Chapter 5

SCHLÄFLI–TYPE MODULAR EQUATIONS

5.1 Introduction


Recently, Mahadeva Naika and Bairy [60] have established several new modular equations in the classical theory for degrees 2, 4, 9, 11, 13, 15, 17, 19, 23, 25, 29, 31, 47 and 71. As an application, they have also obtained several explicit evaluations of Ramanujan-Weber class invariants.

\footnote{Reference [51] and [69] are based on this chapter}
In one of the most celebrated paper [96] Ramanujan offers several new series representation for $1/\pi$ and remarks “There are corresponding theories in which $q$ is replaced by one another of the functions
\[ q_1 = e^{-\pi\sqrt{2}K_1'/K_1}, \quad q_2 = e^{-2\pi K_2'/K_2\sqrt{3}}, \]
\[ q_3 = e^{-2\pi K_3'/K_3} \]
where
\[ K_1 = \, _2F_1 \left( \frac{1}{4}, \frac{3}{4}; 1; \kappa^2 \right), \quad K_2 = \, _2F_1 \left( \frac{1}{3}, \frac{2}{3}; 1; \kappa^2 \right), \quad K_3 = \, _2F_1 \left( \frac{1}{5}, \frac{4}{5}; 1; \kappa^2 \right). ” \]
For $0 < \kappa < 1$, $K_j' = K_j(\kappa')$, where $1 \leq j \leq 3$ and $\kappa' = \sqrt{1 - \kappa^2}$.

Unfortunately, Ramanujan did not developed in detail the theory. Ramanujan gives an outline of theories of elliptic functions to alternate bases corresponding to the classical theory by way of statements of some results on pages 257-262 of his second notebook [97]. On page 259 of his second notebook [97], Ramanujan has recorded several modular equations of degree 2 in the theory of signature 3. For example,
\[ \sqrt[3]{\alpha \beta} + \sqrt[3]{(1 - \alpha)(1 - \beta)} = 1 \quad (5.1.1) \]
and
\[ \sqrt[3]{\frac{\alpha^2}{\beta}} + \sqrt[3]{\frac{(1 - \alpha)^2}{(1 - \beta)}} = \frac{2}{m}. \quad (5.1.2) \]
On page 261 of his second notebook [97]. Ramanujan recorded several modular equations in the theory of signature 4. K. Venkatachaliengar [106], took initial steps and examined some of these results. Most of the claims made by Ramanujan on corresponding theories or alternative theories were proved by Berndt, Bhargava and F. G. Garvan [21]. Berndt, Chan and W. C. Liaw [22] have derived quartic
versions of duplication and dimidiation formulas and also proved many theorems of Ramanujan.

Mahadeva Naika [58] has established several cubic modular equations of degree 2 akin to Ramanujan and as an application he established some new $P$–$Q$ eta-function identities. Adiga, Kim and Mahadeva Naika [1] have also established several cubic modular equations and some $P$–$Q$ eta-function identities. Mahadeva Naika [57] has established some Russell-type cubic mixed modular equations and some $P$–$Q$ eta-function identities. Bhargava, Adiga and Mahadeva Naika [30] have established several new mixed modular equations in the theory of signature 4.

In this Chapter, we established several new Schl"afli–type modular equations in the theory of signature 3 and 4. We introduced cubic class invariants $h_n$ and $H_n$, and explicitly evaluate these class invariants. We also explicitly evaluate cubic singular modulus. We introduced quartic class invariant $l_n$ and $L_n$, and explicitly evaluate these quartic class invariants. We also explicitly evaluate quartic singular modulus.

We shall define a modular equation for the alternative theories.

Let

\[
Z(r) := Z(r; x) := \, _2F_1 \left( \frac{1}{r}, \frac{r-1}{r}; 1; x \right) \tag{5.1.3}
\]

and

\[
g_r := g_r (x) := \exp \left( -\pi \csc \left( \frac{\pi}{r} \right) \frac{2F_1 \left( \frac{1}{r}, \frac{r-1}{r}; 1; 1-x \right)}{2F_1 \left( \frac{1}{r}, \frac{r-1}{r}; 1; x \right)} \right), \tag{5.1.4}
\]

where $r = 2, 3, 4, 6$ and $0 < x < 1$. 
Let $n$ denote a fixed natural number, and assume that

\[
\frac{\binom{1}{r-1}{\frac{1}{r}}}{\binom{1}{r}{\frac{1}{r-1}}; 1; 1 - \alpha} = \frac{\binom{1}{r-1}{\frac{1}{r}}}{\binom{1}{r}{\frac{1}{r-1}}; 1; 1 - \beta}, \quad (5.1.5)
\]

where $r = 2, 3, 4$ or $6$. Then a modular equation of degree $n$ in the theory of elliptic functions of signature $r$ is a relation between $\alpha$ and $\beta$ induced by (5.1.5).

\[
m(r) = \frac{Z(r; \alpha)}{Z(r; \beta)}
\]

is called the multiplier. We also use notations $Z_1 = Z(r; \alpha)$ and $Z_n = Z(r; \beta)$ to indicate that $\beta$ has degree $n$ over $\alpha$. When the context is clear, we omit the argument $r$ in $q_r$, $Z(r)$ and $m(r)$.

### 5.2 Schl"{a}fli–type modular equations in cubic theory

In this section, we established some new Schl"{a}fli–type modular equations in cubic theory (or signature 3). We introduced cubic class invariants $h_n$ and $H_n$ and established modular relations between $h_n$ with $h_{\rho_n}$ and $H_n$ with $H_{\rho_n}$ for $j = 2, 5, 7, 8$ and $11$. We also established several new explicit evaluations of the cubic class invariants and cubic singular modulus $\alpha$.

Throughout this Section for brevity we set $q := q_3$, $z := z(3)$, $m := m(3)$ and $\alpha$ is cubic singular modulus.
5.2.1 Main theorems

In this section, we established several Theorems which are useful to evaluate the values of cubic class invariants and cubic singular modulus.

Theorem 5.2.1. If $\beta$ has degree 2 over $\alpha$ in the theory of signature 3, then

$$\alpha(1 - \beta) + \beta(1 - \alpha) = 3\{\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/3}. \quad (5.2.1)$$

Proof. From [20, Ch. 33, Theorem 6.1, p. 116] if $\alpha = \frac{p(3 + p)^2}{2(1 + p)^3}$ and $\beta = \frac{p^2(3 + p)}{4}$, then, for $0 \leq p < 1$, we have

$$\quad {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \alpha\right) = (1 + p) \quad {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \beta\right). \quad (5.2.2)$$

From the equation (5.2.2), we find

$$1 - \alpha = \frac{(1 - p)^2(2 + p)}{2(1 + p)^3}, \quad (5.2.3)$$

$$1 - \beta = \frac{(1 - p)(2 + p)^2}{4}. \quad (5.2.4)$$

From the equations (5.2.2), (5.2.3) and (5.2.4), we deduce

$$\alpha(1 - \beta) + \beta(1 - \alpha) = \frac{p(1 - p)(3 + p)(2 + p)}{8(1 + p)^3}[(3 + p)(2 + p) + p(1 - p)]$$

$$= 3 \frac{p(1 - p)(3 + p)(2 + p)}{4(1 + p)^2}$$

$$= 3\{\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/3}. \quad (5.2.5)$$
Set
\[ M := \left\{ \alpha \beta \right\}^{1/12} \]  
and
\[ N := \left\{ \frac{\beta}{\alpha} \right\}^{1/12}. \]

where \( \beta \) is of degree \( n \) over \( \alpha \) in the theory of signature 3.

**Theorem 5.2.2.** If \( M \) and \( N \) are defined as in the equations (5.2.6) and (5.2.7), then
\[ N^6 + \frac{1}{N^6} = 3 \left[ \frac{1}{M^2} - M^2 \right] + 2M^6. \]  
where \( \beta \) is of degree 2 over \( \alpha \) in the theory of signature 3.

**Proof.** From the equations (5.2.6) and (5.2.7), we find
\[ MN = \beta^{1/6} \quad \text{and} \quad \frac{M}{N} = \alpha^{1/6}. \]

From [20, Ch. 33, Theorem 7.1(i), p. 120] if \( \beta \) is of degree 2 over \( \alpha \) in the theory of signature 3, then
\[ \left\{ \alpha \beta \right\}^{1/3} + \left\{ (1 - \alpha)(1 - \beta) \right\}^{1/3} = 1. \]

Employing the equation (5.2.9) in the equation (5.2.10), we find
\[ M^2 + N^{12}M^2 - 2M^6N^8 - 3N^6 + 3N^6M^4 = 0. \]

On simplification, we obtain the equation (5.2.8).
Theorem 5.2.3. If $M$ and $N$ are defined as in the equations (5.2.6) and (5.2.7), then
\[
\frac{1}{M^2} \left[ N^{12} + \frac{1}{N^{12}} \right] + 5 \left[ N^6 + \frac{1}{N^6} \right] \left[ 66 + 64M^4 + \frac{15}{M^4} \right] = 320M^{10} + \frac{81}{M^{10}} + 21 \left[ 16M^6 + \frac{15}{M^6} \right] + 20 \left[ 3M^2 + \frac{17}{M^2} \right], \tag{5.2.12}
\]
where $\beta$ is of degree 5 over $\alpha$ in the theory of signature 3.

Proof. From [20, Ch. 33, Theorem 7.6, p. 124] if $\beta$ is of degree 5 over $\alpha$ in the theory of signature 3, we have
\[
\{\alpha \beta \}^{1/3} + \{(1 - \alpha)(1 - \beta)\}^{1/3} + 3\{\alpha \beta (1 - \alpha)(1 - \beta)\}^{1/6} = 1. \tag{5.2.13}
\]
Employing the equation (5.2.6) in the above equation (5.2.13), we find
\[
9M^4a = (1 - M^4 - a)^2, \tag{5.2.14}
\]
where $a := \{(1 - \alpha)(1 - \beta)\}^{1/3}$.
Solving for $a$ and then cubing both sides of the equation (5.2.14) and then using (5.2.9), we obtain the equation (5.2.12). \qed

5.2.2 Schl"afli–type modular equations

In this section, we established several new Schl"afli-type modular equations in cubic theory (or signature 3) by using the known Ramanujan’s modular equations. Set
\[
P := \{\alpha \beta (1 - \alpha)(1 - \beta)\}^{1/12} \tag{5.2.15}
\]
and
\[ Q := \left\{ \frac{\beta (1 - \beta)}{\alpha (1 - \alpha)} \right\}^{1/12}, \] (5.2.16)

where \( \beta \) is of degree \( n \) over \( \alpha \) in the theory of signature 3.

**Theorem 5.2.4.** If \( P \) and \( Q \) are defined as in the equations (5.2.15) and (5.2.16), then
\[ Q^6 + \frac{1}{Q^6} = 3 \left[ \frac{1}{P^2} - 3P^2 \right] + 4P^6, \] (5.2.17)

where \( \beta \) is of degree 2 over \( \alpha \) in the theory of signature 3.

**Proof.** From the equations (5.2.15) and (5.2.16), we find
\[ PQ = \{\beta (1 - \beta)\}^{1/6} \] (5.2.18)

and
\[ \frac{P}{Q} = \{\alpha (1 - \alpha)\}^{1/6}. \] (5.2.19)

From the equations (5.2.18) and (5.2.19), we deduce
\[ \alpha = \frac{1+a}{2}, \text{ where } a := \pm \sqrt{1 - \frac{4P^6}{Q^6}} \] (5.2.20)

and
\[ \beta = \frac{1+b}{2}, \text{ where } b := \pm \sqrt{1 - 4P^6 Q^6}. \] (5.2.21)

Employing the equation (5.2.15) in the equation (5.2.10), we find
\[ \{\alpha \beta\}^{1/3} + \frac{P^4}{\{\alpha \beta\}^{1/3}} = 1. \] (5.2.22)
Cubing both sides of the equation (5.2.22), we find
\[ \alpha \beta + \frac{p^{12}}{\alpha \beta} = (1 - 3p^4). \] (5.2.23)

Employing the equations (5.2.20) and (5.2.21) in the equation (5.2.23), we find
\[ 2p^2 q^{12} + 2a p^2 q^{12} + 2p^2 + 2bp^2 - 8p^8 q^6 - 3q^6 - 3bq^6 - 3aq^6 - 3abq^6 = 0. \] (5.2.24)

Isolating the terms involving \( a \) on one side of the equation (5.2.24) and squaring both sides, we deduce
\[ 3q^6 - p^2 - 9p^4 q^6 - p^2 q^{12} + 4p^8 q^6 = 0. \] (5.2.25)

On simplification, we obtain the equation (5.2.17).

\[ \square \]

**Theorem 5.2.5.** If \( P \) and \( Q \) are defined as in the equations (5.2.15) and (5.2.16), then
\[
\frac{1}{p^2} \left[ q^{12} + \frac{1}{q^{12}} \right] = \left[ \frac{9}{p^{10}} - 320p^{10} \right] + 3 \left[ \frac{288p^6 - 35}{p^6} \right] - 450 \left[ \frac{2p^2 - 1}{p^2} \right],
\] (5.2.26)

where \( \beta \) is of degree 5 over \( \alpha \) in the theory of signature 3.

**Proof.** Employing the equation (5.2.15) in the equation (5.2.13), we deduce
\[ \left( \frac{\alpha \beta}{p^4} \right)^{1/3} = 1 - 3p^2. \] (5.2.27)
Cubing both sides of the equation (5.2.27), we find

\[ \alpha \beta + \frac{P^{12}}{\alpha \beta} = (1 - 3P^2) \left( 6P^4 - 6P^2 + 1 \right). \]  

(5.2.28)

Employing the equations (5.2.20) and (5.2.21) in the equation (5.2.28), we deduce

\[ \left( \frac{(1 + a)(1 + b)}{4} \right)^2 + P^{12} = (1 - 3P^2) \left( 6P^4 - 6P^2 + 1 \right) \left( \frac{(1 + a)(1 + b)}{4} \right). \]  

(5.2.29)

Eliminating \( a \) and \( b \) from the equation (5.2.29), we obtain the equation (5.2.26).

\( \square \)

**Theorem 5.2.6.** If \( P \) and \( Q \) are defined as in the equations (5.2.15) and (5.2.16), then

\[
\frac{1}{P^2} \left[ Q^{24} + \frac{1}{Q^{24}} \right] + \left[ Q^{12} + \frac{1}{Q^{12}} \right] \left\{ 17123392P^{10} + 16489440P^6 \\
+ 569016P^2 + \frac{838332}{P^2} + \frac{48081}{P^6} + \frac{693}{P^{10}} \right\} = \left[ 2883584P^{22} + \frac{243}{P^{22}} \right] \\
+ 216 \left[ 143360P^{18} - \frac{33}{P^{18}} \right] + 396 \left[ 227584P^{14} + \frac{93}{P^{14}} \right] \\
- 198 \left[ 33376P^{10} + \frac{1451}{P^{10}} \right] + 66 \left[ 32128P^{6} - \frac{7065}{P^{6}} \right] \\
- 22 \left[ 858636P^2 + \frac{357377}{P^2} \right].
\]  

(5.2.30)

where \( \beta \) is of degree 11 over \( \alpha \) in the theory of signature 3.

The proof of the equation (5.2.30) is similar to the proof of the equation (5.2.26) except that in place of the equation (5.2.13); we use the result from [20, Ch. 33,
Theorem 7.8, p. 126]

\[
\{\alpha\beta\}^{1/3} + \{(1 - \alpha)(1 - \beta)\}^{1/3} + 6\{\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/6} \\
+ 3\sqrt{3}\{\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/12}\{\alpha\beta\}^{1/6} + \{(1 - \alpha)(1 - \beta)\}^{1/6}\] = 1,
\]

(5.2.31)

where \(\beta\) is of degree 11 over \(\alpha\) in the theory of signature 3.

### 5.2.3 Cubic class invariants \(H_n\) and \(h_n\)

In this section, following Ramanujan we introduced the cubic class invariants \(h_n\) and \(H_n\) and established some properties of these class invariants. We established several modular relations between the cubic class invariants \(H_n\) and \(h_n\). We also established several explicit evaluations of \(H_n\) and \(h_n\) as well. Further, at the end of this Section 5.2.3 we have tabulated some explicit evaluations of cubic singular modulus.

Following Ramanujan, we define cubic class invariants \(h_n\) and \(H_n\) as:

\[
h_n = \{\alpha(1 - \alpha)^{-1}\}^{-1/12},
\]

(5.2.32)

and

\[
H_n = \{\alpha(1 - \alpha)^{-1}\}^{-1/12},
\]

(5.2.33)

where \(q_3 = e^{-2\pi\sqrt{n/3}}\), \(n\) a positive rational number and \(\alpha\) is cubic singular modulus.

**Theorem 5.2.7.** We have

\[
h_nh_{1/n} = 1.
\]

(5.2.34)
Proof. From the equation (5.2.32), we obtain (5.2.34).

**Theorem 5.2.8.** We have

\[ H_n^6 = h_n^6 + \frac{1}{h_n^6}, \]  

(5.2.35)

Proof. From the equations (5.2.32) and (5.2.33), we obtain (5.2.35).

**Theorem 5.2.9.** We have

\[ H_n = H_{1/n}. \]  

(5.2.36)

Proof. From the equation (5.2.33), we obtain (5.2.36).

**Theorem 5.2.10.** If \( X := h_n^2h_{4n}^2 \) and \( Y := \frac{h_n^6}{h_{4n}^6} \), then

\[ Y + \frac{1}{Y} = 3 \left[ X + \frac{1}{X} \right], \]  

(5.2.37)

where \( \beta \) is of degree 2 over \( \alpha \) in the theory of signature 3.

Proof. Employing the equation (5.2.2) in the equation (5.2.32), we find

\[ Y + \frac{1}{Y} = \left\{ \frac{p(1-p)}{(3+p)(2+p)} \right\}^{1/2} + \left\{ \frac{p(1-p)}{(3+p)(2+p)} \right\}^{-1/2} \]

\[ = 3 \left( \frac{2(1+p)}{\sqrt{p(1-p)(3+p)(2+p)}} \right). \]  

(5.2.38)

Employing the equation (5.2.2) in the equation (5.2.32), we find

\[ X + \frac{1}{X} = \left\{ \frac{p(1-p)}{(3+p)(2+p)} \right\}^{1/2} + \left\{ \frac{p(1-p)}{(3+p)(2+p)} \right\}^{-1/2} \]

\[ = \left( \frac{2(1+p)}{\sqrt{p(1-p)(3+p)(2+p)}} \right). \]  

(5.2.39)

From the equations (5.2.38) and (5.2.39), we obtain the required result.
Corollary 5.2.1. We have
\[
h_4 = \left( \frac{\sqrt{3} + 1}{\sqrt{2}} \right)^{1/2}
\]  
\text{(5.2.40)}

and
\[
h_{1/4} = \left( \frac{\sqrt{3} - 1}{\sqrt{2}} \right)^{1/2}.
\]  
\text{(5.2.41)}

Proof. Putting \( n = 1 \) in the equation (5.2.37) and then using the fact \( h_1 = 1 \), we find
\[
\left( h_4^2 + \frac{1}{h_4^2} \right) \left( h_4^4 + \frac{1}{h_4^4} - 1 \right) = 3 \left( h_4^2 + \frac{1}{h_4^2} \right).
\]  
\text{(5.2.42)}

Since \( h_4^2 + \frac{1}{h_4^2} \neq 0 \), we find
\[
h_2^8 - 4h_4^4 + 1 = 0.
\]  
\text{(5.2.43)}

Solving the equation (5.2.43), we obtain the required results.

Corollary 5.2.2. We have
\[
h_2 = \left( \frac{\sqrt{2} + 1}{\sqrt{2}} \right)^{1/6}
\]  
\text{(5.2.44)}

and
\[
h_{1/2} = \left( \sqrt{2} - 1 \right)^{1/6}.
\]  
\text{(5.2.45)}

Proof. Putting \( n = 1/2 \) in the equation (5.2.37), we find
\[
X = 1.
\]  
\text{(5.2.46)}

Employing the equation (5.2.46) in the equation (5.2.37), we find
\[
Y + \frac{1}{Y} = 6.
\]  
\text{(5.2.47)}
Solving the equation (5.2.47), we obtain (5.2.44) and (5.2.45).

**Theorem 5.2.11.** If \( X := h_nh_{25n} \) and \( Y := \frac{h_n}{h_{25n}} \), then

\[
3 \left( X^2 + \frac{1}{X^2} \right) + 5 = \frac{1}{Y^3} - Y^3, \tag{5.2.48}
\]

where \( \beta \) is of degree 5 over \( \alpha \) in the theory of signature 3.

**Proof.** Dividing the equation (5.2.13) by \( \{\alpha \beta\}^{1/3} \), we find

\[
1 + \left\{ \frac{(1 - \alpha)(1 - \beta)}{\alpha \beta} \right\}^{1/3} + 3 \left\{ \frac{(1 - \alpha)(1 - \beta)}{\alpha \beta} \right\}^{1/6} = \frac{1}{\{\alpha \beta\}^{1/3}}. \tag{5.2.49}
\]

Employing the equation (5.2.32) in the equation (5.2.49), we find

\[
1 + X^4 + 3X^2 = \frac{1}{\{\alpha \beta\}^{1/3}}, \tag{5.2.50}
\]

where

\[
\frac{1}{\alpha} = 1 + X^6Y^6 \tag{5.2.51}
\]

and

\[
\frac{1}{\beta} = 1 + \frac{X^6}{Y^6}. \tag{5.2.52}
\]

Cubing both sides of the equation (5.2.50), we find

\[
(1 + X^4 + 3X^2)^3 = \frac{1}{\alpha \beta}. \tag{5.2.53}
\]
Employing the equations (5.2.51) and (5.2.52) in the equation (5.2.53), we find
\[
\left(3X^4Y^3 + X^2 + 5X^2Y^3 + 3Y^3 - Y^6X^2\right) \\
\times \left(3X^4Y^3 - X^2 + 5X^2Y^3 + 3Y^3 + Y^6X^2\right) = 0. \tag{5.2.54}
\]

The first factor does not vanish in the neighbourhood of \( q = e^{-\pi} \). But the second factor vanishes in the neighbourhood of \( q = e^{-\pi} \). By Identity Theorem it vanishes identically. Hence, we obtain the equation (5.2.48).

**Corollary 5.2.3.** We have
\[
h_5 = \left(\frac{11 + 5\sqrt{5}}{2}\right)^{1/6} \tag{5.2.55}
\]

and
\[
h_{1/5} = \left(\frac{-11 + 5\sqrt{5}}{2}\right)^{1/6}. \tag{5.2.56}
\]

**Proof.** Putting \( n = 1/5 \) in the equation (5.2.48), we find
\[
X = 1. \tag{5.2.57}
\]

Employing the equation (5.2.57) in the equation (5.2.48), we find
\[
\frac{1}{Y} - Y = 11. \tag{5.2.58}
\]

Solving the equation (5.2.58), we obtain (5.2.55) and (5.2.56).
Corollary 5.2.4. We have

$$h_{25} = \frac{(\sqrt[3]{10} + 1) + \sqrt{(\sqrt[3]{10} + 1)^2 + 4}}{2}$$  \hspace{1cm} (5.2.59)$$

and

$$h_{1/25} = \frac{-(\sqrt[3]{10} + 1) + \sqrt{(\sqrt[3]{10} + 1)^2 + 4}}{2}. \hspace{1cm} (5.2.60)$$

Proof. Putting $n = 1$ in the equation (5.2.48), we find

$$3 \left( h_{25}^2 + \frac{1}{h_{25}^2} \right) + 5 = h_{25}^3 - \frac{1}{h_{25}^3}. \hspace{1cm} (5.2.61)$$

Putting $a := h_{25} - \frac{1}{h_{25}}$, we deduce

$$a^3 - 3a^2 + 3a - 11 = 0. \hspace{1cm} (5.2.62)$$

Solving the equation (5.2.62), we obtain (5.2.59).

Since the proof of the equation (5.2.60) is similar to the proof of the equation (5.2.59) on putting $n = 1/25$ in the equation (5.2.48). We omit the details.

Theorem 5.2.12. If $X := h_n h_{64n}$ and $Y := \frac{h_n}{h_{64n}}$, then

$$Y^{24} + \frac{1}{Y^{24}} \left[ Y^{12} + \frac{1}{Y^{12}} \right] \left\{ 2592 \left[ X^8 + \frac{1}{X^8} \right] - 26136 \left[ X^4 + \frac{1}{X^4} \right] 
+ 55535 \right\} - Y^6 + \frac{1}{Y^6} \left\{ 2187 \left[ X^{14} + \frac{1}{X^{14}} \right] - 15552 \left[ X^{10} + \frac{1}{X^{10}} \right] \right.$$
\[ +24408 \left( X^6 + \frac{1}{X^6} \right) - 1656 \left( X^2 + \frac{1}{X^2} \right) \right \} = 6561 \left( X^{16} + \frac{1}{X^{16}} \right) \\
-75087 \left( X^{12} + \frac{1}{X^{12}} \right) + 402408 \left( X^8 + \frac{1}{X^8} \right) - 1035288 \left( X^4 + \frac{1}{X^4} \right) \] (5.2.63)
+ 1438824,

where \( \beta \) is of degree 8 over \( \alpha \) in the theory of signature 3.

Proof. From [20, Ch. 33, Theorem 7.11, p. 132], we have

\[ D^4 - DF (5D + 9E) - 2F^2 = 0, \] (5.2.64)

where

\[ D := 1 - \{ \alpha \beta \}^{1/3} - \{ (1 - \alpha) (1 - \beta) \}^{1/3}, \]

\[ E := \{ \alpha \beta \}^{1/3} + \{ (1 - \alpha) (1 - \beta) \}^{1/3}, \]

\[ F := 9 \{ \alpha \beta (1 - \alpha) (1 - \beta) \}^{1/3}, \]

and \( \beta \) is of degree 8 over \( \alpha \) in the theory of signature 3. Since the proof the equation (5.2.63) is similar to that of the equation (5.2.48), except that in place of the equation (5.2.13) the equation (5.2.64) is used. We omit the details. \( \Box \)

Theorem 5.2.13. If \( X := \frac{h_n}{h_{121n}} \) and \( Y := \frac{h_n}{h_{121n}} \), then

\[ Y^6 + \frac{1}{Y^6} = 9\sqrt{3} \left( X^3 + \frac{1}{X^3} \right) + 99 \left( X^4 + \frac{1}{X^4} \right) + 198\sqrt{3} \left( X^3 + \frac{1}{X^3} \right) \]
\[ +759 \left( X^2 + \frac{1}{X^2} \right) + 693\sqrt{3} \left( X + \frac{1}{X} \right) + 1386. \] (5.2.65)

where \( \beta \) is of degree 11 over \( \alpha \) in the theory of signature 3.
Proof. Dividing the equation (5.2.31) by \(\{\alpha\beta\}^{1/3}\), we find

\[
1 + \left\{ \frac{(1 - \alpha) (1 - \beta)}{\alpha \beta} \right\}^{1/3} + 6 \left\{ \frac{(1 - \alpha) (1 - \beta)}{\alpha \beta} \right\}^{1/6} + 3\sqrt{3} \left\{ \frac{(1 - \alpha) (1 - \beta)}{\alpha \beta} \right\}^{1/12} \left( 1 + \left\{ \frac{(1 - \alpha) (1 - \beta)}{\alpha \beta} \right\}^{1/6} \right) = \frac{1}{\{\alpha\beta\}^{1/3}}. \tag{5.2.66}
\]

Employing the equation (5.2.32) in the equation (5.2.66), we find

\[
1 + X^4 + 6X^2 + 3\sqrt{3}X(1 + X^2) = \frac{1}{\{\alpha\beta\}^{1/3}}. \tag{5.2.67}
\]

Cubing both sides of the equation (5.2.67) and then using the equations (5.2.51) and (5.2.52), we deduce

\[
\left[ 1 + X^4 + 6X^2 + 3\sqrt{3}X(1 + X^2) \right]^3 = \left( 1 + X^6y^6 \right) \left( 1 + \frac{X^6}{y^6} \right). \tag{5.2.68}
\]

On simplification, we obtain the required result. \(\square\)

**Corollary 5.2.5.** We have

\[
h_{11} = \left( 10 + 3\sqrt{11} \right)^{1/12} \left( 90\sqrt{3} + 47\sqrt{11} \right)^{1/12} \tag{5.2.69}
\]

and

\[
h_{1/11} = \left( 10 - 3\sqrt{11} \right)^{1/12} \left( 90\sqrt{3} - 47\sqrt{11} \right)^{1/12}, \tag{5.2.70}
\]

Proof. Putting \(n = 1/11\) in the equation (5.2.65), we find

\[
X = 1. \tag{5.2.71}
\]
Employing the equation (5.2.71) in the equation (5.2.65), we find

\[ Y^6 + \frac{1}{Y^6} = 1800\sqrt{3} + 3102. \quad (5.2.72) \]

Solving the equation (5.2.72), we obtain (5.2.69) and (5.2.70).

**Theorem 5.2.14.** If \( X := h_n h_{49n} \) and \( Y := \frac{h_n}{h_{49n}} \), then

\[ 3\sqrt{3} \left( X^3 + \frac{1}{X^3} \right) = \left( Y^4 - \frac{1}{Y^4} \right) - 7 \left( Y^2 - \frac{1}{Y^2} \right), \quad (5.2.73) \]

where \( B \) is of degree 7 over \( A \) in the theory of signature 3.

**Theorem 5.2.15.** If \( X := h_n h_{9n} \) and \( Y := \frac{h_n}{h_{9n}} \), then

\[ 3\sqrt{3} \left[ X^3 + \frac{1}{X^3} \right] + 9 = \frac{1}{Y^6}. \quad (5.2.74) \]

where \( B \) is of degree 3 over \( A \) in the theory of signature 3.

We can prove the above Theorems 5.2.14 and 5.2.15 by using Ramanujan’s modular equations of degree 7 and degree 3 respectively.

**Corollary 5.2.6.** We have

\[ h_3 = \sqrt[6]{3} \left( \frac{\sqrt{3} + 1}{\sqrt{2}} \right)^{1/6} \quad (5.2.75) \]

and

\[ h_{1/3} = \frac{1}{\sqrt[6]{3}} \left( \frac{\sqrt{3} - 1}{\sqrt{2}} \right)^{1/6}. \quad (5.2.76) \]
Corollary 5.2.7. We have
\[ h_9 = \frac{\sqrt{3}}{\sqrt{2} - 1}. \] (5.2.77)

We can prove the above Corollaries 5.2.6 and 5.2.7, by putting \( n = 1/3 \) and \( n = 1 \) in the Theorem 5.2.15 respectively.

Theorem 5.2.16. If \( X := h^2_n h^2_{4n} \) and \( Y := H^2_n H^2_{4n} \), then
\[ Y = X + \frac{1}{X}, \] (5.2.78)

where \( \beta \) is of degree 2 over \( \alpha \) in the theory of signature 3.

Proof. From the equations (5.2.32) and (5.2.33), we find
\[ \alpha^{1/6} = \frac{1}{h_n H_n} \quad \text{and} \quad \beta^{1/6} = \frac{1}{h_{4n} H_{4n}} \] (5.2.79)

and
\[ (1 - \alpha)^{1/6} = \frac{h_n}{H_n} \quad \text{and} \quad (1 - \beta)^{1/6} = \frac{h_{4n}}{H_{4n}}. \] (5.2.80)

Employing the equations (5.2.79) and (5.2.80) in the equation (5.2.10), we obtain (5.2.78). \[ \Box \]

Theorem 5.2.17. If \( X := h^2_n h^2_{25n} \) and \( Y := H^2_n H^2_{25n} \), then
\[ Y = X + \frac{1}{X} + 3. \] (5.2.81)

if \( \beta \) is of degree 5 over \( \alpha \) in the theory of signature 3.
Theorem 5.2.18. If \( X := h_n h_{121n} \) and \( Y := H_n H_{121n} \), then
\[
y^2 = \left[ x^2 + \frac{1}{x^2} \right] + 3 \sqrt{3} \left[ x + \frac{1}{x} \right] + 6, \tag{5.2.82}
\]
where \( \beta \) is of degree 11 over \( \alpha \) in the theory of signature 3.

Theorem 5.2.19. If \( X := h_n^2 h_{64n} \) and \( Y := H_n^2 H_{64n} \), then
\[
\frac{1}{y^2} \left[ x^4 + \frac{1}{x^4} \right] - \frac{4}{y} \left[ x^3 + \frac{1}{x^3} \right] + 2 \left[ x^2 + \frac{1}{x^2} \right] \left[ 3 + \frac{20}{y^2} \right] - \left[ x + \frac{1}{x} \right] \left[ 4y + \frac{3}{y} \right] - 33 = \frac{84}{y^2} - y^2, \tag{5.2.83}
\]
where \( \beta \) is of degree 8 over \( \alpha \) in the theory of signature 3.

Since the proofs of the above Theorems 5.2.17–5.2.19 are similar to the proof of the Theorem 5.2.16, so we omit the details.

Theorem 5.2.20. We have
\[
H_1 = \sqrt[6]{2} \tag{5.2.84}
\]
and
\[
H_4 = \sqrt[12]{54}. \tag{5.2.85}
\]

Proof. Employing Lemma 1.4.28 after replacing \( \alpha \) by \( 1 - \alpha \) in the equation (5.2.10), we find
\[
\{(1 - \alpha) \beta\}^{1/3} + \{\alpha (1 - \beta)\}^{1/3} = 1. \tag{5.2.86}
\]
Cubing both sides of the equation (5.2.86), we find
\[
\{(1 - \alpha) \beta\} + \{\alpha (1 - \beta)\} + 3 \{\alpha \beta (1 - \alpha) (1 - \beta)\}^{1/3} = 1. \tag{5.2.87}
\]
Employing the equation (5.2.1) in the equation (5.2.87), we deduce

\[ 6 \{ \alpha \beta (1 - \alpha)(1 - \beta) \}^{1/3} = 1. \]  \hspace{1cm} (5.2.88)

From the equations (5.2.15) and (5.2.88), we find

\[ P^4 = \frac{1}{6}. \] \hspace{1cm} (5.2.89)

Employing the equation (5.2.89) in the equation (5.2.17), we find

\[ Q^6 + \frac{1}{Q^6} = \frac{29}{3\sqrt{6}}. \] \hspace{1cm} (5.2.90)

Solving the equation (5.2.90) and \( 0 < Q < 1 \), we find

\[ Q = \left\{ \frac{2}{27} \right\}^{1/12}. \] \hspace{1cm} (5.2.91)

From the equations (5.2.89) and (5.2.91), we obtain (5.2.84) and (5.2.85). \( \Box \)

**Theorem 5.2.21.** We have

\[ H_{2/5} = \sqrt{2} \left( 3028871 - 8840\sqrt{117397} \right)^{1/24} \] \hspace{1cm} (5.2.92)

and

\[ H_{10} = \sqrt{2} \left( 3028871 + 8840\sqrt{117397} \right)^{1/24}. \] \hspace{1cm} (5.2.93)

**Proof.** Employing Lemma 1.4.28 after replacing \( \alpha \) by \( 1 - \alpha \) in the equation (5.2.13), we find

\[ \{(1 - \alpha)\beta\}^{1/3} + \{\alpha(1 - \beta)\}^{1/3} = 1 - 3p^2. \] \hspace{1cm} (5.2.94)
Cubing both sides of the equation (5.2.94) and then using the equation (5.2.1), we find

\[ 18p^6 - 21p^4 + 9p^2 - 1 = 0. \quad (5.2.95) \]

Solving the equation (5.2.95) for \( p^2 \), we find

\[ p^2 = \frac{1}{6}. \quad (5.2.96) \]

Employing the equation (5.2.94) in the equation (5.2.26), we find

\[ Q^{12} + \frac{1}{Q^{12}} = \frac{6057742}{729}. \quad (5.2.97) \]

Solving the equation (5.2.97) and \( 0 < Q < 1 \), we find

\[ Q^{12} = \frac{3028871 - 8840\sqrt{117397}}{729}. \quad (5.2.98) \]

From the equations (5.2.96) and (5.2.98), we obtain (5.2.92) and (5.2.93).

**Theorem 5.2.22.** We have

\[ H_2 = 2^{1/4}, \quad (5.2.99) \]
\[ H_3 = \left( \frac{4}{3} \right)^{1/8} \left( \frac{\sqrt{3} + 1}{\sqrt{2}} \right)^{1/3}, \quad (5.2.100) \]
\[ H_5 = 5^{1/4}, \quad (5.2.101) \]
\[ H_{11} = \sqrt{8 + 6\sqrt{3}}. \quad (5.2.102) \]

**Proof.** We obtain Theorem 5.2.22, on using the relation (5.2.35) and the values of \( h_n \) established earlier.
Remark 5.2.1. Different proofs of (5.2.99)–(5.2.101) can be found in [14].

5.2.4 Explicit evaluations of the cubic singular modulus

In this section, we established several explicit evaluations of the cubic singular modulus.

Theorem 5.2.23. We have

\[ h_{2/5} = \left( \frac{3 + \sqrt{5}}{2} \right)^{1/4} \left( \frac{7 + 5\sqrt{2}}{2} \right)^{1/6}, \]  \hspace{1cm} (5.2.103)

\[ h_{10} = \left( \frac{3 + \sqrt{5}}{2} \right)^{1/4} \left( \frac{7 + 5\sqrt{2}}{2} \right)^{1/6}. \]  \hspace{1cm} (5.2.104)

Proof. From the equation (5.2.96), we have

\[ H_{2/5}^2 H_{10}^2 = 6. \]  \hspace{1cm} (5.2.105)

Employing the equation (5.2.105) in the equation (5.2.81), we find

\[ h_{2/5}^2 h_{10}^2 = \frac{3 + \sqrt{5}}{2}. \]  \hspace{1cm} (5.2.106)

Employing the equation (5.2.106) in the equation (5.2.48), we deduce

\[ \frac{h_{10}^3}{h_{2/5}^3} - \frac{h_{2/5}^3}{h_{10}^3} = 14. \]  \hspace{1cm} (5.2.107)

Solving the equation (5.2.107), we find

\[ \frac{h_{2/5}^3}{h_{10}^3} = -7 + 5\sqrt{2}. \]  \hspace{1cm} (5.2.108)
From the equations (5.2.106) and (5.2.108), we obtain (5.2.103) and (5.2.104).

The following are some of the explicit evaluation of cubic singular modulus $\alpha$ obtained by using the values of $h_n$ and $H_n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Cubic Singular Modulus $\alpha_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{\sqrt{2} - 1}{2\sqrt{2}}$</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{2} + 1}{2\sqrt{2}}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{2\sqrt{2}} \left( \frac{\sqrt{3} - 1}{\sqrt{2}} \right)^3$</td>
</tr>
<tr>
<td>$\frac{1}{3}$</td>
<td>$\frac{3\sqrt{3}}{2\sqrt{2}} \left( \frac{\sqrt{3} - 1}{\sqrt{2}} \right)$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1}{3\sqrt{6}} \left( \frac{\sqrt{3} - 1}{\sqrt{2}} \right)^3$</td>
</tr>
<tr>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{3\sqrt{6}} \left( \frac{\sqrt{3} + 1}{\sqrt{2}} \right)^3$</td>
</tr>
</tbody>
</table>
5. SCHLÄFELI–TYPE MODULAR EQUATIONS

| \( \frac{1}{5} \) | \( \frac{1}{5\sqrt{5}} \left( \frac{5\sqrt{5} - 11}{\sqrt{2}} \right) \) |
| \( \frac{1}{11} \) | \( \frac{1}{5\sqrt{5}} \left( \frac{5\sqrt{5} + 11}{\sqrt{2}} \right) \) |
| \( \frac{11}{11} \) | \( \sqrt{ \left( 10 - 3\sqrt{11} \right) \left( 90\sqrt{3} - 47\sqrt{11} \right) } \) |
| \( \frac{1}{11} \) | \( \sqrt{ \left( 10 + 3\sqrt{11} \right) \left( 90\sqrt{3} + 47\sqrt{11} \right) } \) |
| \( \frac{2}{5} \) | \( \left( \frac{3 - \sqrt{5}}{2} \right)^{3/2} \left( \frac{7 + 5\sqrt{2}}{4\sqrt{2}} \right) \left( \frac{3028871 + 8840\sqrt{117397}}{729} \right)^{1/4} \) |
| \( \frac{5}{2} \) | \( \left( \frac{3 + \sqrt{5}}{2} \right)^{3/2} \left( \frac{-7 + 5\sqrt{2}}{4\sqrt{2}} \right) \left( \frac{3028871 + 8840\sqrt{117397}}{729} \right)^{1/4} \) |
| \( \frac{10}{10} \) | \( \left( \frac{3 - \sqrt{5}}{2} \right)^{3/2} \left( \frac{-7 + 5\sqrt{2}}{4\sqrt{2}} \right) \left( \frac{3028871 - 8840\sqrt{117397}}{729} \right)^{1/4} \) |
| \( \frac{1}{10} \) | \( \left( \frac{3 + \sqrt{5}}{2} \right)^{3/2} \left( \frac{7 + 5\sqrt{2}}{4\sqrt{2}} \right) \left( \frac{3028871 - 8840\sqrt{117397}}{729} \right)^{1/4} \) |

**Remark 5.2.2.** Different proofs of explicit evaluation of \( \alpha_1, \alpha_2, \alpha_3 \) and \( \alpha_4 \) can be found in [14].
5.3 Schlafli–type modular equations in the quartic theory

In this Section, we established some new Schlafli–type modular equations in quartic theory (or signature 4). We introduced quartic class invariants \( l_n \) and \( I_n \) which are analogous to Ramanujan–Weber class invariants. We established modular relations between the quartic class invariants \( l_n \) with \( I_{j_n} \) and \( L_n \) with \( L_{j_n} \) for \( j = 3, 5, 7 \) and 11. We also established several new explicit evaluations of the quartic class invariants and quartic singular modulus.

Throughout this Section for brevity we set \( q := q_4, z := z(4), m := m(4) \) and \( \alpha^* \) is quartic singular modulus.

5.3.1 Schlafli–type modular equations in quartic theory

In this section, we established several new Schlafli–type modular equations in the quartic theory using Ramanujan’s modular equations.

Set

\[
P := (\alpha^* \beta^* (1 - \alpha^*) (1 - \beta^*))^{1/24} \tag{5.3.1}
\]

and

\[
Q := \left\{ \frac{\beta^* (1 - \beta^*)}{\alpha^* (1 - \alpha^*)} \right\}^{1/24}, \tag{5.3.2}
\]

where \( \beta^* \) is of degree \( n \) over \( \alpha^* \) in the theory of signature 4.

From the equations (5.3.1) and (5.3.2), we find

\[
PQ = (\beta^* (1 - \beta^*))^{1/12} \tag{5.3.3}
\]
and

\[
\frac{P}{Q} = (\alpha^* (1 - \alpha^*))^{1/12}.
\]  \hspace{1cm} (5.3.4)

From the equations (5.3.3) and (5.3.4), we deduce

\[
\alpha^* = \frac{1 + A}{2}, \quad \text{where} \quad A := \pm \sqrt{1 - \frac{4P^{12}}{Q^{12}}}
\]  \hspace{1cm} (5.3.5)

and

\[
\beta^* = \frac{1 + B}{2}, \quad \text{where} \quad B := \pm \sqrt{1 - 4P^{12}Q^{12}}.
\]  \hspace{1cm} (5.3.6)

**Theorem 5.3.1.** If \( P \) and \( Q \) are defined as in the equations (5.3.1) and (5.3.2), then

\[
\frac{1}{P^3} \left[ Q^{12} + \frac{1}{Q^{12}} \right] + 8 \left[ 24P^9 - \frac{1}{P^9} \right] + 2 \left[ \frac{39}{P^3} - 112P^3 \right] = 0,
\]  \hspace{1cm} (5.3.7)

where \( \beta^* \) is of degree 3 over \( \alpha^* \) in the theory of signature 4.

**Proof.** From [20, Ch. 33, Theorem 10.1, p. 153], we have

\[
\{ \alpha^* \beta^* \}^{1/2} + \{(1 - \alpha^*) (1 - \beta^*)\}^{1/2} + 4 \{ \alpha^* \beta^* (1 - \alpha^*) (1 - \beta^*) \}^{1/4} = 1,
\]  \hspace{1cm} (5.3.8)

where \( \beta^* \) is of degree 3 over \( \alpha^* \) in the theory of signature 4.

Employing the equations (5.3.1) and (5.3.2) in the equation (5.3.8), we find

\[
\{ \alpha^* \beta^* \}^{1/2} + \frac{P^{12}}{\{ \alpha^* \beta^* \}^{1/2}} = 1 - 4P^6.
\]  \hspace{1cm} (5.3.9)
Squaring both sides of the equation (5.3.9), we find

\[ \{ \alpha^* \beta^* \}^2 - \left( \left( 1 - 4p^6 \right)^2 - 2p^{12} \right) \alpha^* \beta^* + P^{24} = 0. \quad (5.3.10) \]

Employing the equations (5.3.1) with \( n = 3 \), (5.3.5) and (5.3.6) in the equation (5.3.10). after eliminating \( \Lambda \) and \( B \), we obtain the equation (5.3.7).

**Theorem 5.3.2.** If \( P \) and \( Q \) are defined as in the equations (5.3.1) and (5.3.2), then

\[
Q^{18} + \frac{1}{Q^{18}} + 3775 \left( Q^6 + \frac{1}{Q^6} \right) + 103680p^{12} \left( Q^6 + \frac{1}{Q^6} \right) + 130 \left( Q^{12} + \frac{1}{Q^{12}} \right) + 64 \left( 1944p^{12} - \frac{1}{p^{12}} \right) + 11900 = 0, \quad (5.3.11)
\]

where \( \beta^* \) is of degree 5 over \( \alpha^* \) in the theory of signature 4.

**Proof.** From [20, Ch. 33, Theorem 10.2, p. 154], we have

\[
\{ \alpha^* \beta^* \}^{1/2} + \{(1 - \alpha^*)(1 - \beta^*)\}^{1/2} + 8\{ \alpha^* \beta^* (1 - \alpha^*)(1 - \beta^*)\}^{1/6} \\
\times \left[ \{ \alpha^* \beta^* \}^{1/6} + \{(1 - \alpha^*)(1 - \beta^*)\}^{1/6} \right] = 1. \quad (5.3.12)
\]

where \( \beta^* \) is of degree 5 over \( \alpha^* \) in the theory of signature 4.

Employing the equations (5.3.1) and (5.3.2) with \( n = 5 \) in the equation (5.3.12), we find

\[
\{ \alpha^* \beta^* \}^{1/2} + \frac{P^{12}}{\{ \alpha^* \beta^* \}^{1/2}} + 8P^4 \left( \{ \alpha^* \beta^* \}^{1/6} + \frac{P^4}{\{ \alpha^* \beta^* \}^{1/6}} \right) = 1. \quad (5.3.13)
\]

Employing the equations (5.3.3), (5.3.4), (5.3.5) and (5.3.6), the above equation
(5.3.13) reduces to

\[
\begin{align*}
(P^{12} + 3775P^{12}Q^{12} + 103680P^{24}Q^{12} + 3775P^{12}Q^{24} + 103680P^{24}Q^{24} + \\
Q^{36}P^{12} - 124416P^{24}Q^{18} + 64Q^{18} - 130Q^{6}P^{12} - 11900P^{12}Q^{18} \\
- 130Q^{30}P^{12}) (P^{12} + 3775P^{12}Q^{12} + 103680P^{24}Q^{12} + 3775P^{12}Q^{24} + \\
103680P^{24}Q^{24} + Q^{36}P^{12} + 124416P^{24}Q^{18} - 64Q^{18} + 130Q^{6}P^{12} + \\
11900P^{12}Q^{18} + 130Q^{30}P^{12}) &= 0.
\end{align*}
\]

By examining the behaviour of first factor near \( q = 0 \), it can be seen that there is a neighbourhood about the origin, where the first factor is not zero. Then the second factor is zero in this neighbourhood. By the Identity Theorem second factor vanishes identically. On simplification, we obtain the equation (5.3.11).

\[\square\]

**Theorem 5.3.3.** If \( P \) and \( Q \) are defined as in the equations (5.3.1) and (5.3.2), then

\[
28 \left[ \frac{1}{Q^{12}} + Q^{12} \right] \left[ 1982880P^{9} + 230832P^{3} + \frac{7329}{P^{3}} + \frac{52}{P^{9}} \right]
\]

\[
+ \frac{1}{P^{3}} \left[ Q^{24} + \frac{1}{Q^{24}} \right] = 188116992P^{21} + 170201088P^{15} + \frac{512}{P^{21}}
\]

\[
- 131072256P^{9} - 42166656P^{3} + \frac{3152954}{P^{3}} + \frac{531552}{P^{9}} - \frac{35392}{P^{15}}.
\]

where \( \beta^* \) is of degree 7 over \( \alpha^* \) in the theory of signature 4.

**Proof.** From [20, Ch. 33, Theorem 10.3, p. 155], we have

\[
\begin{align*}
\{ \alpha^* \beta^* \}^{1/2} + \{(1 - \alpha^*) (1 - \beta^*)\}^{1/2} + 20\{ \alpha^* \beta^* (1 - \alpha^*) (1 - \beta^*)\}^{1/4} \\
+ 8\sqrt{2}\{ \alpha^* \beta^* (1 - \alpha^*) (1 - \beta^*)\}^{1/8} \left[ \{ \alpha^* \beta^* \}^{1/4} + \{(1 - \alpha^*) (1 - \beta^*)\}^{1/4} \right] &= 1,
\end{align*}
\]

(5.3.16)
where $\beta^*$ is of degree 7 over $\alpha^*$ in the theory of signature 4.

Employing the equations (5.3.1) and (5.3.2) with $n = 7$ in the equation (5.3.16), we find

$$\{\alpha^* \beta^*\}^{1/2} + \frac{p_{12}}{\{\alpha^* \beta^*\}^{1/2}} + 20p^6 + 8\sqrt{2}p^3 \left(\{\alpha^* \beta^*\}^{1/4} + \frac{p^6}{\{\alpha^* \beta^*\}^{1/4}}\right) = 1. \tag{5.3.17}$$

Employing the equations (5.3.3), (5.3.4), (5.3.5) and (5.3.6), the equation (5.3.17) reduces to

$$\left(-p^{18} - 1456p^{12}Q^{12} - 42166656p^{24}Q^{24} - 6463296P^{24}Q^{12}
- 205212P^{18}Q^{12} - 55520640p^{30}Q^{12} + 188116992p^{42}Q^{24} + 512Q^{24}
+ 170201088p^{36}Q^{24} - 131072256p^{30}Q^{24} - 6463296P^{24}Q^{36}
- 205212P^{18}Q^{36} - 55520640p^{30}Q^{36} - 1456p^{12}Q^{36} + 531552P^{12}Q^{24}
+ 3152954P^{18}Q^{24} - p^{18}Q^{48} - 35392Q^{24}P^6\right)^2
\left(-p^{18} + 1616P^{12}Q^{12}
- 512Q^{24} - 234588544P^{24}Q^{24} + 33050816P^{24}Q^{12} - 430492P^{18}Q^{12}
- 780266880p^{30}Q^{12} + 1886810112P^{42}Q^{24} - 858451968P^{36}Q^{24}
- 954040576p^{30}Q^{24} + 33050816P^{24}Q^{36} - 430492Q^{18}Q^{36}
+ 1616P^{12}Q^{36} - 780266880p^{30}Q^{36} - 3038112P^{12}Q^{24}
+ 53574714P^{18}Q^{24} - p^{18}Q^{48} + 67008Q^{24}P^6\right)^2 = 0. \tag{5.3.18}$$

By examining the behaviour of second factor near $q = 0$, it can be seen that there is a neighbourhood about the origin, where the second factor is not zero. Then the first factor is zero in this neighbourhood. By the Identity Theorem first factor vanishes identically. On simplification, we obtain the equation (5.3.15). \qed
Theorem 5.3.4. If $P$ and $Q$ are defined as in the equations (5.3.1) and (5.3.2), then

$$
\begin{align*}
\frac{1}{P^3} & \left[ Q^{36} + \frac{1}{Q^{36}} \right] + \left[ Q^{24} + \frac{1}{Q^{24}} \right] \left[ 1899390211799616P^9 \right. \\
& \left. -2981005147296P^3 + \frac{1378118378}{P^3} - \frac{164864}{P^9} \right] + 1301797103468544P^{33} \\
& -4660847651192832P^{27} + 4568332198256640P^{21} - 4743319404552192P^{15} \\
& + 564317587338624P^9 - 2493407003253696P^3 + \frac{1213770519844748}{P^3} \\
& -\frac{99666540942224}{P^9} + \frac{2184867437184}{P^{15}} - \frac{17649625600}{P^{21}} + \frac{52289536}{P^{27}} \\
& -\frac{32768}{P^{33}} \left[ Q^{12} + \frac{1}{Q^{12}} \right] \left[ 3253301270433792P^{21} - \frac{164864}{P^{21}} \\
& +2521615039739136P^9 - 278109156534912P^3 + \frac{18821424472497}{P^3} \\
& -\frac{263374174048}{P^9} + \frac{286658368}{P^{15}} - 5333256318947328P^{15} \right].
\end{align*}
$$

(5.3.19)

where $\beta^*$ is of degree 11 over $\alpha^*$ in the theory of signature 4.

Since the proof of the equation (5.3.19) is similar to the proof of the equation (5.3.15) except that in place of the result (5.3.16), we use the result from [20, Ch. 33, Theorem 10.4, p. 155]

$$
\begin{align*}
& \{\alpha^*\beta^*\}^{1/2} + \{(1 - \alpha^*)(1 - \beta^*)\}^{1/2} + 68\{\alpha^*\beta^*(1 - \alpha^*)(1 - \beta^*)\}^{1/4} \\
& + 16\{\alpha^*\beta^*(1 - \alpha^*)(1 - \beta^*)\}^{1/12} \left[ \{\alpha^*\beta^*\}^{1/3} + \{(1 - \alpha^*)(1 - \beta^*)\}^{1/3} \right] \\
& + 48\{\alpha^*\beta^*(1 - \alpha^*)(1 - \beta^*)\}^{1/6} \left[ \{\alpha^*\beta^*\}^{1/6} + \{(1 - \alpha^*)(1 - \beta^*)\}^{1/6} \right] = 1,
\end{align*}
$$

(5.3.20)
where $\beta^*$ is degree 11 over $\alpha^*$ in the theory of signature 4. Hence we omit the details.

### 5.3.2 Quartic class invariants $L_n$ and $l_n$

In this section, following Ramanujan we defined quartic class invariants $l_n$ and $L_n$ and established some properties of these class invariants. We established several modular relations between the quartic class invariants $L_n$ and $l_n$. We also established several explicit evaluations of $L_n$ and $l_n$. Further, at the end of this Section, we tabulated several new explicit evaluations of quartic singular modulus.

\[
l_n = \left( \alpha^* (1 - \alpha^*)^{-1} \right)^{-1/24}
\]  

(5.3.21)

and

\[
L_n = \left( \alpha^* (1 - \alpha^*) \right)^{-1/24},
\]  

(5.3.22)

where $q_4 = e^{-\pi \sqrt{n}}$, $n$ a positive rational number and $\alpha^*$ is quartic singular modulus.

**Theorem 5.3.5.** We have

\[
l_{2n}l_{2/n} = 1.
\]  

(5.3.23)

**Proof.** From [20, Ch. 33, Theorem 9.9, Theorem 9.10, p. 148], we have

\[
q_4^{1/24} f (-q_4) = \sqrt{Z_4} 2^{-1/4} \alpha^{1/24} (1 - \alpha^*)^{1/12},
\]  

(5.3.24)

\[
q_4^{1/12} f (-q_4^2) = \sqrt{Z_4} 2^{-1/4} \alpha^{1/12} (1 - \alpha^*)^{1/24}.
\]  

(5.3.25)

Employing the equations (5.3.24), (5.3.25) and (1.4.4), we obtain the equation
(5.3.23).

**Theorem 5.3.6.** We have

\[ L_n^{12} = l_n^{12} + \frac{1}{l_n^{12}}. \quad (5.3.26) \]

*Proof.* From the equations (5.3.21) and (5.3.22), we obtain the equation (5.3.26).

**Theorem 5.3.7.** We have

\[ L_{2n} = L_{2/n}. \quad (5.3.27) \]

*Proof.* Employing the equation (5.3.23) in the equation (5.3.26), we obtain the equation (5.3.27).

**Theorem 5.3.8.** We have

\[ \alpha_{2n}^* + \alpha_{2/n}^* = 1. \quad (5.3.28) \]

*Proof.* From the equations (5.3.21), (5.3.22), (5.3.23) and (5.3.27), we obtain the equation (5.3.28).

**Theorem 5.3.9.** If \( X := \frac{j_n^{113}}{j_n^{19�}} \) and \( Y := \frac{j_n^5}{j_n^9�} \), then

\[ \frac{1}{Y} - Y = 2\sqrt{2} \left( X + \frac{1}{X} \right), \quad (5.3.29) \]

where \( \beta^* \) is of degree 3 over \( \alpha^* \) in the theory of signature 4.

*Proof.* The equation (5.3.8) can be written as

\[ 1 + \left\{ \frac{(1 - \alpha^*)(1 - \beta^*)}{\alpha^*\beta^*} \right\}^{1/2} + 4 \left\{ \frac{(1 - \alpha^*)(1 - \beta^*)}{\alpha^*\beta^*} \right\}^{1/4} = \frac{1}{\sqrt{\alpha^*\beta^*}}. \quad (5.3.30) \]
Employing the equation (5.3.21) in the equation (5.3.30), we find

\[
18 + 8X^2 + \frac{8}{X^2} = Y^2 + \frac{1}{Y^2}.
\]  
(5.3.31)

On simplification, we obtain the equation (5.3.29).

**Corollary 5.3.1.** We have

\[
l_6 = \left(\sqrt{2} + 1\right)^{1/6}
\]  
(5.3.32)

and

\[
l_{2/3} = \left(\sqrt{2} - 1\right)^{1/6}.
\]  
(5.3.33)

**Proof.** Putting \( n = 2/3 \) in the equation (5.3.9), we find

\[
X = 1.
\]  
(5.3.34)

Employing the equation (5.3.34) in the equation (5.3.9), we find

\[
\frac{1}{Y} - Y = 4\sqrt{2}.
\]  
(5.3.35)

Solving the equation (5.3.35), we obtain (5.3.32) and (5.3.33).

**Corollary 5.3.2.** We have

\[
l_{18} = \left(\sqrt{3} + \sqrt{2}\right)^{1/3}
\]  
(5.3.36)

and

\[
l_{2/9} = \left(\sqrt{3} - \sqrt{2}\right)^{1/3}.
\]  
(5.3.37)
Proof. Putting \( n = 2 \) in the equation (5.3.9), we find

\[
I_{18}^3 - \frac{1}{I_{18}^3} = 2\sqrt{2}.
\]  

(5.3.38)

On simplification, we obtain (5.3.36).

\[\square\]

**Theorem 5.3.10.** If \( X := l_n^2 l_{25n}^2 \) and \( Y := \frac{I_{25n}^3}{I_{18}^3} \), then

\[
\frac{1}{Y} - Y = 2 \left( X + \frac{1}{X} \right),
\]

(5.3.39)

where \( \beta^* \) is of degree 5 over \( \alpha^* \) in the theory of signature 4.

Proof. The equation (5.3.12) can be written as

\[
1 + A^{1/2} + 8A^{1/6} \left( 1 + A^{1/6} \right) = \frac{1}{\sqrt[6]{\alpha^* \beta^*}},
\]

(5.3.40)

where \( A := \frac{(1 - \alpha^*)(1 - \beta^*)}{\alpha^* \beta^*} \).

Employing the equation (5.3.21) in the equation (5.3.40), we find

\[
\left( Y + 2X + \frac{2}{X} - \frac{1}{Y} \right) \left( Y - 2X - \frac{2}{X} - \frac{1}{Y} \right)
\]

\[
\left( Y^2 + 4X^{4/3} + 10 + \frac{4}{X^{4/3}} + \frac{1}{Y^2} \right) = 0.
\]

(5.3.41)

Since \( Y^2 + 4X^{4/3} + 10 + \frac{4}{X^{4/3}} + \frac{1}{Y^2} \neq 0 \). By examining the behaviour of second factor of the equation (5.3.40) near \( q = 0 \), it can be seen that there is a neighbourhood about the origin, where the second factor is not zero, whereas the first factor is zero in this neighbourhood. By the Identity Theorem first factor vanishes
identically. Hence
\[ Y + 2X + \frac{2}{X} - \frac{1}{Y} = 0. \]

This completes the proof. \(\square\)

**Corollary 5.3.3.** We have

\[ l_{10} = \left( \frac{\sqrt{5} + 1}{2} \right)^{1/2} \]  \hspace{1cm} (5.3.42)

and

\[ l_{2/5} = \left( \frac{\sqrt{5} - 1}{2} \right)^{1/2}. \]  \hspace{1cm} (5.3.43)

**Proof.** Putting \( n = 2/5 \) in the equation (5.3.39), we find

\[ X = 1. \]  \hspace{1cm} (5.3.44)

Employing the equation (5.3.44) in the equation (5.3.39), we find

\[ \frac{1}{Y} - Y = 4. \]  \hspace{1cm} (5.3.45)

Solving the equation (5.3.45), we obtain (5.3.42) and (5.3.43). \(\square\)

**Theorem 5.3.11.** If \( X := l_{n,49n}^4 l_{49n}^1 \) and \( Y := \frac{l_n^4}{l_{49n}^4} \), then

\[ \frac{1}{Y} + Y = 2\sqrt{2} \left( X + \frac{1}{X} \right) + 7, \]  \hspace{1cm} (5.3.46)

where \( \beta^* \) is of degree 7 over \( \alpha^* \) in the theory of signature 4.
5. Schläfli—Type Modular Equations

Proof. The equation (5.3.16) can be written as

\[ 1 + A^{1/2} + 20A^{1/4} + 8\sqrt{2}A^{1/8} \left( 1 + A^{1/4} \right) = \frac{1}{\sqrt{\alpha^* \beta^*}}, \quad (5.3.47) \]

where \( A := \frac{(1 - \alpha^*)(1 - \beta^*)}{\alpha^* \beta^*} \).

Employing the equation (5.3.21) in the equation (5.3.47), we find

\[
\begin{align*}
&\left( I_{14}^{16} + 8I_{14}^{2}I_{49n}^{2} + 64I_{14}^{8}I_{49n}^{8} + 2\sqrt{2}I_{n}^{8}I_{49n}^{8} + 2\sqrt{2}l_{14}^{15}l_{49n}^{7} + 7l_{49n}^{4}l_{n}^{12} + l_{49n}^{16} \\
&\quad + 2\sqrt{2}l_{n}^{8}l_{49n}^{8} + 28l_{49n}^{11}l_{n}^{11} \sqrt{2} + 2l_{n}^{15} \sqrt{2}l_{49n}^{7} + 28 \sqrt{2}l_{n}^{5}l_{49n}^{5} + 7l_{n}^{12}l_{49n}^{12} \\
&\quad + 8l_{49n}^{14}l_{n}^{14} \right) \left( -l_{n}^{8} - l_{49n}^{8} + 2\sqrt{2}l_{n}l_{49n} + 7l_{n}^{4}l_{49n}^{4} + 2\sqrt{2}l_{n}^{7}l_{49n}^{7} \right) = 0.
\end{align*}
\]

(5.3.48)

By examining the behaviour of first factor of the equation (5.3.48) near \( q = 0 \), it can be seen that there is a neighbourhood about the origin, where the first factor is not zero, whereas the second factor is zero in this neighbourhood. By the Identity Theorem second factor vanishes identically. Hence

\[
-l_{n}^{8} - l_{49n}^{8} + 2\sqrt{2}l_{n}l_{49n} + 7l_{n}^{4}l_{49n}^{4} + 2\sqrt{2}l_{n}^{7}l_{49n}^{7} = 0.
\]

On simplification, we obtain the equation (5.3.46). \( \square \)

Corollary 5.3.4. We have

\[
I_{14} = \sqrt{\frac{1 + \sqrt{2} + \sqrt{2}\sqrt{2} - 1}{2}}, \quad (5.3.49)
\]

and

\[
I_{2/7} = \sqrt{\frac{1 + \sqrt{2} - \sqrt{2}\sqrt{2} - 1}{2}}. \quad (5.3.50)
\]
Proof. Putting \( n = 2/7 \) in the equation (5.3.46), we find

\[
\frac{1}{Y} + Y = 4\sqrt{2} + 7. \tag{5.3.51}
\]

Solving the equation (5.3.51), we obtain (5.3.49) and (5.3.50).

\[\Box\]

**Theorem 5.3.12.** If \( X := l_n l_{121n} \) and \( Y := \frac{l_n^6}{l_{121n}^6} \), then

\[
\frac{1}{Y} - Y = 2\sqrt{2} \left( \frac{2}{X^3} + \frac{11}{X} + \frac{22}{X} + 22X + 11X^3 + 2X^5 \right), \tag{5.3.52}
\]

where \( \beta^* \) is of degree 11 over \( \alpha^* \) in the theory of signature 4.

**Proof.** The equation (5.3.20) can be written as

\[
1 + A^{1/2} + 68A^{1/4} + 16A^{1/4} \left( 1 + A^{1/4} \right) + 48A^{1/6} \left( 1 + A^{1/6} \right) = \frac{1}{\sqrt{\alpha^* \beta^*}}, \tag{5.3.53}
\]

where \( A := \frac{(1 - \alpha^*) (1 - \beta^*)}{\alpha^* \beta^*} \).

Employing the equation (5.3.21) in the equation (5.3.53), we find

\[
1672 \left( \frac{1}{l_n^{10} l_{121n}^{10}} + l_n^{10} l_{121n}^{10} \right) + 32 \left( \frac{1}{l_n^{12} l_{121n}^{12}} + l_n^{12} l_{121n}^{12} \right) + 352 \left( \frac{1}{l_n^{8} l_{121n}^{8}} + l_n^{8} l_{121n}^{8} \right) + 4576 \left( \frac{1}{l_n^{4} l_{121n}^{4}} + l_n^{4} l_{121n}^{4} \right) + 8096 \left( \frac{1}{l_n^{2} l_{121n}^{2}} + l_n^{2} l_{121n}^{2} \right) - \frac{l_n^{12}}{l_{121n}^{12}} - \frac{l_n^{12}}{l_{121n}^{12}} - 9746 = 0. \tag{5.3.54}
\]

On simplification, we obtain the equation (5.3.52). \[\Box\]

**Corollary 5.3.5.** We have

\[
l_{22} = \sqrt{1 + \sqrt{2}} \quad \tag{5.3.55}
\]
and
\[ l_{2/11} = \sqrt{-1 + \sqrt{2}}. \tag{5.3.56} \]

**Proof.** Putting \( n = 2/11 \) in the equation (5.3.39), we find
\[ \frac{1}{Y} - Y = 140\sqrt{2}. \tag{5.3.57} \]

Solving the equation (5.3.57), we obtain (5.3.55) and (5.3.56).

**Theorem 5.3.13.** If \( P := l_{n\sigma_n}^6 l_{n\sigma_n}^6 \) and \( Q := l_{n\sigma_n}^6 l_{n\sigma_n}^6 \), then
\[ \left( P + \frac{1}{P} \right) + 4 = Q, \tag{5.3.58} \]
where \( \beta^* \) is of degree 3 over \( \alpha^* \) in the theory of signature 4.

**Proof.** From the equations (5.3.21) and (5.3.22), we find
\[ (\alpha^*)^{1/12} = \frac{1}{l_n L_n} \quad \text{and} \quad (\beta^*)^{1/12} = \frac{1}{l_{n\lambda} L_{\lambda n}}, \tag{5.3.59} \]
and
\[ (1 - \alpha^*)^{1/12} = \frac{l_n}{L_n} \quad \text{and} \quad (1 - \beta^*)^{1/12} = \frac{l_{n\lambda}}{L_{\lambda n}}. \tag{5.3.60} \]

Employing the equations (5.3.59) and (5.3.60) in the equation (5.3.8), we obtain (5.3.58).

**Theorem 5.3.14.** If \( P := l_{25n}^{12} l_{25n}^{12} \) and \( Q := L_{n\sigma_{25n}}^6 L_{n\sigma_{25n}}^6 \), then
\[ \left( P^3 + \frac{1}{P^3} \right) + 8 \left( P + \frac{1}{P} \right) = Q, \tag{5.3.61} \]
where \( \beta^* \) is of degree 5 over \( \alpha^* \) in the theory of signature 4.
Theorem 5.3.15. If $P := L_{149n}^{1413}$ and $Q := L_{49n}^{214},$ then
\[
\left( P^2 + \frac{1}{P^2} \right) + 8 \sqrt{2} \left( P + \frac{1}{P} \right) + 20 = Q, \tag{5.3.62}
\]
where $\beta^*$ is of degree 7 over $\alpha^*$ in the theory of signature 4.

Theorem 5.3.16. If $P = L_{121n}^{121}$ and $Q = L_{121n}^{6},$ then
\[
\left( P^3 + \frac{1}{P^3} \right) + 16 \left( P^2 + \frac{1}{P^2} \right) + 48 \left( P + \frac{1}{P} \right) + 68 = Q, \tag{5.3.63}
\]
where $\beta^*$ is of degree 11 over $\alpha^*$ in the theory of signature 4.

Since the proofs of Theorems 5.3.14–5.3.16 are similar to the proof of Theorem 5.3.13, we omit the details.

The following are some new explicit evaluations of $L_n.$ For brevity, we prove $L_2.$

<table>
<thead>
<tr>
<th>$n$</th>
<th>Quartic Class Invariant $L_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\frac{1}{\sqrt{2}}$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{1}{\sqrt{6}}$</td>
</tr>
<tr>
<td>10</td>
<td>$\frac{1}{2} \sqrt{18}$</td>
</tr>
<tr>
<td>14</td>
<td>$\frac{1}{2} \left( \sqrt{11} + 8 \sqrt{2} \right)$</td>
</tr>
<tr>
<td>18</td>
<td>$\sqrt[12]{98}$</td>
</tr>
<tr>
<td>----</td>
<td>----------------</td>
</tr>
<tr>
<td>22</td>
<td>$\sqrt[12]{198}$</td>
</tr>
<tr>
<td>30</td>
<td>$\sqrt[12]{6 \left(5 + 4\sqrt{2}\right)^2}$</td>
</tr>
<tr>
<td>$\frac{10}{3}$</td>
<td>$\sqrt[12]{6 \left(-5 + 4\sqrt{2}\right)^2}$</td>
</tr>
<tr>
<td>42</td>
<td>$\sqrt[12]{6 \left(5 + 4\sqrt{2}\right)^2}$</td>
</tr>
<tr>
<td>$\frac{6}{7}$</td>
<td>$\sqrt[12]{6 \left(7\sqrt{3} - 8\sqrt{2}\right)^2}$</td>
</tr>
<tr>
<td>70</td>
<td>$\sqrt[12]{126 \left(8\sqrt{2} + 5\sqrt{5}\right)^2}$</td>
</tr>
<tr>
<td>$\frac{10}{7}$</td>
<td>$\sqrt[12]{126 \left(8\sqrt{2} - 5\sqrt{5}\right)^2}$</td>
</tr>
<tr>
<td>78</td>
<td>$\sqrt[12]{6 \left(75 + 52\sqrt{2}\right)^2}$</td>
</tr>
<tr>
<td>$\frac{26}{3}$</td>
<td>$\sqrt[12]{6 \left(75 - 52\sqrt{2}\right)^2}$</td>
</tr>
</tbody>
</table>
Proof of $l_2$. Putting $n = 1$ in the equation (5.3.23), we find

$$l_2 = 1. \quad (5.3.64)$$

Employing the equation (5.3.64) in the equation (5.3.26) with $n = 2$, we obtain the required result. \qed

Remark. Baruah and Berndt [14] found the evaluations of $L_n$ for $n = 6, 10, 14, 18$ and $22$.

5.3.3 Explicit evaluations of the quartic singular modulus

In this section, we established several new explicit evaluations of the quartic singular modulus using the values of $l_n$ and $L_n$.

<table>
<thead>
<tr>
<th>n</th>
<th>Quartic Singular Modulus $\alpha_n^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\frac{2}{3}$</td>
<td>$\frac{(\sqrt{2} + 1)^2}{6}$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{(\sqrt{2} - 1)^2}{6}$</td>
</tr>
<tr>
<td>$\frac{2}{9}$</td>
<td>$\frac{(\sqrt{3} + \sqrt{2})^4}{98}$</td>
</tr>
</tbody>
</table>
|     | \[
\frac{(\sqrt{3} - \sqrt{2})^4}{98}
\] |
|-----|--------------------------------------------------|
| \(\frac{2}{5}\) | \[
\frac{1}{18} \left(\frac{\sqrt{5}+1}{2}\right)^6
\] |
| \(\frac{10}{11}\) | \[
\frac{1}{18} \left(\frac{\sqrt{5} - 1}{2}\right)^6
\] |
| \(\frac{2}{7}\) | \[
\frac{(8\sqrt{2} - 11)}{14} \left(\frac{1 + \sqrt{2} + \sqrt{2\sqrt{2} - 1}}{2}\right)^6
\] |
| \(\frac{14}{11}\) | \[
\frac{(8\sqrt{2} - 11)}{14} \left(\frac{1 + \sqrt{2} - \sqrt{2\sqrt{2} - 1}}{2}\right)^6
\] |
| \(\frac{2}{11}\) | \[
\frac{(\sqrt{2} - 1)^6}{198}
\] |
| \(\frac{22}{11}\) | \[
\frac{(\sqrt{2} + 1)^6}{198}
\] |
| \(30\) | \[
\frac{(\sqrt{3} - 2)^2 (\sqrt{10} - 3)^2 (4\sqrt{2} - 5)^2}{294}
\] |
| \(\frac{2}{15}\) | \[
\frac{(\sqrt{5} + 2)^2 (\sqrt{10} + 3)^2 (4\sqrt{2} - 5)^2}{294}
\] |
| \(\frac{10}{3}\) | \[
\frac{(\sqrt{5} + 2)^2 (\sqrt{10} - 3)^2 (4\sqrt{2} + 5)^2}{294}
\] |
|   | \[
\frac{(\sqrt{5} - 2)^2 (\sqrt{10} + 3)^2 (4\sqrt{2} + 5)^2}{294}\]
|---|---|
| 6/5 | \[
\frac{(2\sqrt{2} - 7)^2 (7\sqrt{3} - 8\sqrt{2})^2}{2166} \left(\frac{\sqrt{7} - \sqrt{3}}{2}\right)^6
\]
| 6/7 | \[
\frac{(2\sqrt{2} - 7)^2 (7\sqrt{3} + 8\sqrt{2})^2}{2166} \left(\frac{\sqrt{7} - \sqrt{3}}{2}\right)^6
\]
| 70 | \[
\frac{(\sqrt{2} - 1)^6 (8\sqrt{2} - 5\sqrt{5})^2}{353934} \left(\frac{3 - \sqrt{5}}{2}\right)^6
\]
| 10/7 | \[
\frac{(\sqrt{2} - 1)^6 (8\sqrt{2} + 5\sqrt{5})^2}{353934} \left(\frac{3 - \sqrt{5}}{2}\right)^6
\]
| 78 | \[
\frac{(\sqrt{26} - 5)^2 (75 - 52\sqrt{2})^2}{282534} \left(\frac{\sqrt{13} - 3}{2}\right)^6
\]

**Proof of $\alpha_n^\ast$.** From the equations (5.3.21) and (5.3.22), we find

\[
\alpha_n^\ast = \frac{1}{l_n^{12} L_n^{12}}. \tag{5.3.65}
\]

Putting $n = 2$ in the equation (5.3.65) and then using the values of $l_2$ and $L_2$, we arrive at the required result. \qed

**Remark.** Baruah and Berndt [14] found the evaluations of $\alpha_n^\ast$ for $n = 6, 18, 10, 14$ and 22.