4. Two Phase Batch Arrival Retrial Queue with Impatient Customers, Server Breakdown and Modified Vacation

A batch arrival retrial queue with balking and reneging is considered. If an arriving batch finds the server free, then one of the customers receives the service immediately and others join the orbit. If the server is blocked the arriving batch joins the orbit. All the arriving customers receive the first essential service, whereas only some of them opt the second optional service. The server is subject to breakdown while it is working. The repair of the failed server starts instantaneously. After completion of the repair, the server continues the service of the interrupted customer. The server is not allowed to accept new customers until the interrupted customer leaves the system. Whenever the orbit is empty the server takes atmost J vacations repeatedly until atleast one customer appears in the orbit on return from a vacation. The stability condition of the system is derived. Using generating function technique, the steady state distributions of the server state and the number of customers in the orbit are obtained along with performance measures. Reliability indices of the system are obtained. Special cases are discussed and numerical results are provided.

4.1 Model Description

Consider a single server queueing system in which customers arrive in batches according to compound Poisson process with rate $\lambda$. The batch size $Y$ is a random variable with distribution function $P(Y = k) = C_k$, $k = 1, 2, \ldots$, the probability generating function $C(z)$ and first two moments $m_1$ and $m_2$.

If the server is free, then one of the arriving customers receives the service immediately and others join the orbit. If the server is busy, down or on vacation then all the customers join the orbit with probability $p$ or leave the system with probability $\overline{p} (= 1 - p)$. The customer access from the orbit to the server is governed by an arbitrary law with distribution function $A(x)$ and
Laplace Stieltjes transform $A^*(s)$. If a primary customer arrives first, then the retrial customer cancels the attempt for service and either returns to its position in the orbit with probability $q$ or leaves the system with probability $\bar{q} (= 1 - q)$.

There are two heterogeneous phases of service first essential and second optional. Essential service is provided to all the arriving customers. As soon as the essential service is completed the customer may leave the system with probability $r_0$ or opt for the optional service with probability $\bar{r}_0 (= 1 - r_0)$. The service time of $i^{th}$ phase follows an arbitrary distribution with distribution function $B_i(x)$, Laplace Stieltjes transform $B_i^*(s)$ and the first two moments $\mu_{i1}$ and $\mu_{i2}$, $i = 1, 2$.

The server is subject to breakdown while it is working. It is assumed that the lifetime of the server in $i^{th}$ phase is exponential with rate $\alpha_i$. The repair time of the server failed during $i^{th}$ phase service is generally distributed with distribution function $R_i(x)$, Laplace Stieltjes transform $R_i^*(s)$ and the first two moments $\gamma_{i1}$, $\gamma_{i2}$, $i = 1, 2$.

When the server fails during $i^{th}$ phase service, the interrupted customer remains in the service position with probability $1 - \theta_i$ or leaves the service area with probability $\theta_i$ and keeps returning at times exponentially distributed with rate $\tau_i$, $i = 1, 2$. If the interrupted customer is not in the service position, then after the completion of the repair, the server waits for the same customer to continue the service. This waiting time of the server is termed as reserved time. The server is not allowed to accept new customers until the interrupted customer leaves the system. Whenever the system becomes empty, the server leaves for a vacation of random length. On return from vacation if there is no customer in the orbit, the server takes another vacation. This pattern continues until the server return from vacation finds at least one customer in the system. Number of consecutive vacations is limited to $J$. At the end of $J^{th}$ vacation even if the system is empty the server stays in the system. The
vacation times are generally distributed with distribution function $V(x)$, Laplace Stieltjes transform $V^*(s)$ and the first two moments $\mu_1$ and $\mu_2$.

Fig. 4.1 Two Phase Batch Arrival Retrial Queue with Impatient Customers, Server Breakdown and Modified Vacation

The state of the system at time $t$ can be described by the Markov process $\{X(t), t \geq 0\} = \{S(t), N(t), \xi_1(t), \xi_2(t)\}$, where $S(t)$ denote the server state 0, 1, 2, 3, 4, 5, 6, j+6 according as the server being idle, busy in first essential service, busy in second optional service, under repair during essential service, under repair during optional service, in reserved time during essential service, in reserved time during optional service and in $j^{th}$ vacation $(1 \leq j \leq J)$. $N(t)$ denotes the number of customers in the orbit at time $t$. Define the supplementary variables $\xi_1(t)$ and $\xi_2(t)$ as follows.

If $S(t) = 0$, $\xi_1(t)$ = elapsed retrial time

If $S(t) = 1, 2$, $\xi_1(t)$ = elapsed service time

If $S(t) = 3, 4$, $\xi_1(t)$ = elapsed service time and $\xi_2(t)$ = elapsed repair time
If \( S(t) = 5, 6, \xi_1(t) = \) elapsed service time and \( \xi_2(t) = \) elapsed reserved time

If \( S(t) = j + 6, \xi_1(t) = \) elapsed vacation time, \( 1 \leq j \leq J. \)

The functions \( \eta(x), \mu_1(x), \mu_2(x), \beta_1(x), \beta_2(x) \) and \( \nu(x) \) are the conditional completion rates for repeated attempts, first essential service, second optional service, repair during essential service, repair during optional service and vacation respectively. Then

\[
\eta(x) = \frac{a(x)}{1 - A(x)}, \mu_1(x) = \frac{b_1(x)}{1 - B_1(x)}, \mu_2(x) = \frac{b_2(x)}{1 - B_2(x)}, \beta_1(x) = \frac{r_1(x)}{1 - R_1(x)}, \\
\beta_2(x) = \frac{r_2(x)}{1 - R_2(x)} \text{ and } \nu(x) = \frac{\nu(x)}{1 - V(x)}.
\]

4.2 Stability Condition

**Theorem 4.1**

The inequality

\[
p \lambda m_1 [\mu_1 (1 + \alpha_1 (\gamma_{11} + \frac{\theta_1}{\tau_1})) + \bar{r}_0 \mu_{21} (1 + \alpha_2 (\gamma_{21} + \frac{\theta_2}{\tau_2}))] + (1 - A^*(\lambda)) m_1 < 1 + \bar{q} (1 - A^*(\lambda))
\]

is the necessary and sufficient condition for the system to be stable.

**Proof**

Let \( S^{(i)} \) be the generalized service time of \( i^{th} \) customer in service.

Then \( \{S^{(i)}\} \) are independently and identically distributed with Laplace transform

\[
B_1^*[s + \alpha_1 - \alpha_1 \left( \frac{s(1 - \theta_1) + \tau_1}{s + \tau_1} \right) R_1^*(s)] + \bar{r}_0 B_2^*[s + \alpha_2 - \alpha_2 \left( \frac{s(1 - \theta_2) + \tau_2}{s + \tau_2} \right) R_2^*(s)]
\]

\[
E[S^{(i)}] = \mu_1 (1 + \alpha_1 (\gamma_{11} + \frac{\theta_1}{\tau_1})) + \bar{r}_0 \mu_{21} (1 + \alpha_2 (\gamma_{21} + \frac{\theta_2}{\tau_2})).
\]
Let $P(B)$ be the probability that the system is blocked and $P(I)$ be the probability that the system is idle.

Let $E[S^{(i)}]$ be the expected blocked time and $E[I]$ be the expected idle time. Then

$$P(B) = \frac{E[S^{(i)}]}{E[S^{(i)}] + E[I]} ; \quad P(I) = \frac{E[I]}{E[S^{(i)}] + E[I]}$$

When the system is blocked, the arrival rate at the retrial queue is $p \lambda m_1 P(B)$.

When the server is idle, the arrival rate at the retrial queue is

$$\frac{(1 - A^*(\lambda)) (m_1 - 1) P(I)}{E(I)}.$$

The exit rate from the retrial queue by entering service is $\frac{A^*(\lambda) P(I)}{E(I)}$.

The exit rate from the retrial queue by leaving the system is

$$\frac{(1 - A^*(\lambda))(1 - q) P(I)}{E(I)}.$$

The system is stable if and only if the total arrival rate is less than the exit rate. Hence

$$p \lambda m_1 \left[ \frac{E[S^{(i)}]}{E[S^{(i)}] + E[I]} + \frac{(1 - A^*(\lambda)) (m_1 - 1)}{E[S^{(i)}] + E(I)} \right] < \frac{(1 - A^*(\lambda))(1 - q)}{E[S^{(i)}] + E(I)} + \frac{A^*(\lambda)}{E[S^{(i)}] + E(I)}.$$

Consequently

$$p \lambda m_1 \left[ \mu_{11} (1 + \alpha_1 \left( \frac{\theta_1}{\tau_1} \right) ) + \bar{r}_0 \mu_{21} \left( 1 + \alpha_2 \left( \frac{\theta_2}{\tau_2} \right) \right) \right] + (1 - A^*(\lambda)) m_1 < 1 + \bar{q} (1 - A^*(\lambda))$$

is the necessary and sufficient condition for the system to be stable.
4.3 Definitions and Equations Governing the System

For the process \( \{X(t), t \geq 0\} \) define the following probabilities

\[
\begin{align*}
I_0(t) &= P\{S(t) = 0, X(t) = 0\} \\
I_n(x, t) \, dx &= P\{S(t) = 0, X(t) = n, x \leq \xi_1(t) < x + dx\}, \quad x \geq 0, \ n \geq 1 \\
P_n(x, t) \, dx &= P\{S(t) = 1, X(t) = n, x \leq \xi_1(t) < x + dx\}, \quad x \geq 0, \ n \geq 0 \\
Q_n(x, t) \, dx &= P\{S(t) = 2, X(t) = n, x \leq \xi_1(t) < x + dx\}, \quad x \geq 0, \ n \geq 0 \\
F_{1,i,n}(x, y, t) \, dx \, dy &= P\{S(t) = 3, X(t) = n, x \leq \xi_1(t) < x, \ y \leq \xi_2(t) < y + dy\}, \\
&\quad x \geq 0, \ n \geq 0, \ i = 0, 1 \\
F_{1,2,i,n}(x, y, t) \, dx \, dy &= P\{S(t) = 4, X(t) = n, x \leq \xi_1(t) < x, \ y \leq \xi_2(t) < y + dy\}, \\
&\quad x \geq 0, \ n \geq 0, \ i = 0, 1 \\
F_{2,1,n}(x, y, t) \, dx \, dy &= P\{S(t) = 5, X(t) = n, x \leq \xi_1(t) < x, \ y \leq \xi_2(t) < y + dy\}, \\
&\quad x \geq 0, \ n \geq 0 \\
F_{2,2,n}(x, y, t) \, dx \, dy &= P\{S(t) = 6, X(t) = n, x \leq \xi_1(t) < x, \ y \leq \xi_2(t) < y + dy\}, \\
&\quad x \geq 0, \ n \geq 0 \\
V_{j,n}(x, t) \, dx &= P\{S(t) = j + 6, X(t) = n, x \leq \xi_1(t) < x + dx\}, \\
&\quad x \geq 0, \ n \geq 0, \ 1 \leq j \leq J
\end{align*}
\]

\( i = 0 \) means the interrupted customer is in service position and \( i = 1 \) means the customer is not in the service position.

The system of steady state equations that governs the model under consideration is

\[
\begin{align*}
\lambda \, I_0 &= \int_{0}^{\infty} V_{j,0}(x) \, v(x) \, dx \quad (4.1) \\
\frac{d}{dx} I_n(x) &= - (\lambda + \eta(x)) \, I_n(x), \quad n \geq 1 \quad (4.2)
\end{align*}
\]
\[ \frac{d}{dx} P_0(x) = - (p \lambda + \alpha_1 + \mu_1(x)) P_0(x) + \int_0^\infty F_{1,1,0,0}(x, y) \beta_1(y) \, dy \]
\[ + \tau_1 \int_0^\infty F_{2,1,0}(x, y) \, dy \]  
(4.3)

\[ \frac{d}{dx} P_n(x) = - (p \lambda + \alpha_1 + \mu_1(x)) P_n(x) + \int_0^\infty F_{1,1,0,n}(x, y) \beta_1(y) \, dy \]
\[ + \tau_1 \int_0^\infty F_{2,1,n}(x, y) \, dy + p \lambda \sum_{k=1}^n C_k P_{n-k}(x), \quad n \geq 1 \]  
(4.4)

\[ \frac{d}{dx} Q_0(x) = - (p \lambda + \alpha_2 + \mu_2(x)) Q_0(x) + \int_0^\infty F_{1,2,0,0}(x, y) \beta_2(y) \, dy \]
\[ + \tau_2 \int_0^\infty F_{2,2,0}(x, y) \, dy \]  
(4.5)

\[ \frac{d}{dx} Q_n(x) = - (p \lambda + \alpha_2 + \mu_2(x)) Q_n(x) + \int_0^\infty F_{1,2,0,n}(x, y) \beta_2(y) \, dy \]
\[ + \tau_2 \int_0^\infty F_{2,2,n}(x, y) \, dy + p \lambda \sum_{k=1}^n C_k Q_{n-k}(x), \quad n \geq 1 \]  
(4.6)

\[ \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) F_{1,1,i,n}(x, y) = - (p \lambda + \beta_1(y)) F_{1,1,i,n}(x, y) + p \lambda \sum_{k=1}^n C_k F_{1,1,i,n-k}(x, y), \quad i = 0, 1 ; \ n \geq 0 \]  
(4.7)

\[ \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) F_{1,2,i,n}(x, y) = - (p \lambda + \beta_2(y)) F_{1,2,i,n}(x, y) + p \lambda \sum_{k=1}^n C_k F_{1,2,i,n-k}(x, y), \quad i = 0, 1 ; \ n \geq 0 \]  
(4.8)

\[ \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) F_{2,1,0}(x, y) = - (p \lambda + \tau_1) F_{2,1,0}(x, y) \]  
(4.9)

\[ \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) F_{2,1,n}(x, y) = - (p \lambda + \tau_1) F_{2,1,n}(x, y) + p \lambda \sum_{k=1}^n C_k F_{2,1,n-k}(x, y), \quad n \geq 1 \]  
(4.10)

\[ \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) F_{2,2,0}(x, y) = - (p \lambda + \tau_2) F_{2,2,0}(x, y) \]  
(4.11)
\[
\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) F_{2,2,n}(x, y) = -(p \lambda + \tau_2) F_{2,2,n}(x, y) + p \lambda \sum_{k=1}^{n} C_k F_{2,2,n-k}(x, y), \quad n \geq 1
\]

(4.12)

\[
\frac{\partial}{\partial x} V_{j,n}(x) = -(p \lambda + v(x)) V_{j,n}(x) + p \lambda \sum_{k=1}^{n} C_k V_{j,n-k}(x), \quad n \geq 0 ; 1 \leq j \leq J
\]

(4.13)

with boundary conditions

\[
I_n(0) = \sum_{j=1}^{J} \int_{0}^{\infty} V_{j,n}(x) v(x) \, dx + r_0 \int_{0}^{\infty} P_n(x) \mu_1(x) \, dx + \int_{0}^{\infty} Q_n(x) \mu_2(x) \, dx,
\]

\[n \geq 1\]

(4.14)

\[
P_0(0) = \lambda C_1 I_0 + \int I_1(x) \eta(x) \, dx + \lambda \bar{q} \int_{0}^{\infty} I_1(x) \, dx
\]

(4.15)

\[
P_n(0) = \lambda C_{n+1} I_0 + \lambda q \sum_{k=1}^{n} C_k \int_{0}^{\infty} I_{n-k+1}(x) \, dx + \lambda \bar{q} \sum_{k=1}^{n+1} C_k \int_{0}^{\infty} I_{n-k+2}(x) \, dx
\]

\[+ \int_{0}^{\infty} I_{n+1}(x) \eta(x) \, dx, \quad n \geq 1\]

(4.16)

\[
Q_n(0) = \bar{r}_0 \int_{0}^{\infty} P_n(x) \mu_1(x) \, dx, \quad n \geq 0
\]

(4.17)

\[
F_{1,1,0,n}(x, 0) = \alpha_1 (1 - \theta_1) P_n(x), \quad n \geq 0
\]

(4.18)

\[
F_{1,1,1,n}(x, 0) = \alpha_1 \theta_1 P_n(x), \quad n \geq 0
\]

(4.19)

\[
F_{1,2,0,n}(x, 0) = \alpha_2 (1 - \theta_2) Q_n(x), \quad n \geq 0
\]

(4.20)

\[
F_{1,2,1,n}(x, 0) = \alpha_2 \theta_2 Q_n(x), \quad n \geq 0
\]

(4.21)

\[
F_{2,1,n}(x, 0) = \int_{0}^{\infty} F_{1,1,1,n}(x, y) \beta_1(y) \, dy, \quad n \geq 0
\]

(4.22)

\[
F_{2,2,n}(x, 0) = \int_{0}^{\infty} F_{1,2,1,n}(x, y) \beta_2(y) \, dy, \quad n \geq 0
\]

(4.23)

\[
V_{1,n}(0) = r_0 \int_{0}^{\infty} P_0(x) \mu_1(x) \, dx + \int_{0}^{\infty} Q_0(x) \mu_2(x) \, dx, \quad n = 0
\]

(4.24)
\[ V_{j,n}(0) = \int_{0}^{\infty} V_{j-1,0}(x) \nu(x) \, dx, \quad n = 0; \ j = 2, 3, \ldots J \quad (4.25) \]

The normalising condition is

\[
I_0 + \sum_{n=1}^{\infty} \int I_n(x) \, dx + \sum_{n=0}^{\infty} \int P_n(x) \, dx + \sum_{n=0}^{\infty} \int Q_n(x) \, dx
\]

\[
+ \sum_{i=0}^{1} \sum_{n=0}^{\infty} \sum_{k=1}^{2} \int F_{1,k,i,n}(x, y) \, dy \, dx + \sum_{n=0}^{\infty} \sum_{k=1}^{2} \int F_{2,k,n}(x, y) \, dx \, dy
\]

\[
+ \sum_{j=1}^{J} \sum_{n=0}^{\infty} \int V_{j,n}(x) \, dx = 1 \quad (4.26)
\]

4.4 Generating Functions of the Orbit Length

Define the probability generating functions

\[ I(x, z) = \sum_{n=1}^{\infty} I_n(x) z^n; \quad P(x, z) = \sum_{n=0}^{\infty} P_n(x) z^n; \quad Q(x, z) = \sum_{n=0}^{\infty} Q_n(x) z^n; \]

\[ F_{1,i,j}(x, y, z) = \sum_{n=0}^{\infty} F_{1,i,j,n}(x, y) z^n, \quad i = 0, 1; \ j = 1, 2 \]

\[ F_{2,j}(x, y, z) = \sum_{n=0}^{\infty} F_{2,j,n}(x, y) z^n; \quad j = 1, 2 \]

\[ V_{j}(x, z) = \sum_{n=0}^{\infty} V_{j,n}(x) z^n, \quad 1 \leq j \leq J \]

Theorem 4.2

The stationary distribution of the process \{"X(t), t \geq 0\" has the following generating functions

\[ I(z) = \frac{I_0 z (1 - A^*(\lambda)) [z (T(z) - 1) + C(z) T_2(z)]}{z^2 - T_1(z) T_2(z)} \quad (4.27) \]
\[
P(z) = \frac{\lambda I_0 \left[ z \, C(z) + T_1(z) \, (T(z) - 1) \right] \left[ 1 - B_1^*(k_1 \, (p \lambda - p \lambda \, C(z))) \right]}{\left[ z^2 - T_1(z) \, T_2(z) \right] \left[ k_1 \, (p \lambda - p \lambda \, C(z)) \right]} (4.28)
\]

\[
Q(z) = \frac{\lambda I_0 \, \bar{r}_0 \, B_1^*(k_1 \, (p \lambda - p \lambda \, C(z))) \left[ z \, C(z) + T_1(z) \, (T(z) - 1) \right]}{\left[ z^2 - T_1(z) \, T_2(z) \right] \left[ 1 - B_2^* \, (k_2 \, (p \lambda - p \lambda \, C(z))) \right]} (4.29)
\]

\[
F_{1,1,0}(z) = \frac{I_0 \, \alpha_1 \left( 1 - \theta_1 \right) \left[ T_1(z) \, (T(z) - 1) + z \, C(z) \right]}{\left[ z^2 - T_1(z) \, T_2(z) \right] \left( p - p \, C(z) \right) \left( k_1 \, (p \lambda - (1 - C(z))) \right)} (4.30)
\]

\[
F_{1,1,1}(z) = \frac{I_0 \, \alpha_1 \, \alpha_1 \, \left[ T_1(z) \, (T(z) - 1) + z \, C(z) \right] \left[ 1 - B_1^* \, (k_1 \, (p \lambda - p \lambda \, C(z))) \right]}{\left[ z^2 - T_1(z) \, T_2(z) \right] \left( p - p \, C(z) \right) \left( k_1 \, (p \lambda - (1 - C(z))) \right)} (4.31)
\]

\[
F_{1,2,0}(z) = \frac{I_0 \, \bar{r}_0 \, \alpha_2 \, \left( 1 - \theta_2 \right) \left[ T_1(z) \, (T(z) - 1) + z \, C(z) \right] B_1^*(k_1 \, (p \lambda - p \lambda \, C(z)))}{\left[ z^2 - T_1(z) \, T_2(z) \right] \left( p - p \, C(z) \right) \left( k_1 \, (p \lambda - (1 - C(z))) \right)} (4.32)
\]

\[
F_{1,2,1}(z) = \frac{I_0 \, \bar{r}_0 \, \alpha_2 \, \alpha_2 \, \left[ T_1(z) \, (T(z) - 1) + z \, C(z) \right] B_1^*(k_1 \, (p \lambda - p \lambda \, C(z)))}{\left[ z^2 - T_1(z) \, T_2(z) \right] \left( p - p \, C(z) \right) \left( k_1 \, (p \lambda - (1 - C(z))) \right)} (4.33)
\]

\[
F_{2,1}(z) = \frac{\lambda I_0 \, \alpha_1 \, \theta_1 \, \left[ T_1(z) \, (T(z) - 1) + z \, C(z) \right] \left[ 1 - B_1^* \, (k_1 \, (p \lambda - p \lambda \, C(z))) \right]}{\left[ z^2 - T_1(z) \, T_2(z) \right] \left( k_1 \, (p \lambda - p \lambda \, C(z)) \right) \left( p - p \, C(z) \right) \left( p - p \lambda \, C(z) + \tau_1 \right)} (4.34)
\]

\[
F_{2,2}(z) = \frac{\lambda I_0 \, \bar{r}_0 \, \alpha_2 \, \alpha_2 \, \left[ T_1(z) \, (T(z) - 1) + z \, C(z) \right] B_1^*(k_1 \, (p \lambda - p \lambda \, C(z)))}{\left[ z^2 - T_1(z) \, T_2(z) \right] \left( k_1 \, (p \lambda - p \lambda \, C(z)) \right) \left( p - p \lambda \, C(z) + \tau_2 \right)} (4.35)
\]

\[
V_j(z) = \frac{I_0}{\left[ V^*(p \lambda) \right]^{j-j_1+1}} \frac{1 - V^* \left( p \lambda \, (1 - C(z)) \right)}{p - p \, C(z)}, \, 1 \leq j \leq J (4.36)
\]
\[ I_0 = \frac{D_1}{D_2} \quad (4.37) \]

where

\[ T_1(z) = \frac{1 - (V^*(p \lambda))^j}{(1 - V^*(p \lambda))(V^*(p \lambda))^j} \left( V^*(p \lambda - p \lambda C(z)) - 1 \right) \]

\[ T_2(z) = z \ A^*(\lambda) + C(z) (1 - q + q z) (1 - A^*(\lambda)) \text{ and} \]

\[ T_3(z) = B_1^*(k_1 (p \lambda - p \lambda C(z))) [r_0 + r_0 \ B_2^*(k_2 (p \lambda - p \lambda C(z)))] \]

\[ D_1 = 2 - [p \lambda m_1 \mu_{11} (1 + \alpha_1 (\gamma_{11} + \frac{\theta_1}{\tau_1})) + r_0 p \lambda m_1 \mu_{21} (1 + \alpha_2 (\gamma_{21} + \frac{\theta_2}{\tau_2})) - [A^*(\lambda) + (1 - A^*(\lambda)) (m_1 + q)] \]

\[ D_2 = [\lambda \mu_{11} (1 + \alpha_1 (\gamma_{11} + \frac{\theta_1}{\tau_1})) + r_0 \lambda \mu_{21} (1 + \alpha_2 (\gamma_{21} + \frac{\theta_2}{\tau_2}))] [(1 - A^*(\lambda)) \bar{q} + N + \bar{p} m_1 A^*(\lambda)] + 1 + (1 - A^*(\lambda)) (N - q) + \frac{N}{p m_1} D_1 \]

and

\[ N = \frac{1 - (V^*(p \lambda))^j}{(1 - V^*(p \lambda))(V^*(p \lambda))^j} \ p \lambda m_1 v_1 \]

**Proof**

Multiplying equation (4.2) by \( z^n \) and summing over \( n \) for \( n = 1, 2, 3, \ldots \) we get

\[ \left( \frac{\partial}{\partial x} + \lambda + \eta(x) \right) I(x, z) = 0 \quad (4.38) \]

Solution of the partial differential equation (4.38) is given by

\[ I(x, z) = I(0, z) e^{-\lambda x} (1 - A(x)) \quad (4.39) \]
Multiplying equations (4.3) to (4.6) by $z^n$, and summing over $n = 0, 1, 2, \ldots$, we get

$$
\begin{align*}
\left[ \frac{d}{dx} + p \lambda - p \lambda C(z) + \alpha_1 + \mu_1(x) \right] P(x, z) &= \int_0^\infty F_{1,1,0}(x, y, z) \beta_1(y) \, dy \\
&\quad + \tau_1 \int_0^\infty F_{2,1}(x, y, z) \, dy \quad (4.40) \\
\left[ \frac{d}{dx} + p \lambda - p \lambda C(z) + \alpha_2 + \mu_2(x) \right] Q(x, z) &= \int_0^\infty F_{1,2,0}(x, y, z) \beta_2(y) \, dy \\
&\quad + \tau_2 \int_0^\infty F_{2,2}(x, y, z) \, dy \quad (4.41)
\end{align*}
$$

Performing similar operations on equations (4.7) to (4.13) and solving the corresponding partial differential equations, we get

$$
\begin{align*}
F_{1,1,0}(x, y, z) &= F_{1,1,0}(x, 0, z) \, e^{-\rho(1-C(z))y} \left( 1 - R_1(y) \right) \quad (4.42) \\
F_{1,1,1}(x, y, z) &= F_{1,1,1}(x, 0, z) \, e^{-\rho(1-C(z))y} \left( 1 - R_1(y) \right) \quad (4.43) \\
F_{1,2,0}(x, y, z) &= F_{1,2,0}(x, 0, z) \, e^{-\rho(1-C(z))y} \left( 1 - R_2(y) \right) \quad (4.44) \\
F_{1,2,1}(x, y, z) &= F_{1,2,1}(x, 0, z) \, e^{-\rho(1-C(z))y} \left( 1 - R_2(y) \right) \quad (4.45) \\
F_{2,1}(x, y, z) &= F_{2,1}(x, 0, z) \, e^{-[\rho(1-C(z))^1 + \tau_1]y} \quad (4.46) \\
F_{2,2}(x, y, z) &= F_{2,2}(x, 0, z) \, e^{-[\rho(1-C(z))^2 + \tau_2]y} \quad (4.47) \\
V_j(x, z) &= V_j(0, z) \, e^{-\rho(1-C(z))^x} \left( 1 - V(x) \right), \quad 1 \leq j \leq J \quad (4.48)
\end{align*}
$$

Multiplying equation (4.14) by $z^n$, summing over all possible values of $n$ and using equations (4.1), (4.24) and (4.25), we get

$$
I(0, z) = \sum_{j=1}^J \int_0^\infty [V_j(x, z) - V_{j,0}(x)] \, v(x) \, dx + r_0 \int_0^\infty [P(x, z) - P_0(x)] \, \mu_1(x) \, dx
$$
\[
+ \int_0^\infty [Q(x, z) - Q_0(x)] \mu_2(x) \, dx
\]

\[
= \sum_{j=1}^J \int_0^\infty V_j(x, z) \nu(x) \, dx + r_0 \int_0^\infty P(x, z) \mu_1(x) \, dx + \int_0^\infty Q(x, z) \mu_2(x) \, dx
\]

\[- \sum_{j=1}^J V_{j0}(0) - \lambda I_0 \quad (4.49)\]

Multiplying equation (4.16) by \(z^n\), summing over \(n\) and adding with equation (4.15), we get

\[
P(0, z) = \frac{\lambda}{z} C(z) I_0 + \frac{1}{z} \int_0^\infty I(x, z) \eta(x) \, dx + \frac{\lambda q}{z} C(z) \int_0^\infty I(x, z) \, dx
\]

\[+ \frac{\lambda q}{z^2} C(z) \int_0^\infty I(x, z) \, dx \quad (4.50)\]

Similarly equations (4.17) to (4.23) yield the following equations

\[
Q(0, z) = r_0 \int_0^\infty P(x, z) \mu_1(x) \, dx \quad (4.51)
\]

\[
F_{1,1,0}(x, 0, z) = \alpha_1 (1 - \theta_1) P(x, z) \quad (4.52)
\]

\[
F_{1,1,1}(x, 0, z) = \alpha_1 \theta_1 P(x, z) \quad (4.53)
\]

\[
F_{1,2,0}(x, 0, z) = \alpha_2 (1 - \theta_2) Q(x, z) \quad (4.54)
\]

\[
F_{1,2,1}(x, 0, z) = \alpha_2 \theta_2 Q(x, z) \quad (4.55)
\]

\[
F_{2,1}(x, 0, z) = \int_0^\infty F_{1,1,1}(x, y, z) \beta_1(y) \, dy \quad (4.56)
\]

\[
F_{2,2}(x, 0, z) = \int_0^\infty F_{1,2,1}(x, y, z) \beta_2(y) \, dy \quad (4.57)
\]

Using equations (4.42) and (4.46) in equation (4.40), we obtain
Using equations (4.44) and (4.47) in equation (4.41), we get

\[
\left[ \frac{d}{dx} + p \lambda - p \lambda C(z) + \alpha_1 + \mu_1(x) \right] P(x, z) = F_{1,1,0}(x, 0, z) R_1^*(p \lambda - p \lambda C(z)) \\
+ \tau_1 F_{2,1}(x, 0, z) / (p \lambda (1 - C(z)) + \tau_1)
\] (4.58)

Substituting the expressions of \( F_{1,1,1}(x, y, z) \) and \( F_{1,2,1}(x, y, z) \) obtained from equations (4.43) and (4.45) in equations (4.56) and (4.57), we have

\[
F_{2,1}(x, 0, z) = F_{1,1,1}(x, 0, z) R_1^*(p \lambda - p \lambda C(z)) \\
F_{2,2}(x, 0, z) = F_{1,2,1}(x, 0, z) R_2^*(p \lambda - p \lambda C(z))
\] (4.59)

At \( n = 0 \), equation (4.13) becomes

\[
\frac{\partial}{\partial x} V_{j,0}(x) = -(p \lambda + v(x)) V_{j,0}(x)
\]

\[
\left[ \frac{\partial}{\partial x} + p \lambda + v(x) \right] V_{j,0}(x) = 0
\] (4.62)

On solving equation (4.62), we get

\[
V_{j,0}(x) = V_{j,0}(0) e^{-p\lambda x} (1 - V(x)), \quad 1 \leq j \leq J
\] (4.63)

Putting \( j = J \) in equation (4.63), multiplying both sides by \( v(x) \) and integrating with respect to \( x \) from 0 to \( \infty \), we get

\[
\int_{0}^{\infty} V_{J,0}(x) v(x) \, dx = V_{J,0}(0) \int_{0}^{\infty} e^{-p\lambda x} (1 - V(x)) \, v(x) \, dx
\]

\[
= V_{J,0}(0) V^*(p \lambda)
\]
From equation (4.1), we have

\[ V_{J,0}(0) = \frac{\lambda I_0}{V^*(p \lambda)} \]  

(4.64)

Putting \( j = J \) in equation (4.25) and using equation (4.63) we obtain

\[ V_{J,0}(0) = \int_0^\infty V_{J-1,0}(x) \nu(x) \, dx \]

\[ = \int_0^\infty V_{J-1,0}(0) e^{-p \lambda x} (1 - V(x)) \, dx \]

\[ = V_{J-1,0}(0) V^*(p \lambda) \]

Now using equation (4.64), we get

\[ V_{J-1,0}(0) = \frac{\lambda I_0}{[V^*(p \lambda)]^2} \]  

(4.65)

Applying similar procedure for \( J - 1, J - 2, \ldots, 1 \) recursively, we get

\[ V_j(0, z) = V_{j,0}(0) = \frac{\lambda I_0}{[V^*(p \lambda)]^{J-j+1}}, \quad j = 1, 2, \ldots, J - 1 \]  

(4.66)

Substituting equation (4.66) in equation (4.48), we obtain

\[ V_j(x, z) = \frac{\lambda I_0}{[V^*(p \lambda)]^{J-j+1}} e^{-p \lambda (1 - C(z))x} (1 - V(x)), \quad 1 \leq j \leq J \]  

(4.67)

Integrating equation (4.67) with respect to \( x \) we get equation (4.36).

Using equations (4.52), (4.53) and (4.60) in equation (4.58) and solving for \( P(x, z) \) we get

\[ P(x, z) = P(0, z) e^{-k_1(p \lambda - p \lambda C(z))x} (1 - B_1(x)) \]  

(4.68)
Solving equation (4.59) by using equations (4.54), (4.55) and (4.61), we obtain

\[ Q(x, z) = Q(0, z) \frac{e^{-k_2(p \lambda - p \lambda \cdot C(z))x}}{1 - B_2(x)} \]  

(4.69)

where

\[ k_i(u) = u + \alpha_i - \frac{\alpha_i R_i^*(u)[u(1 - \theta_i) + \tau_i]}{u + \tau_i}, \quad i = 1, 2 \]

Using equations (4.51) and (4.68), equation (4.69) yields

\[ Q(x, z) = \bar{r}_0 P(0, z) B^*_1(k_1 (p \lambda - p \lambda \cdot C(z))) e^{-k_2(p \lambda - p \lambda \cdot C(z))x} (1 - B_2(x)) \]  

(4.70)

Using equations (4.67), (4.68) and (4.70) in equation (4.49), we get

\[ I(0, z) = \sum_{j=1}^{\infty} \frac{\lambda I_0}{(V^*(p \lambda))^j - \lambda I_0} V^*(p \lambda \cdot C(z)) + r_0 P(0, z) B^*_1(k_1 (p \lambda - p \lambda \cdot C(z))) \]

\[ + \bar{r}_0 P(0, z) B^*_1(k_1 (p \lambda - p \lambda \cdot C(z))) B^*_2(k_2 (p \lambda - p \lambda \cdot C(z))) \]

\[ - \sum_{j=1}^{\infty} V_{j,0}(0) - \lambda I_0 \]  

(4.71)

\[ = P(0, z) [r_0 B^*_1(k_1 (p \lambda - p \lambda \cdot C(z))) + \bar{r}_0 B^*_1(k_1 (p \lambda - p \lambda \cdot C(z))) B^*_2(k_2 (p \lambda - p \lambda \cdot C(z)))] + T_1(z) - \lambda I_0 \]  

(4.72)

Using equation (4.72) in equation (4.39), we get

\[ I(x, z) = [P(0, z) B^*_1(k_1 (p \lambda - p \lambda \cdot C(z))) (r_0 + \bar{r}_0 B^*_2(k_2 (p \lambda - p \lambda \cdot C(z)))) \]

\[ + T_1(z) - \lambda I_0] e^{-\lambda x} (1 - A(x)) \]  

(4.73)

Using equation (4.39) in equation (4.50) and simplifying we get

\[ P(0, z) = \frac{\lambda}{z} C(z) I_0 + \frac{I(0, z)}{z^2} [z A^*(\lambda) + C(z) (1 - q + qz)] \]  

(4.74)

Substituting equation (4.72) in equation (4.74) and simplifying we get
\[ P(0, z) = \frac{\lambda I_0 [T_2(z)(T_1(z) - 1) + zC(z)]}{z^2 - T_2(z)T_3(z)} \]  

(4.75)

Substituting \( P(0, z) \) in the expressions of \( \theta(x, z) \), \( P(x, z) \) and \( Q(x, z) \) which are in equations (4.73), (4.68) and (4.70) and integrating with respect to \( x \) we get equations (4.27) to (4.29).

Using equations (4.68) and (4.70) in equations (4.52) to (4.55), we obtain

\[ F_{1,1,0}(x, 0, z) = P(0, z) \alpha_1 (1 - \theta_1) e^{-k_1(p \beta - p \lambda C(z))x} (1 - B_1(x)) \]  

(4.76)

\[ F_{1,1,1}(x, 0, z) = P(0, z) \alpha_1 \theta_1 e^{-k_1(p \beta - p \lambda C(z))x} (1 - B_1(x)) \]  

(4.77)

\[ F_{1,2,0}(x, 0, z) = \tilde{r}_0 P(0, z) \alpha_2 (1 - \theta_2) B_1^+(k_1 (p \lambda - p \lambda C(z))) e^{-k_2(p \beta - p \lambda C(z))x} (1 - B_2(x)) \]  

(4.78)

\[ F_{1,2,1}(x, 0, z) = \tilde{r}_0 P(0, z) \alpha_2 \theta_2 B_1^+(k_1 (p \lambda - p \lambda C(z))) e^{-k_2(p \beta - p \lambda C(z))x} (1 - B_2(x)) \]  

(4.79)

Substituting equations (4.77) and (4.79) in equations (4.60) and (4.61), we get

\[ F_{2,1}(x, 0, z) = P(0, z) \alpha_1 \theta_1 R_1^+(p \lambda - p \lambda C(z)) e^{-k_1(p \beta - p \lambda C(z))x} (1 - B_1(x)) \]  

(4.80)

\[ F_{2,2}(x, 0, z) = \tilde{r}_0 P(0, z) \alpha_2 \theta_2 B_1^+(k_1 (p \lambda - p \lambda C(z))) R_2^+(p \lambda - p \lambda C(z)) e^{-k_2(p \beta - p \lambda C(z))x} (1 - B_2(x)) \]  

(4.81)

Using equations (4.76) to (4.81), the equations (4.42) to (4.47) respectively yield

\[ F_{1,1,0}(x, y, z) = P(0, z) \alpha_1 (1 - \theta_1) e^{-k_1(p \beta - p \lambda C(z))x} e^{-p \lambda(1 - C(z))x} (1 - B_1(x)) (1 - R_1(y)) \]  

(4.82)

\[ F_{1,1,1}(x, y, z) = P(0, z) \alpha_1 \theta_1 e^{-k_1(p \beta - p \lambda C(z))x} e^{-p \lambda(1 - C(z))x} (1 - B_1(x)) (1 - R_1(y)) \]  

(4.83)

\[ F_{1,2,0}(x, y, z) = P(0, z) \tilde{r}_0 \alpha_2 (1 - \theta_2) B_1^+(k_1 (p \lambda - p \lambda C(z))) e^{-k_2(p \beta - p \lambda C(z))x} \]
\[ e^{-p \lambda (1 - C(z))} x (1 - B_2(x)) (1 - R_2(y)) (4.84) \]

\[ F_{1,2,1}(x, y, z) = P(0, z) \bar{r}_0 \alpha_2 \theta_2 B_1^*(k_1 (p \lambda - p \lambda C(z))) e^{-k_2 (p \lambda - p \lambda C(z))} x (1 - B_2(x)) (1 - R_2(y)) (4.85) \]

\[ F_{2,1}(x, y, z) = P(0, z) \alpha_1 \theta_1 R_1^*(p \lambda - p \lambda C(z)) e^{-k_1 (p \lambda - p \lambda C(z))} x (1 - B_1(x)) (4.86) \]

\[ F_{2,2}(x, y, z) = P(0, z) \bar{r}_0 \alpha_2 \theta_2 B_1^* (k_1 (p \lambda - p \lambda C(z))) R_2^*(p \lambda - p \lambda C(z)) e^{-k_2 (p \lambda - p \lambda C(z))} x e^{-[p \lambda (1 - C(z)) + \tau_1]} y (1 - B_2(x)) (4.87) \]

Substituting the expression of \( P(0, z) \) in \( F_{1,k,i}(x, y, z) \), \( F_{2,k}(x, y, z) \), \( k = 1, 2 \); \( i = 0, 1 \) which are in equations (4.82) to (4.87) and integrating with respect to \( x \) and \( y \) we get the required results in (4.30) to (4.35). Using normalising condition (4.26), \( I_0 \) can be obtained as in equation (4.37).

**Corollary 4.1**

The probability generating function of the number of customers in the orbit is

\[ P_q(z) = \frac{I_0 [(z C(z) - T_2(z)) (1 - (1 - p + p C(z)) T_3(z)) + z A^* (\lambda) p (1 - C(z)) + (z - C(z) T_3(z)) + T_1(z) (T_2(z) - z^2 (1 - p (1 - C(z)) (1 - A^* (\lambda))))]}{(z^2 - T_2(z) T_3(z)) (p - p C(z))} (4.88) \]

**Proof**

The probability generating function of the number of customers in the orbit is

\[ P_q(z) = I_0 + I(z) + P(z) + Q(z) + \sum_{k=1}^{2} \left[ \sum_{l=0}^{1} F_{1,k,l}(z) + F_{2,k}(z) \right] + \sum_{j=1}^{1} V_j(z) \]
By substituting equations (4.27) to (4.37) and simplifying we get equation (4.88).

**Corollary 4.2**

The probability generating function of the number of customers in the system is

\[
\begin{align*}
P_S(z) &= I_0 [(z C(z) - T_2(z))(z - (z - p + p C(z)) T_3(z)) \\
&\quad + z A^*(\lambda) p (1 - C(z))(z - C(z) T_3(z)) \\
&\quad + T_1(z) (T_2(z) (z + T_3(z) (1 - z)) - z^2 (1 - p (1 - C(z)) (1 - A^*(\lambda))))]
\end{align*}
\]

\[\text{(4.89)}\]

**Proof**

The probability generating function of the number of customers in the system is given by

\[
P_S(z) = I_0 + I(z) + z \left[ P(z) + Q(z) + \sum_{k=1}^{2} \left[ \sum_{i=0}^{k-1} F_{1,k,i}(z) + F_{2,k}(z) \right] \right] + \sum_{j=1}^{J} V_j(z)
\]

with the help of equations (4.27) to (4.37) and by direct substitution we get equation (4.89).

### 4.5 Performance Measures

- The server is idle in non-empty system with probability

\[
I = \lim_{z \to 1} I(z) = I_0 (1 - A^*(\lambda)) [m_1 + p \lambda m_1 \mu_{11} (1 + \alpha_1 (\frac{\theta_1}{\tau_1} + \gamma_{11})) + \tilde{r}_0 \ p \lambda m_1 \mu_{21}]
\]
\( (1 + \alpha_2 (\gamma_{21} + \frac{\theta_2}{\tau_2})) + N - 1 \) / \( D_1 \) \hspace{1cm} (4.90)

- The server is busy with probability

\[
S = \lim_{z \to 1} (P(z) + Q(z)) \\
= \lambda I_0 (\mu_{11} + \tilde{r}_0 \mu_{21}) [1 + m_1 + N - (A^*(\lambda) + (1 - A^*(\lambda)) (m_1 + q))] / D_1
\] \hspace{1cm} (4.91)

- The server is under repair with probability

\[
F = \lim_{z \to 1} (F_{1,1}(z) + F_{1,2}(z)) \\
= \lambda I_0 (\alpha_1 \mu_{11} \gamma_{11} + \tilde{r}_0 \alpha_2 \mu_{21} \gamma_{21}) [1 + m_1 + N - (A^*(\lambda) + (1 - A^*(\lambda)) (m_1 + q))] / D_1
\] \hspace{1cm} (4.92)

- The server is on vacation with probability

\[
V = \lim_{z \to 1} \sum_{j=1}^{J} V_j(z) \\
= \frac{1 - (V^*(p \lambda))^j}{(1 - V^*(p \lambda))(V^*(p \lambda))^j} \lambda I_0 v_1
\] \hspace{1cm} (4.93)

- The mean queue length is given by

\[
L_q = \lim_{z \to 1} \frac{d}{dz} P_q(z)
\]

Let \( N_r(z) \) and \( D_r(z) \) be the numerator and denominator of \( P_q(z) \). Since \( N_r(1) = D_r(1) = N_r'(1) = D_r'(1) = 0 \). Using L Hospital rule, we get

\[
L_q = \frac{D_r''(1) N_r'''(1) - N_r''(1) D_r'''(1)}{3D_r''(1)^2}
\] \hspace{1cm} (4.94)

where
\[ Nr''(1) = 2 \left[ (1 - A^*(\lambda)) \left( N (\overline{p} m_1 - \overline{q}) - 2 \overline{p} \overline{q} \ m_1 \right) - (1 + N + p \ m_1 + A^*(\lambda) \ (g_1 + \overline{r}_0 \ g_2)) \right] \]

\[ Nr''(1) = 3 \ (h_3 + h_4 + h_5) \]

\[ Dr''(1) = -2 \ p \ m_1 \ D_1 \]

\[ Dr''(1) = -3 \ p \ (m_1 \ g_3 + m_2 \ D_1) \]

\[ g_i = p \ \lambda \ \mu_i \ (1 + \alpha_i \ (\gamma_{i1} + \frac{\theta_i}{\tau_i})), \ i = 1, 2 \]

\[ g_3 = 2 - (h_1 + 2 \ \overline{r}_0 \ g_1 \ g_2 + \overline{r}_0 \ h_2) - 2(g_1 + \overline{r}_0 \ g_2) \ (A^*(\lambda) + (1 - A^*(\lambda)) \ (q + m_1)) \]

\[ - (1 - A^*(\lambda)) \ (2 \ q \ m_1 + m_2) \]

\[ h_i = \mu_i \ [p \ \lambda \ \mu_2 \ + \alpha_i \ (p \ \lambda \ \mu_2 \ \gamma_{i1} + p^2 \ \lambda^2 \ \mu_1^2 \ \gamma_{i2} + \frac{p \ \lambda \ \mu_2 \ \theta_i}{\tau_i} \]

\[ + \frac{2 \ p^2 \ \lambda^2 \ \mu_1^2 \ \theta_i}{\tau_i} \ (\gamma_{i1} + \frac{1}{\tau_i})] + p^2 \ \lambda^2 \ \mu_1^2 \ \mu_2 \ (1 + \alpha_i \ (\gamma_{i1} + \frac{\theta_i}{\tau_i}))^2, \ i = 1, 2 \]

\[ h_3 = (h_1 + \overline{r}_0 \ (2 \ g_1 \ g_2 + h_2) + 2 \ p \ m_1 \ (g_1 + \overline{r}_0 \ g_2) + p \ m_2) \]

\[ ((1 - A^*(\lambda)) \ (m_1 - \overline{q} - m_1) - (g_1 + \overline{r}_0 \ g_2 + p \ m_1) \]

\[ (m_2 + 2 \ m_1 - (1 - A^*(\lambda)) \ (2 \ q \ m_1 + m_2)) \]

\[ h_4 = \frac{N}{m_1 \ \nu_1} \ (m_2 \ \nu_1 + p \ \lambda \ \mu_2 \ \nu_2) \ (A^*(\lambda) + (1 - A^*(\lambda)) \ (q + \overline{p} \ m_1) - 2) \]

\[ + N \ ((1 - A^*(\lambda)) \ (2 \ q \ m_1 + \overline{p} \ m_2 - 4 \ p \ m_1) - 2) \]

and

\[ h_5 = p \ A^*(\lambda) \ [m_1 \ (h_1 + \overline{r}_0 \ (2 \ g_1 \ g_2 + h_2) + 2 \ m_1 \ (g_1 + \overline{r}_0 \ g_2) + m_2) \]

\[ - (2 \ m_1 + m_2) \ (1 - (g_1 + \overline{r}_0 \ g_2) - m_1)] \]

• The mean number of customers in the system is
\[ L_S = L_q + S + F \]  
(4.95)

### 4.6 Reliability Indices

**Theorem 4.3**

The steady state availability of the server is given by

\[
A = 1 - \left[ \lambda (1 + m_1 + N - (A^*(\lambda)) + (1 - A^*(\lambda)) (m_1 + q)) \right] (\alpha_1 \mu_{11} \gamma_{11} + \beta_0 \alpha_2 \mu_{21} \gamma_{21}) + (N D_1 / (p m_1)) / D_2
\]  
(4.96)

**Proof**

\[
A = I_0 + \lim_{z \to 1} [I(z) + P(z) + Q(z) + F_{2,1}(z) + F_{2,2}(z)]
\]

Substituting the expressions of \( I(z) \), \( P(z) \), \( Q(z) \), \( F_{2,1}(z) \) and \( F_{2,2}(z) \) we obtain the result in (4.96).

**Theorem 4.4**

The steady state failure frequency of the server is

\[
F = \frac{\lambda [1 + m_1 + N - (A^*(\lambda) + (1 - A^*(\lambda)) (m_1 + q))] [\alpha_1 \mu_{11} + \beta_0 \alpha_2 \mu_{21}]}{D_2}
\]  
(4.97)

**Proof**

\[
F = \alpha_1 \lim_{z \to 1} P(z) + \alpha_2 \lim_{z \to 1} Q(z)
\]

Using equations (4.28) and (4.29) and by direct calculation we get equation (4.97).

### 4.7 Special Cases

**Case (i)**
If \( C(z) \rightarrow z \), \( r_0 = 1 \), \( p = 1 \), \( q = 1 \), \( \theta_1 \rightarrow 0 \) (single arrival, no optional service, no balking and reneging and no reserved time) then the model reduces to a retrial queue with server breakdown and modified vacation. In this case

\[
I_0 = \frac{A^*(\lambda) - \lambda \mu_{11}(1 + \alpha \gamma_{11})}{A^*(\lambda) + N}
\]

\[
P_d(z) = \frac{I_0 [T_4(z)(A^*(\lambda) + z(1 - A^*(\lambda))) + A^*(\lambda)(z-1)]}{z - (A^*(\lambda) + z(1 - A^*(\lambda)))B'_1(\lambda - \lambda z + \alpha - \alpha R^*(\lambda - \lambda z))}
\]

where \( N = \frac{1 - (V^*(\lambda))^j}{(1 - V^*(\lambda))(V^*(\lambda))^j} \lambda v_1 \)

and \( T_4(z) = \frac{1 - (V^*(\lambda))^j}{(1 - V^*(\lambda))(V^*(\lambda))^j}(V^*(\lambda - \lambda z) - 1) \)

The above results coincide with the results obtained in Chen et al. (2010).

**Case (ii)**

If \( r_0 = 1 \), \( p = 1 \), \( q = 1 \), \( \alpha = 0 \) (no optional service, no balking and reneging and no breakdown) then the model becomes a batch arrival retrial queue with modified vacation. In this case

\[
I_0 = \frac{1 - m_1(\lambda \mu_{11} + (1 - A^*(\lambda)))}{A^*(\lambda) + N}
\]

\[
P_d(z) = \frac{I_0 (z-1)[T_5(z)(A^*(\lambda) + C(z)(1 - A^*(\lambda))) + A^*(\lambda)(C(z) - 1)]}{[z - (A^*(\lambda) + C(z)(1 - A^*(\lambda)))B'_1(\lambda - \lambda C(z))] (C(z) - 1)}
\]

where \( N = \frac{1 - (V^*(\lambda))^j}{(1 - V^*(\lambda))(V^*(\lambda))^j} \lambda v_1 \)
and \( T_5(z) = \frac{1 - (V^*(\lambda))^j}{(1 - V^*(\lambda))(V^*(\lambda))^j} (V^*(\lambda - \lambda_c(z)) - 1) \)

The above results agree with that of Chang and Ke (2009).

### 4.8 Practical Justification of the Model

An e-mail server is a server program running in a networked environment, capable of exchanging user’s messages in the form of e-mails (essential service). Every e-mail server belongs to a particular network domain and has a unique identifier. Each e-mail server has many clients who seek the mail service of the e-mail server. Each e-mail request (send / receive) forms a customer in the system. It is possible for clients to send / receive e-mails to clients in another domain (a remote domain). To service such requests the e-mail server will have to forward the request to the e-mail server in the other domain (optional service).

Under normal conditions, the server selects a request for service while the remaining are enqueued (orbit). In case the server is busy or the bandwidth is insufficient, then the incoming request and also the enqueued request may be cancelled (balking and reneging).

Servers may undergo hardware or software failure at any time and therefore need to be repaired.

When there are no incoming requests, the server performs one or more maintenance activities (vacation) such as hardware upgradation, software updates, increasing the capacity of the server and modernization. After the maintenance activities are completed the server waits for new messages.

### 4.9 Numerical Results

Numerical examples are presented to study the effect of the parameters on the system characteristics. It is assumed that the retrial time, \( i^{th} \) phase service time, repair time of the server failed during \( i^{th} \) phase service
and vacation time are exponentially distributed with respective parameters $\eta_i$, $\mu_i$, $\beta_i$ and $\nu$ where $i = 1, 2$.

The following arbitrary values are selected for the parameters in such a way that stability condition holds $\lambda = 1$, $\alpha_1 = 0.4$, $\alpha_2 = 0.4$, $r_0 = 0.5$, $r_1 = 0.5$, $p = 0.4$, $q = 0.6$, $J = 5$, $\theta_1 = 0.4$, $\theta_2 = 0.4$, $\tau_1 = 0.5$, $\tau_2 = 0.5$, $C_1 = 0.5$, $C_2 = 0.5$, $\mu_1 = 5$, $\mu_2 = 3$, $\beta_1 = 3$, $\beta_2 = 3$, $\nu = 5$ and $\eta = 3$.

Table 4.1 shows the dependence of the availability ($A$) and failure frequency ($F$) on the joining probability $p$, failure rate $\alpha_1$ and retrial rate $\eta$. It is noted that availability increases as $p$ increases and decreases as $\eta$ and $\alpha_1$ increase. Failure frequency increases with increase in $p$ and $\alpha_1$ and decreases with increase in $\eta$.

Effect of the parameters on the performance measures $I_0$ - the probability that the server is idle in empty system, $I$ - the probability that the server is idle in non-empty system, $S$ - the probability that the server is busy, $F$ - the probability that the server is under repair, $V$ - the probability that the server is on vacation, $L_S$ - the mean system size are displayed in Fig. 4.2 to 4.7.

From Fig. 4.2, we observe the effect of $\lambda$, $\mu_1$, $\eta$, $\alpha_1$, $\beta_1$ and $\nu$ on $I_0$. $I_0$ decreases for increasing $\lambda$ and $\alpha_1$ and increases for increasing values of $\mu_1$, $\eta$, $\beta_1$ and $\nu$.

In Fig. 4.3, the variation of $I$ with respect to the parameters $\lambda$, $\mu_1$, $\eta$, $\alpha_1$, $\beta_1$ and $\nu$ is displayed. $I$ increases with increase in $\lambda$ and $\alpha_1$ and decreases for increase in $\mu_1$, $\eta$ and $\nu$. The influence of $\beta_1$ on $I$ is negligible.

Fig. 4.4 indicates the variation of $S$ with respect to the parameters $\lambda$, $\mu_1$, $\eta$, $\alpha_1$, $\beta_1$ and $\nu$. The probability that the server is busy increases with increase in $\lambda$, $\beta_1$ and $\nu$ and decreases with increase in $\mu_1$, $\eta$ and $\alpha_1$. 
The values of $F$ by varying the parameters $\lambda$, $\mu_1$, $\eta$, $\alpha_1$, $\beta_1$ and $\nu$ are displayed as surface diagram in Fig. 4.5. The surfaces show a downward trend for $F$ against the parameters $\mu_1$, $\eta$ and $\beta_1$ and an upward trend for the parameters $\lambda$, $\alpha_1$ and $\nu$.

From Fig. 4.6 it is clear that $V$ decreases for increasing values of $\lambda$, $\alpha_1$ and $\nu$ and increases for increase in $\mu_1$, $\eta$ and $\beta_1$.

Fig. 4.7 illustrates the dependence of $L_s$ on the parameters $\lambda$, $\mu_1$, $\eta$, $\alpha_1$, $\beta_1$ and $\nu$. $L_s$ increases with increase in $\lambda$ and $\alpha_1$ and decreases with increase in $\mu_1$, $\eta$, $\beta_1$ and $\nu$.

**Table 4.1 Reliability Indices for varying Values of $p$, $\alpha_1$ and $\eta$**

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Fig. 4.2 Effect of various Parameters on $I_0$
(a) Effect of $(\mu_1, \lambda)$ on $I$

(b) Effect of $(\eta, \alpha_1)$ on $I$

(c) Effect of $(\beta_1, \nu)$ on $I$

Fig. 4.3 Effect of various Parameters on $I$
(a) Effect of $(\mu_1, \lambda)$ on $S$

(b) Effect of $(\eta, \alpha_1)$ on $S$

(c) Effect of $(\beta_1, \nu)$ on $S$

Fig. 4.4 Effect of various Parameters on $S$
Fig. 4.5 Effect of various Parameters on F

(a) Effect of ($\mu_1$, $\lambda$) on F

(b) Effect of ($\eta$, $\alpha_1$) on F

(c) Effect of ($\beta_1$, $\nu$) on F
(a) Effect of $\mu_1, \lambda$ on $V$

(b) Effect of $\eta, \alpha_1$ on $V$

(c) Effect of $\beta_1, v$ on $V$

Fig. 4.6 Effect of various Parameters on $V$
Fig. 4.7 Effect of various Parameters on $L_S$