$. Any $g \in A$ can be expressed in the form

$$g = e + \sum_{i=1}^{r} \alpha_i (f_i - e).$$

Since $A$ is affine, for each $i$, the line joining $e$ and $f_i$ lies in $A$ and hence in $E_n$. This implies that

$$e + f_i = ef_i + fi_e.$$ 

Hence we have

$$f_i - e = (f_i e - e) + (ef_i - e) = (1 - e)(f_i - e) + e(f_i - e)(1 - e) \in S^l_e \oplus S^r_e.$$
and so \( g \in E + S^l_e \oplus S^r_e \).

The following result helps us to determine the elements in \( E + S^r_e \oplus S^l_e \) which are in \( E_n \).

**Lemma 5.1.2.** Let \( t_1 \in S^l_e \) and \( t_2 \in S^r_e \). Then \( E + t_1 + t_2 \in E_n \) if and only if \( t_1 t_2 = t_2 t_1 = 0 \).

**Proof.** Let \( f = e + t_1 + t_2 \). Now using Lemma 3.2.1 we see that

\[
   f^2 = f + t_1 t_2 + t_2 t_1.
\]

If \( t_1 t_2 = t_2 t_1 = 0 \) then \( f^2 = f \) and so \( f \in E_n \). Conversely, if \( f \in E_n \) then \( f^2 = f \) and so \( t_1 t_2 + t_2 t_1 = 0 \). Pre-multiplying this by \( e \) and using Lemma 3.2.1 we get \( t_2 t_1 = 0 \), and post-multiplying by \( e \) we get \( t_1 t_2 = 0 \).

We also have the following result.

**Lemma 5.1.3.** If \( t_1, t'_1 \in S^l_e \) and \( t_2, t'_2 \in S^r_e \) are such that \( f = e + t_1 + t_2 \) and \( g = e + t'_1 + t'_2 \) are in \( E_n \), then the line joining \( f \) and \( g \) lies in \( E_n \) if and only if

\[
   t_1 t'_2 + t'_1 t_2 = 0, \quad t_2 t'_1 + t'_2 t_1 = 0.
\]

**Proof.** In the proof of Lemma 5.1.1 we have seen that the line joining \( f \) and \( g \) lies completely in \( E_n \) if and only if

\[
   f + g = fg + gf.
\]

Using Lemma 3.2.1 and using the facts \( t_1 t'_1 = t'_1 t_1 = t_2 t'_2 = t'_2 t_2 = 0 \) we note that the above condition is equivalent to

\[
   t_1 t'_2 + t'_1 t_2 + t_2 t'_1 + t'_2 t_1 = 0.
\]

Now if the conditions in Eq.(5.1.2) are satisfied then the condition in Eq.(5.1.3) is also satisfied so that the line joining \( f \) and \( g \) lies completely in \( E_n \). Conversely, if the line joining \( f \) and \( g \) lies in \( E_n \) then Eq.(5.1.3) is satisfied. Pre- and post-multiplying Eq.(5.1.3) by \( e \) we get the equations in Eq.(5.1.2).
Motivated by the above lemmas, we define, for $X_1 \subseteq S_t^l$ and $X_2 \subseteq S_t^r$,

$$K_2(X_1) = \{t_2 \in S_t^r : t_2X_1 = X_1t_2 = \{0\}\}$$  \hspace{1cm} (5.1.4)

$$K_1(X_2) = \{t_1 \in S_t^l : t_1X_2 = X_2t_1 = \{0\}\}.$$  \hspace{1cm} (5.1.5)

($K$ is for "killing"!) The next lemma summarizes the properties of the functions $K_1$ and $K_2$.

**Lemma 5.1.4.** Let $K_1$ and $K_2$ be as defined by Eq.(5.1.4) and Eq.(5.1.5) and let $X_1, Y_1 \subseteq S_t^l$, $X_2 \subseteq S_t^r$.

1. $K_2(X_1)$ is a linear subspace of $S_t^r$.

2. If $X_1 \subseteq Y_1$ then $K_2(Y_1) \subseteq K_2(X_1)$.

3. $K_2(X_1 \cup Y_1) = K_2(X_1) \cap K_2(Y_1)$.

4. If Span($X_1$) denotes the linear span of $X_1$, then

$$K_2(X_1) = K_2(\text{Span}(X_1)).$$

5. $X_2 \subseteq K_2(K_1(X_2))$ and $X_1 \subseteq K_1(K_2(X_1))$.

6. $K_1K_2K_1(X_2) = K_1(X_2)$, $K_2K_1K_2(X_1) = K_2(X_1)$.

Similar properties hold for $K_1$.

**Proof.** The results (1)–(5) are obvious. To prove (6), we start with (5), namely, $X_1 \subseteq K_1K_2(X_1)$. Applying (2) to this we get $K_2K_1K_2(X_1) \subseteq K_2(X_1)$. Now applying (5) to $K_2(X_1)$ we have $K_2(X_1) \subseteq K_2K_1(K_2(X_1))$. From the two inclusions we have (6). \hfill $\square$

**Remark 5.1.1.** $K_1$ and $K_2$, which are functions defined on subsets of $S_t^r$ and $S_t^l$ respectively, can be given another interpretation. Let $\rho$ be the relation between the elements of $S_t^l$ and $S_t^r$ defined as follows:

$$t_1 \rho t_2 \iff t_1t_2 = t_2t_1 = 0.$$
Then, for $X_1 \subseteq S^t_v$, $K_2(X_1)$ is the set of all $t_2 \in S^t_v$ such that $t_1 \rho t_2$ for all $t_1 \in X_1$. $K_2(X_1)$ is the “polar” of $X_1$ under the relation $\rho$. Similarly, for $X_2 \subseteq S^t_v$, $K_1(X_2)$ is the “polar” of $X_2$ under $\rho$. The properties of $K_1$ and $K_2$ given in Lemma 5.1.4 now readily follow from the properties of polar sets. (For details of polarities, see pp.122-124 in [Bir79].)

**Lemma 5.1.5.** In the notations introduced above we have

$$K_2(S^t_v) = \{0\}, \quad K_1(S^t_v) = \{0\}.$$

**Proof.** Assume that $K_2(S^t_v) \neq \{0\}$ and let $0 \neq t_2^0 \in K_2(S^t_v)$.

First we note that $R(e) \setminus N(t_2^0) \neq \emptyset$. Otherwise $R(e) \subseteq N(t_2^0)$, and by Lemma 3.2.1 we also have $N(e) \subseteq N(t_2^0)$. These imply that $V \subseteq N(t_2^0)$ which in turn implies that $t_2^0 = 0$ contradicting our assumption that $t_2^0 \neq 0$.

We next note that $N(e) \setminus R(t_2^0) \neq \emptyset$. Otherwise we have $N(e) \subseteq R(t_2^0)$. Also if $t_1$ is any element in $S^t_v$, then by Lemma 3.2.1 we have $R(e) \subseteq N(t_1)$. Further, since $t_2^0 \in K_2(S^t_v)$, we have $t_2^0 t_1 = 0$ and so $R(t_2^0) \subseteq N(t_1)$ and hence $N(e) \subseteq N(t_1)$. Thus $V \subseteq N(t_1)$ implying that $t_1 = 0$. This is clearly a contradiction.

Choose $x \in N(e) \setminus R(t_2^0)$ and $y \in R(e) \setminus N(t_2^0)$. Let $U = \text{Span}\{\{x\}\}$ and choose a subspace $W$ containing $R(e)$ such that $V = U \oplus W$. Define $t_1$ by

$$zt_1 = \begin{cases} y & \text{if } z = x \\ 0 & \text{if } z \in W \end{cases}$$

Clearly, for this $t_1$ we have

$$R(t_1) = \text{Span}\{\{y\}\} \subseteq R(e) \subseteq N(t_1)$$

and so $t_1 \in S^t_v$. Further, since $y \notin N(t_2^0)$, we have $yt_2^0 \neq 0$ and therefore $xt_1t_2^0 = yt_2^0 \neq 0$. This implies that $t_1t_2^0 \neq 0$ which contradicts the fact that $t_2^0 \in K_2(S^t_v)$.

We conclude that $K_2(S^t_v) = \{0\}$. The proof of $K_1(S^t_v) = \{0\}$ is similar. \qed

The next few results characterize certain maximal affine spaces in $E_n$. 

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Theorem 5.1.6. If \( X_2 \subseteq S^e_r \) is an arbitrary nonempty set then
\[
A = e + K_1(X_2) \oplus K_2 K_1(X_2)
\]
is a maximal affine subspace of \( E_n \).

Proof. In view of Lemma 5.1.2 and by the definition of the functions \( K_1 \) and \( K_2 \) we see that \( A \subseteq E_n \). Since \( K_1(X_2) \) and \( K_2 K_1(X_2) \) are linear spaces, \( A \) is an affine space. Assume that \( A \) is not maximal. Let \( A' \) be an affine space contained in \( E_n \) and properly containing \( A \). Choose \( g \in A' \) such that \( g \notin A \). By Lemma 5.1.1, both \( A \) and \( A' \) are subspaces of \( e + S^l_e \oplus S^r_e \). So we can find \( t'_1 \in S^l_e \) and \( t'_2 \in S^r_e \) such that \( g = e + t'_1 + t'_2 \). Since \( g \notin A \), either \( t'_1 \notin K_1(X_2) \) or \( t'_2 \notin K_2 K_1(X_2) \).

If \( t_2 \in K_1(X_2) \), then \( e + t_1 \in A \) and so \( e + t_1 \in A' \). Since \( g \) is also in \( A' \), and since \( A' \) is an affine space, the line joining \( e + t_1 \) and \( g \) lies in \( A' \). By Lemma 5.1.3, this implies that \( t_2 t_1 = t_1 t_2 = 0 \) and so \( t_2 \in K_2 K_1(X_2) \). Also if \( t_2 \in K_2 K_1(X_2) \) then by a similar argument we have \( t_2 t_1 = t_1 t_2 = 0 \) implying that \( t_1 \in K_1(X_2) \).

Thus both of the conditions \( t'_1 \notin K_1(X_2) \) and \( t'_2 \notin K_2 K_1(X_2) \) are not satisfied. This contradiction implies that \( A \) is a maximal affine subspace of \( E_n \).

Corollary 5.1.7. If \( X_1 \subseteq S^l_e \) is an arbitrary nonempty set then
\[
A = e + K_1 K_2(X_1) \oplus K_2(X_1)
\]
is a maximal affine subspace of \( E_n \).

Proof. This follows from Theorem 5.1.6 by taking \( X_2 = K_2(X_1) \) and then using Lemma 5.1.4.

Corollary 5.1.8. \( E(L_e) \) and \( E(R_e) \) are maximal affine subspaces of \( E_n \).

Proof. If we choose \( X_2 = S^r_e \) then by Lemma 5.1.5 we have \( K_1(X_2) = \{0\} \) and \( K_1 K_2(X_2) = S^l_e \). Hence
\[
e + K_1(X_2) \oplus K_2 K_1(X_2) = e + 0 \oplus S^l_e = E(L_e)
\]
which implies that \( E(L_e) \) is a maximal affine subspace of \( E_n \). A similar argument holds for \( E(R_e) \).
Corollary 5.1.9. If $X_1 \subseteq S_e^l$ and $X_2 \subseteq S_e^r$ be nonempty sets, then $A = e + X_1 \oplus X_2$ is a maximal affine subspace of $E_n$ containing $e$ if and only if

$$K_1(X_2) = X_1, \quad K_2(X_1) = X_2.$$ 

Proof. Suppose that $K_1(X_2) = X_1$ and $K_2(X_1) = X_2$. Then

$$K_2K_1(X_2) = K_2(X_1) = X_2$$

so that

$$A = e + X_1 \oplus X_2$$

$$= e + K_1(X_2) \oplus K_2K_1(X_2),$$

and hence, by Proposition 5.1.6, $A$ is maximal.

Conversely, let $A$ be maximal. Assume that $K_1(X_2) \neq X_1$. Since $A \subseteq E_n$, by Lemma 5.1.2, we have $X_2X_1 = X_1X_2 = 0$ and so $X_1 \subseteq K_1(X_2)$. Since $K_1(X_2) \neq X_1$, we can find $t'_1 \in K_1(X_2) \setminus X_1$. Clearly $t'_1X_2 = X_2t'_1 = 0$. Setting

$$X'_1 = \text{Span}(X_1 \cup \{t'_1\}), \quad X'_2 = X_2$$

we see that $X'_1X'_2 = X'_2X'_1 = 0$ and so $A' = e + X'_1 \oplus X'_2$ is an affine subspace of $E_n$ containing $e$ and properly containing $A$. This contradicts the maximality of $A$. Thus $K_1(X_2) = X_1$ and by a similar argument we have $K_2(X_1) = X_2$. \qed

Remark 5.1.2. Even if $X_1$ and $X_2$ are linear spaces, both the equalities

$$K_1(X_2) = X_1, \quad K_2(X_1) = X_2$$

are necessary for the maximality of $A$.

Remark 5.1.3. It is interesting to observe that, in general, the converse of Theorem 5.1.6 is not true. There may be maximal affine spaces containing $e$ and contained in $E_n$ which are not in the form specified in Theorem 5.1.6.

Problems regarding the dimensions of maximal affine spaces in $E_n$, are treated in the next section.
5.2 Rectangular Bands in $E(k)$

Recall the definition of a rectangular band given in Section 1.1.4. Recall also the property of a rectangular band given in Proposition 1.1.13.

5.2.1 Preliminary Results

Since a rectangular band has only one $\mathcal{B}$-class, any rectangular band contained in $E_n$ must be contained in $E(k)$ for some $k$. Let $B \subseteq E(k)$ be a rectangular band.

Let $e \in B$ and $L_e^B \sqcup R_e^B$ be the $\mathcal{L}$-class [$\mathcal{R}$-class] in $B$ containing $e$. Obviously $L_e^B \subseteq E(L_e)$ [$R_e^B \subseteq E(R_e)$] and $B = L_e^B R_e^B$. Proposition 1.1.13 implies that any $a \in B$ can be written in the form $(e + t_1)(e + t_2)$ with $t_1 \in S_e^I$ and $t_2 \in S_e^J$. Since $R_e^B L_e^B = e$, we must have $t_2 t_1 = 0$. We state this as a lemma.

Lemma 5.2.1. In the notations introduced above, if $e + t_1 \in L_e^B$ and $e + t_2 \in R_e^B$ then $t_2 t_1 = 0$. Consequently, if $X \subseteq E(L_e)$ and $Y \subseteq E(R_e)$ are such that whenever $e + t_1 \in X$ and $e + t_2 \in Y$ we have $t_2 t_1 = 0$ then $B = XY$ is a rectangular band.

Proof. We need only prove the second part. First of all we observe that if $a = (e + t_1)(e + t_2) \in XY$ then, by Lemma 3.2.1 and by the hypothesis on $X$ and $Y$, we have $(e + t_2)(e + t_1) = e$. This implies that $a^2 = a$ so that $E(XY) = XY$. Next if $a_i = (e + t_1^{(i)})(e + t_2^{(i)}) \in XY$ for $i = 1, 2, 3$ then, we also have

$$a_1 a_2 a_3 = (e + t_1^{(1)})(e + t_2^{(3)}) = a_1 a_3.$$ 

It follows that $XY$ is a rectangular band. \(\square\)

Now let $X \subseteq E(L_e)$ and $Y \subseteq E(R_e)$ be such that $XY$ is a rectangular band. Then the product $\text{Aspan}(X \cup \{e\}) \text{Aspan}(Y \cup \{e\})$, where ‘Aspan’ denotes the ‘affine span’, is also a rectangular band. To prove this, note that every element of $\text{Aspan}(X \cup \{e\})$ is of the form $e + t_1$ where

$$t_1 = \alpha_1 t_1^{(1)} + \cdots + \alpha_r t_1^{(r)} \quad \text{with} \quad e + t_1^{(i)} \in X$$

and every element of $\text{Aspan}(Y \cup \{e\})$ is of the form $e + t_2$ where

$$t_2 = \beta_1 t_2^{(1)} + \cdots + \beta_s t_2^{(s)} \quad \text{with} \quad e + t_2^{(j)} \in Y.$$
Since $XY$ is a rectangular band we have $t_2^{(j)}t_1^{(i)} = 0$ for all $i, j$. Hence $t_2t_1 = 0$ and so $\text{Aspan}(X \cup \{e\}) \text{Aspan}(Y \cup \{e\})$ is a rectangular band.

To obtain the structure of maximal rectangular bands in $E(k)$, we define

$$
k_2(X_1) = \{t_2 \in S_e^r : t_2X_1 = \{0\}\}
$$

and

$$
k_1(X_2) = \{t_1 \in S_e^l : X_2t_1 = \{0\}\},
$$

where $X_1 \subseteq S_e^l$ and $X_2 \subseteq S_e^r$.

**Remark 5.2.1.** It is easy to see that if, in Lemma 5.1.4, we replace $K_1$ by $k_1$ and $K_2$ by $k_2$ then the resulting statements are all true. We also have the following relations between $k$ and $K$:

$$
K_1(X_2) \subseteq k_1(X_2), \quad K_2(X_1) \subseteq k_2(X_1). \tag{5.2.1}
$$

**Remark 5.2.2.** If we define the relation $\rho$ between $S_e^l$ and $S_e^r$ by $t_1 \rho t_2$ if and only if $t_2t_1 = 0$, then $k_1(X_2)$ is the polar of $X_2$ and $k_2(X_1)$ is the polar of $X_1$ under this relation (see Remark 5.1.1).

### 5.2.2 Maximal Rectangular Bands

The next theorem establishes the structure of maximal rectangular bands in $E_n$.

**Theorem 5.2.2.** For any nonempty set $X_2 \subseteq S_e^r$, the set

$$
B = (e + k_1(X_2))(e + k_2k_1(X_2))
$$

is a maximal rectangular band in $E_n$. Conversely, any maximal rectangular band in $E_n$ is in this form.

**Proof.** If $t_1 \in k_1(X_2)$ and $t_2 \in k_2k_1(X_2)$ then, by the definition of the functions $k_1$ and $k_2$, we see that $t_2t_1 = 0$ and so, by Lemma 5.2.1, the set $B$ is a rectangular band.

Assume that $B$ is not maximal and let $B'$ be a rectangular band properly containing $B$. Clearly

$$
e + k_1(X_2) = L_e^B \subseteq L_e^{B'}, \quad e + k_2k_1(X_2) = R_e^B \subseteq R_e^{B'}.$$
Let \((e + t'_1)(e + t'_2) \in B' \setminus B\). Then either \(t'_1 \not\in k_1(X_2)\) or \(t'_2 \not\in k_2k_1(X_2)\). We show that both cases are impossible.

Suppose \(t'_1 \not\in k_1(X_2)\). Since \(e + t'_1 \in L^B_e\) and \(R^B_e \subseteq R^B_e\), by Lemma 5.2.1, we have \((k_2k_1(X_2))t'_1 = 0\) and so \(t'_1 \in k_1k_2k_1(X_2)\). But, by Remark 5.2.1, \(k_1k_2k_1(X_2) = k_1(X_2)\) and therefore \(t'_1 \in k_1(X_2)\) which is a contradiction. Let \(t'_2 \not\in k_2k_1(X_2)\). Since \(e + t'_2 \in R^B_e\) and \(L^B_e \subseteq L^B_e\) we have \(t'_2(k_1(X_2)) = 0\) which implies that \(t'_2 \in k_2k_1(X_2)\). This is again a contradiction.

To prove the converse, let \(B = L^B_eR^B_e\) be a maximal rectangular band in \(E_n\). We write

\[X_1 = \{t_1 : e + t_1 \in L^B_e\}, \quad X_2 = \{t_2 : e + t_2 \in R^B_e\}\]

so that \(B = (e + X_1)(e + X_2)\). By Remark 5.2.1 we have \(X_1 \subseteq k_1(X_2)\) and \(X_2 \subseteq k_2(X_1)\). This implies that \((e + X_1)(e + k_2(X_1))\) is a rectangular band containing \(B\).

The maximality of \(B\) now implies that \(k_2(X_1) = X_2\). In a similar way we also see that \(k_1(X_2) = X_1\). Hence \(B\) must be in the form specified in the proposition. □

The properties of \(k_1\) and \(k_2\) referred to in Remark 5.2.1 now imply the following corollaries.

**Corollary 5.2.3.** For any nonempty set \(X_1 \subseteq S^t_e\), the set

\[B = (e + k_1k_2(X_1))(e + k_2(X_1))\]

is a maximal rectangular band in \(E_n\). Conversely, any maximal rectangular band in \(E_n\) is in this form.

**Corollary 5.2.4.** If \(X_1 \subseteq S^t_e\) and \(X_2 \subseteq S^t_e\) be nonempty sets, then \(B = (e + X_1)(e + X_2)\) is a maximal rectangular band containing \(e\) if and only if

\[k_1(X_2) = X_1, \quad k_2(X_1) = X_2.\]

**Theorem 5.2.5.** Let \(B = (e + X_1)(e + X_2)\) with \(X_1 \subseteq S^t_e\) and \(X_2 \subseteq S^t_e\) be a maximal rectangular band in \(E(k)\). Then \(X_1\) and \(X_2\) are linear spaces such that \(\dim X_1 + \dim X_2 = k(n - k)\), and conversely.
Proof. By Corollary 5.2.4, we have
\[ k_1(X_2) = X_1, \quad k_2(X_1) = X_2. \] (5.2.2)

Remark 5.2.1 implies that \(X_1\) and \(X_2\) are linear spaces.

We write
\[ U = \text{Span} \left( \cup_{t_2 \in X_2} R(t_2) \right), \quad \dim U = r. \]

By Lemma 3.2.1 we have \(R(t_2) \subseteq N(e)\) and so \(U \subseteq N(e)\). Choose some fixed subspace \(W\) of \(V\) such that \(N(e) = U \oplus W\).

For any \(t_1 \in S_e^1\) we have \(R(t_1) \subseteq R(e)\) and so the restriction \(t_1|W\) is in \(L(W, R(e))\). We show that the map \(t_1 \mapsto t_1|W\) is an isomorphism from \(X_1\) onto \(L(W, R(e))\). This map is clearly linear. So it is enough to show that it is a bijection.

For any \(t_1 \in X_1\) and \(t_2 \in X_2\) we have, by Lemma 5.2.1, \(t_2 t_1 = 0\) and so \(R(t_2) \subseteq N(t_1)\). Hence \(U \subseteq N(t_1)\). By Lemma 3.2.1, we already have \(R(e) \subseteq N(t_1)\). Thus we get \(R(e) \oplus U \subseteq N(t_1)\). From this, and also from the fact that \(V = R(e) \oplus U \oplus W\), it follows that whenever \(t'_1, t''_1 \in X_1\) are such that \(t'_1|W = t''_1|W\) then \(t'_1 = t''_1\). Hence the map \(t_1 \mapsto t_1|W\) is an injection.

To show that the map is a surjection, choose \(t^0 \in L(W, R(e))\). Define \(t_1\) by
\[ xt_1 = \begin{cases} 0 & \text{if } x \in R(e) \oplus U \\ xt^0 & \text{if } x \in W \end{cases} \]

Clearly, \(R(t_1) \subseteq R(e) \subseteq N(t_1)\), and so \(t_1 \in S_e^1\). Since \(U \subseteq N(t_1)\) we must have \(R(t_2) \subseteq N(t_1)\), that is \(t_2 t_1 = 0\), for all \(t_2 \in Y\). Therefore \(t_1 \in k_1(X_2)\). Eq.(5.2.2) now implies that \(t_1 \in X_1\). Obviously we also have \(t_1|W = t^0\). Therefore the map \(t_1 \mapsto t_1|W\) is a surjection also.

Since \(X_1\) and \(L(W, R(e))\) are isomorphic as vector spaces, we have
\[ \dim(X_1) = \dim(W) \dim(R(e)) = (n - (k + r))k. \] (5.2.3)

Next, for any \(t_2 \in X_2\) we have \(R(t_2) \subseteq U\) and so \(t_2 R(e) \subseteq L(R(e), U)\). We now show that the map \(t_2 \mapsto t_2|R(e)\) is an isomorphism from \(X_2\) onto \(L(R(e), U)\).
Since the map is clearly linear it is enough to prove that it is a bijection.

If \( t', t'' \in Y \) are such that \( t'|_R(e) = t''|_R(e) \) then, since \( N(e) \subseteq N(t') \) and \( N(e) \subseteq N(t'') \), we have \( t' = t'' \). So the map is an injection. To prove that it is a surjection, choose \( t^0 \in L(R(e), U) \) arbitrarily. Define \( t_2 \) by

\[
xt_2 = \begin{cases} 
0 & \text{if } x \in N(e) \\
xt^0 & \text{if } x \in R(e) 
\end{cases}
\]

Obviously, \( R(t_2) \subseteq U \subseteq N(e) \subseteq N(t_2) \) and therefore, \( t_2 \in S^r_\infty \). For every \( t_1 \in X_1 \) we have \( U \subseteq N(t_1) \). Since \( t^0 \in L(R(e), U) \) we must have \( t_2 t_1 = 0 \) for every \( t_1 \in X_1 \) and so \( t_2 X_1 = 0 \) implying that \( t_2 \in k_2(X_1) = X_2 \) (Eq.(5.2.2)). It is clear that \( t^0 = t_2|_R(e) \). Hence the map \( t_2 \mapsto t_2|_R(e) \) is a surjection also.

Since the vector spaces \( X_2 \) and \( L(R(e), U) \) are isomorphic we get

\[
\dim(X_2) = \dim(R(e)) \dim(U) = kr. \tag{5.2.4}
\]

From Eq.(5.2.3) and Eq.(5.2.4) we get

\[
\dim(e + X_1) + \dim(e + X_2) = \dim(X_1) + \dim(X_2) = k(n - k).
\]

Proof. We prove the inequality for a maximum difference from which the result follows for any affine space.

To prove the converse, let \( B = (e + X_1)(e + X_2) \) be a rectangular band in \( E(k) \) such that \( \dim(X_1) + \dim(X_2) = k(n - k) \). By Theorem 5.2.2, \( (e + k_1(X_2))(e + k_2k_1(X_2)) \) is a maximal rectangular band and by what we have proved above, \( \dim(k_1(X_2)) + \dim(k_2k_1(X_2)) = k(n - k) \). Since \( B \) is a rectangular band, \( X_1 \subseteq k_1(X_2) \) and \( X_2 \subseteq k_2(X_1) \). It follows that

\[
\dim(k_1(X_2)) + \dim(k_2k_1(X_2)) = \dim(X_1) + \dim(X_2) \leq \dim(k_1(X_2)) + \dim(X_2).
\]

From this we get \( \dim(k_2k_1(X_2)) \leq \dim(X_2) \). Since, by Remark 5.2.1, we have \( X_2 \subseteq k_2k_1(X_2) \) we also have \( \dim(X_2) \leq \dim(k_2k_1(X_2)) \). Therefore the two dimensions are equal and so \( X_2 = k_2k_1(X_2) \). We now have \( \dim(k_1(X_2)) = \dim(X_1) \) which implies that \( X_1 = k_1(X_2) \). Thus \( (e + X_1)(e + X_2) = (e + k_1(X_2))(e +
$k_2k_1(X_2)$ showing, by Theorem 5.2.2, that $B$ is maximal.

**Remark 5.2.3.** The above discussion can be summarized as follows. Every maximal rectangular band contained in $E(k)$ is the point-wise product of two affine spaces, which are contained in $E(L_e)$ and $E(R_e)$ respectively, the sum of whose dimensions is $k(n - k)$. We cannot choose the affine spaces arbitrarily. They are to be chosen in such a way that the condition of Lemma 5.2.1 is satisfied. As special cases we note that $E(L_e)$ and $E(R_e)$ are maximal rectangular bands in $E(k)$.

**Remark 5.2.4.** Maximal rectangular bands are point-wise products of affine spaces. Hence it is obvious that they are contractible manifolds.

The rest of this section is devoted to a discussion of the dimensionality of maximal affine spaces contained in $E_n$. We have considered only those affine spaces which are in the form $e + X_1 \oplus X_2$. We begin with a result, which is a corollary to Theorem 5.2.5, which gives an upper bound for the dimension of affine spaces of this form.

**Corollary 5.2.6.** Let $A = e + X_1 \oplus X_2$, with $X_1 \subseteq S^1_e$ and $X_2 \subseteq S^2_e$, be an affine space contained in $E(k)$. Then $\dim(A) \leq k(n - k)$.

**Proof.** We prove the inequality for a maximal affine space from which the result follows for any affine space.

Let $e \in A$ and $A = e + X_1 \oplus X_2$ be a maximal affine space in $E(k)$. By Corollary 5.1.9, we must have $K_1(X_2) = X_1$ and $K_2(X_1) = X_2$ and so, by Lemma 5.1.4, $X_1$ and $X_2$ are linear spaces. Hence

$$\dim(A) = \dim(X_1) + \dim(X_2).$$

But, by Eq.(5.2.1), $X_1 = K_1(X_2) \subseteq k_1(X_2)$ and from the observations in Remark 5.2.1, we have $X_2 \subseteq k_2k_1(X_2)$. Since, by Theorem 5.2.2, $(e + k_1(X_2))(e + k_2k_1(X_2))$ is a maximal rectangular band, by Theorem 5.2.5, we have

$$\dim(X_1) + \dim(X_2) \leq \dim(k_1(X_2)) + \dim(k_2k_1(X_2)) = k(n - k).$$

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We next show that the only affine spaces having the maximal dimension and which are in the form \( e + X_1 \oplus X_2 \) are \( E(L_e) \) and \( E(R_e) \). To do this we require the following result.

**Lemma 5.2.7.** Let \( X_2 \subseteq S^r_e \). If \( k_1(X_2) = K_1(X_2) \) then we have \( K_1(X_2) = \{0\} \) or \( K_1(X_2) = S^l_e \).

**Proof.** Let \( N_2 = \bigcap_{t_2 \in X_2} N(t_2) \) and \( W_2 = \text{Span} (\cup_{t_2 \in X_2} R(t_2)) \). In view of Lemma 3.2.1 we have the following inclusions:

\[
\{0\} \subseteq W_2 \subseteq N(e) \subseteq N_2 \subseteq V. \tag{5.2.5}
\]

Further, if \( t_1 \in K_1(X_2) \) then, for any \( t_2 \in X_2 \) we have \( t_1 t_2 = t_2 t_1 = 0 \) and hence \( R(t_1) \subseteq N(t_2) \) and \( R(t_2) \subseteq N(t_1) \). This implies the following:

\[
R(t_1) \subseteq N_2 \text{ and } W_2 \subseteq N(t_1) \text{ for any } t_1 \in K_1(X_2). \tag{5.2.6}
\]

We prove the lemma by considering several cases.

(a) Suppose \( W_2 = \{0\} \). Then \( R(t_2) = \{0\} \) for every \( t_2 \in X_2 \), and so \( X_2 = \{0\} \).

Hence by Lemma 5.1.5 we have \( K_1(X_2) = S^l_e \).

(b) Let \( W_2 = N(e) \). If \( t_1 \in K_1(X_2) \) then by Eq.(5.2.6) we have \( W_2 \subseteq N(t_1) \), that is, \( N(e) \subseteq N(t_1) \). But, by Lemma 3.2.1, we have \( R(e) \subseteq N(t_1) \). From these it follows that \( V \subseteq N(t_1) \) and so \( t_1 = 0 \). Thus \( K_1(X_2) = \{0\} \).

(c) Let \( N_2 = N(e) \). Again, let \( t_1 \in K_1(X_2) \). By Eq.(5.2.6) we have \( R(t_1) \subseteq N_2 \), that is, \( R(t_1) \subseteq N(e) \). Again by Lemma 3.2.1, we have \( R(t_1)R(e) \). These imply that \( R(t_1) = \{0\} \) and so \( t_1 = 0 \). Hence \( K_1(X_2) = \{0\} \) in this case also.

(d) Finally let \( N_2 = V \). Then \( N(t_2) = V \) for every \( t_2 \in X_2 \). This means that \( t_2 = 0 \) for every \( t_2 \in X_2 \). Hence \( X_2 = \{0\} \) which then implies that \( K_1(X_2) = S^l_e \).

The above conclusions are valid whatever be \( X_2 \). Let us now assume that \( k_1(X_2) = K_1(X_2) \). We now show that, in this case, one of the above four possibilities must definitely occur. To prove this, assume the contrary and let the following
inclusions all be proper.

\[ \{0\} \subset W_2 \subset N(e) \subset N_2 \subset V \]

Since the inclusions are proper, we can find \( W' \neq \{0\} \) such that \( N(e) = W_2 \oplus W' \). We now choose \( 0 \neq x \in W' \) and find \( W'' \) such that \( W' = \text{Span}(\{x\}) \oplus W'' \). Again since the inclusion \( N_2 \subset V \) is proper, \( R(e) \) cannot be a subset of \( N_2 \); for, if \( R(e) \) is also contained in \( N_2 \) then \( V = R(e) \oplus N(e) \subseteq N_2 \) implying that \( N_2 = V \). Choose \( y \in R(e) \setminus N_2 \). Then define \( t_1 \) by

\[
z t_1 = \begin{cases} 
y & \text{if } z = x \\
0 & \text{if } z \in R(e) \oplus W_2 \oplus W'' \end{cases}
\]

Obviously, \( R(t_1) \subset R(e) \subseteq N(t_1) \) and so \( t_1 \in S^r_e \). We also have \( W_2 \subseteq N(t_1) \) so that \( t_2 t_1 = 0 \) for all \( t_2 \in X_2 \). Therefore this \( t_1 \) must be in \( k_1(X_2) \). Since \( y \notin N_2 \), there is some \( t_2 \in X_2 \) for which \( y \notin N(t_2) \). For this \( t_2 \) we have \( y t_2 \neq 0 \) and so \( x t_1 t_2 = y t_2 \neq 0 \) which means that \( t_1 t_2 \neq 0 \) and so \( t_1 \notin K_1(X_2) \). This contradicts our hypothesis that \( k_1(X_2) = K_1(X_2) \). Therefore the inclusions referred to above cannot all be proper.

**Theorem 5.2.8.** If \( A = e + X_1 \oplus X_2 \), with \( X_1 \subseteq S^l_e \) and \( X_2 \subseteq S^r_e \), is an affine subspace of \( E(k) \) having dimension \( k(n-k) \) then \( A = E(L_e) \) or \( A = E(R_e) \).

**Proof.** By Corollary 5.2.6, the maximal dimension of an affine space contained in \( E(k) \) is \( k(n-k) \). Hence \( A \) must be a maximal affine subspace of \( E(k) \).

By Corollary 5.2.6, we have \( k(n-k) = \dim(X_1) + \dim(X_2) \). By Corollary 5.1.9, \( X_1 = K_1(X_2) \). Also \( K_1(X_2) \subseteq k_1(X_2) \). Hence \( \dim(K_1(X_2)) \leq \dim(k_1(X_2)) \). Also, since \( X_2 \subseteq k_2 k_1(X_2) \), we have \( \dim(X_2) \leq \dim(k_2 k_1(X_2)) \).

But, at the same time, by Theorems 5.2.2 and 5.2.5 we have

\[ \dim(k_2(X_2)) + \dim(k_2 k_1(X_2)) = k(n-k) \]

Therefore, we also have

\[ k(n-k) = \dim(X_1) + \dim(X_2) \]
These imply that \( \dim K_1(X_2) = \dim k_1(X_2) \) and therefore \( k_1(X_2) = K_1(X_2) \).

Therefore, by Lemma 5.2.7, \( K_1(X_2) = \{0\} \) or \( K_1(X_2) = S^l_\varepsilon \), that is, \( X_1 = \{0\} \) or \( X_1 = S^l_\varepsilon \). In the former case \( X_2 = S^c_\varepsilon \) and in the latter case \( X_2 = \{0\} \). Thus

\[ A = e + S^c_\varepsilon = E(\mathcal{R}_\varepsilon) \text{ or } A = e + S^l_\varepsilon = E(\mathcal{L}_\varepsilon). \]

\[ 5.3 \quad E(k) : A Generalized Hyperboloid \]

The aim of this section is to show that the space \( E(k) \) has geometrical properties which are exact analogues of the properties of a hyperboloid of one sheet.

**Properties of Hyperboloids**

To begin with, let us summarize the most important properties of a hyperboloid of one sheet as stated in the classical text [Bel60].

H1. The hyperboloid has two systems of generators.

H2. Through each point of the hyperboloid there passes one generator of each system.

H3. No two generators of the same system intersect.

H4. Any two generators belonging to two different systems intersect, except when they are parallel.

H5. The locus of the intersection of perpendicular generators is the curve of intersection of the hyperboloid and the director sphere.

H6. The tangent plane at a point of the hyperboloid meets the hyperboloid in the two generators through the point.

H7. Any plane through a generating line is the tangent plane at some point of the generator.
5.3.1 Generators of $E(k)$

Unless otherwise specified it will be assumed throughout that $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Clearly, $E(k)$ is equal to the union $\bigcup_{e \in E(k)} E(L_e)$. $E(L_e)$’s are affine spaces all having the same dimension $k(n - k)$. So we can think of $E(k)$ as a space generated by these affine spaces. The concept of an affine space is an exact generalization of the concept of a straight line. On $E(1)$ with $n = 2$ (which is a hyperboloid of one sheet), the generators are straight lines. Hence the family of affine spaces $\{E(L_e) : e \in E(k)\}$ can be considered as constituting a system of generators on the space $E(k)$. Similarly, the family $\{E(R_e) : e \in E(k)\}$ defines another system of generators on $E(k)$ (see Theorem 2.4.8).

With this interpretation of the term ‘generators’ we see that the following theorem is completely true.

**Theorem 5.3.1.** We have:

1. The space $E(k)$ has two systems of generators.

2. Through each point of $E(k)$ there passes one generator of each system.

3. No two generators of the same system intersect.

This theorem generalises the properties H1, H2 and H3 of a hyperboloid of one sheet to $E(k)$. However, the property H4 has no simple generalisation.

5.3.2 Perpendicular Generators

In this subsection we obtain a partial generalisation of the property H5 to the space $E(k)$. We show here that the intersections of ‘perpendicular’ generators of $E(k)$ lie on a sphere. To introduce the concept of perpendicularity, we require the concept of an inner product. Hence, we first define an inner product in $S_n$.

For $a \in S_n$, let $a^*$ be the Hilbert-space adjoint of $a$ (see Section 1.2.4). Recall that an idempotent $e$ is a perpendicular projection if $e^* = e$. The set of perpendicular projections in $E(k)$ is denoted by $P(k)$. We also require the trace function which is defined in Section 1.2.5.

The following lemma gives the traces of the elements of $S_e^I$ and $S_e^r$. 

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Lemma 5.3.2. If \( t \in S^t_e \cup S_e^t \) then \( \text{tr}(t) = 0 \).

**Proof.** If \( t \in S^t_e \cup S_e^t \) then \( t^2 = 0 \) and so every characteristic value of \( t \) is zero and hence \( \text{tr}(t) = 0 \).

We now introduce an inner product in \( S_n \) as follows:

\[
\langle a, b \rangle = \text{tr}(a^*b)
\]

The norm defined by this inner product is the Frobenius norm (see [Lim81]; see also pp.218-219 [MF88]).

Lemma 5.3.3. If \( f \in E(k) \) then \( ||f|| \geq \sqrt{k} \), and \( ||f|| = \sqrt{k} \) if and only if \( f \in P(k) \).

**Proof.** First of all we note that if \( e \in P(k) \) then (see Lemma 1.2.14)

\[
||e||^2 = \langle e, e \rangle = \text{tr}(e^*e) = \text{tr}(ee) = \text{tr}(e) = k.
\]

Now let \( f \) be any element in \( E(k) \) let \( e \in P(k) \) be such that \( e \not\parallel f \). Hence, by Proposition 1.2.12, \( e \not\parallel f^* \). Therefore, by using the properties of the trace function (see Proposition 1.2.15), we have

\[
||f||^2 = \text{tr}(f^*f) = \text{tr}[(e + (f^* - e))(e + (f - e))] = \text{tr}[e + (f^* - e)(f - e)] = k + ||f - e||^2.
\]

It follows that \( ||f|| \geq \sqrt{k} \) and \( ||f|| = \sqrt{k} \) if and only if \( ||f - e|| = 0 \), that is, \( f = e \).

This lemma implies that all the perpendicular projections in \( E(k) \) lie on the sphere with centre 0 and radius \( \sqrt{k} \). The next lemma shows that all perpendicular projections in \( E_n \) (irrespective of their ranks) lie on the fixed sphere with centre \( \frac{1}{2} \) and radius \( \frac{1}{2}\sqrt{n} \).

Lemma 5.3.4. If \( f \in E_n \) then \( ||f - \frac{1}{2}|| \geq \frac{\sqrt{n}}{2} \), and \( ||f - \frac{1}{2}|| = \frac{1}{2}\sqrt{n} \) if and only if \( f \) is a perpendicular projection.
Proof. Let $e \in \mathcal{P}(k)$ be such that $e \not\in f$. Hence, by Proposition 1.2.12, $e \not\in \mathcal{R} f^*$. Then, as in the proof of Lemma 5.3.3, we get

$$
\|f - \frac{1}{2}\|^2 = \|(f - e) + (e - \frac{1}{2})\|^2 = \|f - e\|^2 + \frac{n}{4}.
$$

(Note that $\text{tr}(1) = n$.) It follows that $\|f - \frac{1}{2}\| \geq \frac{\sqrt{n}}{2}$ and $\|f - \frac{1}{2}\| = \frac{\sqrt{n}}{2}$ if and only if $\|f - e\| = 0$, that is, $f = e$.

Since $1 - e \in \mathcal{E}_n$ whenever $e \in \mathcal{E}_n$, the centre $\frac{1}{2}$ of the fixed sphere referred to above is the centre of symmetry of the whole space $\mathcal{E}_n$. However $\frac{1}{2}$ is not a centre of symmetry for $E(k)$ because $1 - e$ is, in general, not in $E(k)$ when $e$ is in $E(k)$. However $1 - e$ is in $E(n - k)$. If $n$ is an even integer and $k = \frac{n}{2}$, then $E(k) = E(n - k)$ so that, in this special case, $\frac{1}{2}$ is indeed a centre of $E(k)$. This case occurs when $n = 2$.

The next result implies that every point on the line joining 0 and 1 is at a constant distance from all points in $\mathcal{P}(k)$.

**Lemma 5.3.5.** If $e \in \mathcal{P}(k)$ and $\alpha$ a scalar then $\|e - \alpha\|$ is independent of $e$. Moreover this is minimum whenever $\alpha = \frac{k}{n}$.

**Proof.** Let $\alpha^*$ denote the complex conjugate of $\alpha$. We have

$$
\|e - \alpha\|^2 = \text{tr}[(e - \alpha^*)(e - \alpha)]
= \text{tr}[e - (\alpha + \alpha^*)e + |\alpha|^2]
= \text{tr}[(1 - \alpha - \alpha^*)e + |\alpha|^2]
= \text{tr}[(1 - \alpha - \alpha^*)e] + \text{tr}[|\alpha|^2]
= (1 - \alpha - \alpha^*)k + |\alpha|^2 n
$$

which shows that $\|e - \alpha\|$ is independent of $e$.

To find the value of $\alpha$ for which $\|e - \alpha\|$ is minimum, we write $\alpha = \delta + i\epsilon$ with $\delta, \epsilon \in \mathbb{R}$ and $i = \sqrt{-1}$. Then we have

$$
(1 - \alpha - \alpha^*)k + |\alpha|^2 n = (1 - 2\delta)k + (\delta^2 + \epsilon^2)n.
$$
It is easy to see that this is minimum when \( \delta = \frac{k}{n} \) and \( \epsilon = 0 \). Thus, \( \|e - \alpha\| \) is minimum when \( \alpha = \frac{k}{n} \). The minimum distance is \( \sqrt{\frac{k(n-k)}{n}} \).

**Theorem 5.3.6.** Let \( e \in E(k) \). Then \( e \in P(k) \) if and only if \( E(L_e) \) and \( E(R_e) \) are mutually perpendicular.

**Proof.** Any elements \( f \in E(L_e) \) and \( g \in E(R_e) \) can be written in the form (see Proposition 3.2.2)

\[
f = e + (1 - e)s_1e, \quad g = e + es_2(1 - e), \quad \text{with} \quad s_1, s_2 \in \mathbb{S}_n.
\]

Since \( e^* = e \) we have

\[
(f - e)^*(g - e) = [(1 - e)s_1e]^*[es_2(1 - e)] = [es_1^*(1 - e)][es_2(1 - e)] = 0,
\]

and so

\[
(f - e, g - e) = \text{tr}(((f - e)^*(g - e)) = \text{tr}(0) = 0.
\]

This shows that every vector in \( E(L_e) \) is perpendicular to every vector in \( E(R_e) \) and so \( E(L_e) \) and \( E(R_e) \) are mutually perpendicular.

To prove the converse, let \( e \in E(k) \) be such that \( E(L_e) \) and \( E(R_e) \) are mutually perpendicular.

Let \( \{u_1, \ldots, u_n\} \) be an orthonormal basis for \( V \) such that \( \{u_1, \ldots, u_k\} \) is a basis for \( R(e) \). Relative to this basis \( e \) can be represented by a matrix of the form

\[
\begin{pmatrix}
I & 0 \\
A & 0
\end{pmatrix}
\]

where \( I \) is the unit matrix of order \( k \) and \( A \) is a matrix of order \( (n - k) \times k \) (see Remark 3.2.1). Any element \( f \in E(L_e) \) is of the form \( (\gamma) \).

By direct computations using matrices we can see that, relative to the orthonormal basis referred to above, any element \( g \in E(R_e) \) is of the form \( \left( \begin{array}{c}
-IQ^A \\
AQ^A
\end{array} \right) \), where \( Q \) is a matrix of order \( k \times (n - k) \).

By multiplying out the matrices we see that

\[
\text{tr}[(f - e)^*(g - e)] = \text{tr}(RAQA),
\]

where \( R = (P - A)^* \). By the hypothesis on \( e \), we must have \( \text{tr}(RAQA) = 0 \). This must be true for all \( P \) and \( Q \). Let \( A \neq 0 \). Then, it is easy to see that we can choose...
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\( P \) and \( Q \) such that \( \text{tr}(RAQA) \neq 0 \). This is a contradiction. Therefore we must have \( A = 0 \). Hence \( e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) which is in \( P(k) \).

From Lemmas 5.3.4 and 5.3.6, we have the following corollary.

**Corollary 5.3.7.** The perpendicular generators of \( E(k) \) intersect on the sphere with centre at \( \frac{1}{2} \) and radius \( \frac{1}{2} \sqrt{n} \).

### 5.3.3 The Tangent Space of \( E(k) \)

The property \( H6 \) can be reformulated as follows:

\( H6' \). The tangent plane at a point of a hyperboloid is the plane containing the generators of the hyperboloid through the point.

Since the tangent plane can be identified as the tangent space and since the plane containing the generators is the affine space generated by the generators, the above statement can be presented in the following form:

\( H6'' \). The tangent space at a point of a hyperboloid is the affine space generated by the generators of the hyperboloid through the point.

We show below that the above statement is valid for \( E(k) \) also.

In this subsection we assume that \( K = \mathbb{R} \). This is to facilitate the computation of the tangent space. We adopt the following definition of the tangent space to a manifold (see p.9 in [GP74]):

- Let \( x \) be a point in a manifold \( M \) which is sitting in \( \mathbb{R}^r \) for some positive integer \( r \). The tangent space of \( M \) at \( x \) is the flat space that best approximates \( M \) at \( x \). Let \( U \) be an open set in some \( \mathbb{R}^s \) and \( \phi : U \to M \) be a local parameterization of \( M \) around \( x \) and assume that \( \phi(0) = x \). Let \( d\phi_0 : \mathbb{R}^s \to \mathbb{R}^r \) be the derivative of \( \phi \) at 0 and \( \hat{T}_x(M) \) be the image of \( d\phi_0 \) in \( \mathbb{R}^r \). Then \( T_x(M) = x + \hat{T}_x(M) \) is the tangent space to \( M \) at \( x \).

The theorem below proves that the statement \( H6'' \) is valid for \( E(k) \) also.

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Theorem 5.3.8. The tangent space of $E(k)$ at $e \in E(k)$ is

$$T_e(E(k)) = e + S_e^c \oplus S_e^r.$$  

Proof. To determine the tangent space at $e \in E(k)$, we have to introduce a local parameterisation around $e$. This we do as explained in Section 3.4.3.

We choose some ordered basis $B$ of $V$ such that the first $k$ vectors in $B$ is a basis for $R(e)$ and the remaining $n-k$ vectors is a basis for $N(e)$. Relative to this basis, $e$ is represented by $(b_q)$. Let $P$ be a $k \times (n-k)$ matrix and $Q$ a $(n-k) \times k$ matrix. Let $W'$ be the space spanned by the rows of $[I \ P]$ and $N'$ be the space spanned by the rows of $[-Q \ I]$. Now $W' \oplus N' = V$ if and only if $u = (-Q \ P)$ is nonsingular. In this case the unique $g \in E(k)$ with $R(g) = W'$ and $N(g) = N'$ is given by $g = u^{-1}eu$.

Identifying $(P, Q)$ as an element in $\mathbb{R}^{2k(n-k)}$, let

$$U = \{(P, Q) : u = (-Q \ P) \text{ is nonsingular}\}.$$  

It is obvious that $U$ is an open set in $\mathbb{R}^{2k(n-k)}$. The map

$$\phi : U \rightarrow E(k), \quad (P, Q) \mapsto u^{-1}eu$$

is a local parameterization of $E(k)$ around $e$. Moreover, we have $\phi(0) = e$.

Let $P = [p_{ij}]$ and $Q = [q_{ij}]$, and let $t$ represent one of the $p_{ij}$’s or $q_{ij}$’s. Then using the fact that

$$\frac{\partial u^{-1}}{\partial t} = -u^{-1} \frac{\partial u}{\partial t} u^{-1},$$

we have

$$\frac{\partial g}{\partial t} = u^{-1}e \begin{pmatrix} O & \hat{P} \\ -\hat{Q} & O \end{pmatrix} - u^{-1} \begin{pmatrix} O & \hat{P} \\ -\hat{Q} & O \end{pmatrix} u^{-1}eu,$$

where dots indicate differentiation with respect to $t$. Note that if $t = p_{ij}$, then $\hat{P}$ is the constant matrix with 1 in the $(i, j)$th position and 0’s at the remaining positions and $-\hat{Q}$ is a zero matrix. Also if $t = q_{ij}$ then $\hat{P}$ is a zero matrix and $-\hat{Q}$ is a constant matrix with $-1$ in the $(i, j)$th position and 0’s at the remaining positions.
Now let \( \left( \frac{\partial g}{\partial t} \right)_0 \) denote the value of \( \frac{\partial g}{\partial t} \) when \((P, Q) = (0,0)\). Since \( u = 1 \) when \((P, Q) = (0,0)\) we have

\[
\left( \frac{\partial g}{\partial t} \right)_0 = e \begin{pmatrix} 0 & \dot{P} \\ -\dot{Q} & 0 \end{pmatrix} - \begin{pmatrix} 0 & \dot{P} \\ -\dot{Q} & 0 \end{pmatrix} e = \begin{pmatrix} 0 & \dot{P} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -\dot{Q} & 0 \end{pmatrix}.
\]

These partial derivatives define the derivative \( d\phi_0 \). The image of \( d\phi_0 \) is the set of all points in \( \mathbb{R}^{n^2} \) which can be expressed in the form

\[
\sum_t t \left( \frac{\partial g}{\partial t} \right)_0.
\]

From the observations regarding the nature of the matrices \( \dot{P} \) and \(-\dot{Q} \) we see that these points can be expressed in the form

\[
\begin{pmatrix} O & \dot{P} \\ \dot{O} & O \end{pmatrix} - \begin{pmatrix} O & 0 \\ -\dot{Q} & 0 \end{pmatrix}.
\]

(5.3.1)

This is clearly equal to

\[ eu(1-e) + (1-e)(-u)e, \]

and it is obviously in \( S^e_T \oplus S^l_T \). Every point in \( S^l_T \oplus S^r_T \) is in the form Eq.(5.3.1) and hence is in the image of \( d\phi_0 \). This completes the proof of the theorem.

The next result helps us to sharpen the picture of tangent spaces of \( E(k) \).

**Lemma 5.3.9.** If \( e \in P(k) \) then \( e - \frac{1}{2} \perp T_e(E(k)) \).

**Proof.** In view of Theorem 5.3.8, any element in \( T_e(E(k)) \) is of the form \( a = e + t_1 + t_2 \) with \( t_1 \in S^l_T \) and \( t_2 \in S^r_T \). Now,

\[
\text{tr}((e - \frac{1}{2})(e - a)) = \frac{1}{2}(\text{tr}(t_2) - \text{tr}(t_1)),
\]

which is 0 by Lemma 5.3.2. Hence \( e - \frac{1}{2} \perp e - a \) for every \( a \) in \( T_e(E(k)) \).
5.3.4 The Tangent Space of $P(k)$

**Theorem 5.3.10.** If $e \in P(k)$, then

$$T_e(P(k)) = \left\{ \frac{1}{2}(f + f^*) : f \in E(R_e) \right\}.$$  

**Proof.** Let $e \in P(k)$. Choose an orthonormal basis for $V$ such that the first $k$ vectors in the basis form a basis for $R(e)$ and the remaining $(n-k)$-vectors form a basis for $N(e)$. Relative this basis $e = (I \ O)$ and elements of $E(R_e)$ are of the form $f = (I \ O)$. The set

$$\{e' \in P(k) : e' \ L f \text{ for some } f \in E(R_e)\}$$

is a neighbourhood of $e$ in $P(k)$. Clearly the elements of this neighbourhood can be written the form $e' = u^{-1}eu$ where $u = ( I \ P )$. The map $\phi : P \mapsto e'$ is a local parameterization of $P(k)$ (see Remark 3.4.6).

Let $t$ denote one of the elements of the matrix $P$. Since

$$\frac{\partial e'}{\partial t} = u^{-1}e\frac{\partial u}{\partial t}u^{-1}u^{-1}\frac{\partial u}{\partial t}u^{-1}eu,$$

and since $u = 1$, $e' = e$ when $P = 0$ we have (see proof of Theorem 5.3.8)

$$\left( \frac{\partial e'}{\partial t} \right)_{P=0} = \begin{pmatrix} O & \dot{P} \\ \dot{P}^* & O \end{pmatrix},$$

where dots denote partial differentiation with respect to $t$. These partial derivatives define the derivative $d\phi_0$.

Again, as in the Proof of Theorem 5.3.8, we see that the elements of the tangent space of $P(k)$ at $e$ are of the form

$$e + \sum t \left( \frac{\partial e'}{\partial t} \right)_{P=0}.$$  

These can be expressed in the form $(I \ P)$. This is clearly equal to $\frac{1}{2}(f + f^*)$. \( \square \)

Since $T_e(P(k))$ is a subspace of $T_e(E(k))$, Lemma 5.3.9 implies the following.
Corollary 5.3.11. If \( e \in P(k) \), then tangent space \( T_e(P(k)) \) is perpendicular to the diameter through \( e \) of the sphere with centre \( \frac{1}{2} \) and radius \( \frac{1}{2} \sqrt{n} \).

The next theorem implies that the tangent space \( T_e(P(k)) \) is, in some sense, the perpendicular bisector of the ‘angle’ between \( E(L_e) \) and \( E(R_e) \).

Theorem 5.3.12. If \( e \in P(k) \), then \( E(L_e) \mid E(R_e) \mid \) is the reflection of \( E(R_e) \mid E(L_e) \) in the tangent space \( T_e(P(k)) \).

Proof. Let \( a = \frac{1}{2}(f + f^*) \), where \( f \in E(R_e) \), be any point in \( T_e(P(k)) \). We now write \( f = e + t \) with \( t \in S_e^r \) so that \( t^* \in S_e^l \). Using Proposition 1.2.15 we have

\[
\langle a - e, a - f \rangle = \text{tr}(\frac{1}{2}(t + t^*)^* (t^* - t)) = \frac{1}{4} \text{tr}(tt^* - t^*t) = 0.
\]

This shows that \( a - e \) is perpendicular to \( a - f \). We next show that \( a \) is the only point in \( T_e(P(k)) \) having this property so that we can think of \( a \) as the foot of the perpendicular from \( f \in E(R_e) \) to \( T_e(P(k)) \).

So let \( f \in E(R_e) \) be given. Let \( b \in T_e(P(k)) \) be such that \( (w - b) \perp (f - b) \), that is \( \langle w - b, f - b \rangle = 0 \), for every \( w \in T_e(P(k)) \). Here we write

\[
f = e + t^0, \quad b = e + \frac{1}{2}(t + t^*), \quad w = e + \frac{1}{2}(t' + t^*)
\]

with \( t^0, t, t' \in S_e^r \). Now using the various properties of the trace function and also properties of elements of \( S_e^r \) we get

\[
0 = \langle w - b, f - b \rangle \\
= \text{tr}(((w - b)(f - b)) \\
= \frac{1}{4} \text{tr}(((t' - t) + (t^* - t^*))^* (t^0 - t^*)) \\
= \frac{1}{4} \text{tr}((t^* - t^*)t^0 - (t' - t)t^*) \\
= \frac{1}{4} \left[ \text{tr}((t^* - t^*)t^0) - \text{tr}((t' - t)t^*) \right] \\
= \frac{1}{4} \text{tr}((t^* - t^*)(t^0 - t)).
\]

Since \( t' \) is arbitrary so is \( t^* - t^* \). Expressing the above equation in matrix form, and using Proposition 1.2.15, we note that the above equality is possible only if \( t^0 = t \).
which means that \( b = \frac{1}{2} (f + f^*) \).

Now note that \( \frac{1}{2} (f + f^*) \) is the midpoint of the line segment joining \( f \) and \( f^* \). The theorem follows from this.

\[ \square \]

Chapter 6

Homotopy Properties of \( E_n \)

In this chapter we focus our attention on the homotopy properties of the space of idempotent endomorphisms. We begin our study by determining the fundamental group of the space \( E(k) \). It turns out that, if the underlying vector space \( V \) is complex, the fundamental group of \( E(k) \) is trivial. So in this case there is nothing much to talk about.

In the real case, if the dimension \( n \) of \( V \) is greater than 2, the fundamental group of \( E(k) \) is the two element group \( \mathbb{Z}_2 \). This result is proved in Section 6.3. Prove that, if \( n = 2 \), the fundamental group of \( E(k) \) is the additive group of \( \mathbb{R}^2 \) (see Proposition 2.4.13).

Every \( E \)-cycle in \( E_n \) represents a polygonal path. Hence, if \( n > 2 \) and \( K = \mathbb{R} \), then, given a polygonal cycle \( \gamma \) in \( E(k) \), either \( \gamma \) is null homotopic or the square of \( \gamma \) is null homotopic. One of the main results of this chapter concerns the determination of the cycles which are null homotopic and the cycles which are not null homotopic. As a first step in this direction we show in Section 6.2 that every chain in \( E_n \) can be mapped to a polygonal path in \( \text{GL}(V) \) having a particular property. Then in the next Section we consider a classification of cycles into positive and negative cycles. These facts are then used in Section 6.4 to determine which cycles in \( E(k) \) (when \( K = \mathbb{R} \)) are null homotopic and which cycles are not. In this section we have also included a discussion of the homotopy properties of \( K \)-chains in \( E_n \).

In Section 6.5 we show that every path in \( E(k) \) is homotopic to a chain. Homotopies of arbitrary chains are also discussed in this section. The complex case is...