Shannon’s measure of information plays a very important role for measuring uncertainty in probability distributions and also for measuring diversity in plants and animals in Biology. But this measure does not deal with growth models other than exponential. Since there are families of distributions other than exponential and there are laws of population growth other than exponential, we can not confine ourselves to exponential families only and consequently, Shannon’s measure may not be much applicable. Thus we need a parametric models which are suitable for all types of distribution.

Shannon [107] introduced the ordinary mean codeword length and established bounds in terms of entropy. In this chapter, bounds on generalized mean codeword length are obtained by considering parametric measures of information with utility distribution which are suitable for all types of of distributions. The bounds obtained in this chapter are not only new but also generalizes some well established results available in the literature of information theory.

3.1 Introduction

Let \( P = (p_1, p_2, \ldots, p_n), 0 \leq p_i \leq 1, \sum_{i=1}^{n} p_i = 1 \) be the probability distribution associated with a finite system of events \( X = (x_1, x_2, \ldots, x_n) \) representing the realization of some experiment. The different events \( x_i \) depend upon the experimenters goal or upon some qualitative characteristics of the physical system taken in to account; ascribe to each event \( x_i \) a non negative number \( u_i (> 0) \) directly proportional to its importance and call \( u_i \) the utility of the event \( x_i \). Then the weighted entropy [17] of the experiment \( X \) is defined as

\[
(3.1.1) \quad H(P; U) = -\sum_{i=1}^{n} u_i p_i \log p_i
\]

Now let us suppose that the experimenter asserts that the probability of the \( i^{th} \) outcome \( x_i \) is \( q_i \), whereas the true probability is \( p_i \), with \( \sum_{i=1}^{n} p_i = 1 = \sum_{i=1}^{n} q_i \). Thus, we have two utility information schemes

\[
(3.1.2) \quad S = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ p_1 & p_2 & \cdots & p_n \\ u_1 & u_2 & \cdots & u_n \end{bmatrix}, \quad 0 \leq p_i \leq 1, \quad u_i > 0, \quad \sum_{i=1}^{n} p_i = 1
\]

of a set of \( n \) events after an experiment, and
\[(3.1.3) \quad S^* = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \\ q_1 & q_2 & \ldots & q_n \\ u_1 & u_2 & \ldots & u_n \end{bmatrix}, \quad 0 \leq q_i \leq 1, \quad u_i > 0, \quad \sum_{i=1}^{n} q_i = 1 \]

of the same set of \( n \) events before the experiment.

In both the schemes (3.1.2) and (3.1.3) the utility distribution is the same because we assume that the utility \( u_i \) of an outcome \( x_i \) is independent of its probability of occurrence \( p_i \), or predicted probability \( q_i \). \( u_i \) is only a ‘utility’ or value of the outcome \( x_i \) for an observer relative to some specified goal (refer to [87]).

The quantitative- qualitative measure of inaccuracy [119] associated with the above schemes

\[(3.1.4) \quad I (P, Q; U) = -\sum_{i=1}^{n} u_i p_i \log q_i \]

Guiasu and Picard [51] considered the problem of encoding the letter output by the source (3.1.2) by means of a single letter prefix code with codewords \( c_1, c_2, \ldots, c_n \) having length \( l_1, l_2, \ldots, l_n \) satisfying Kraft [80] inequality

\[(3.1.5) \quad \sum_{i=1}^{n} D^{-l_i} \leq 1 \]

\( D \) being the size of the code alphabet. They defined the useful mean length \( L (U) \) of the code as

\[(3.1.6) \quad L (U) = \frac{\sum_{i=1}^{n} u_i p_i l_i}{\sum_{i=1}^{n} u_i p_i} \]

and obtained bounds for it.

Taneja and Tuteja [119] considered the codeword mean length given in (3.1.6) and obtained bounds in terms of (3.1.4), under the condition

\[(3.1.7) \quad \sum_{i=1}^{n} p_i q_i^{-1} D^{-l_i} \leq 1 \]

\( D \) is the size of the code alphabet. It is easy to see that for \( p_i = q_i \forall i = 1, 2, \ldots, n \) (3.1.7) reduces to Kraft [80] inequality.

Longo [87], Gurdial and Pessoa [53], Autar and Khan [6], Jain and Tuteja [63], Taneja et al [120], Bhatia [22], Hooda and Bhaker [59], Singh, Kumar and Tuteja [118] and Khan et al [78] considered the problem of information measures and used it studying the bounds.

In the next section, generalized ‘useful’ codeword mean length are considered
and bounds have been obtained in terms of generalized ‘useful’ inaccuracy measure of order $\alpha$ and type $\beta$. The beauty of these results is that it generalizes the results which exists in the literature of information theory and the measures considered here are suitable for the distributions other than exponential. This work is published in “International journal of pure and applied Mathematics”, Vol 32 (4), PP 467-474 (2006)(Baig and Rayees [8]). All the logarithms used in this chapter are with base D, where D is the size of the code alphabet.

3.2. Generalized measures of information and their bounds

Consider a function

$$I_\alpha^\beta (P, Q; U) = \frac{1}{D^\frac{1}{\alpha - \alpha}} \left[ 1 - \left( \sum_{i=1}^{n} \frac{u_i p_i^\alpha q_i^{-\beta}}{\sum_{i=1}^{n} u_i p_i^\alpha} \right)^{\frac{1}{\alpha}} \right]$$

where $\alpha > 0 \neq 1, \beta > 0, p_i \geq 0, \sum_{i=1}^{n} p_i \leq 1, i = 1, 2, ..., n$. D is the size of the code alphabet.

**Remark (3.2.1).**

1. When $\alpha \to 1, \beta = 1$ and distribution is complete, the measure (3.2.1) reduces to measure of ‘useful’ inaccuracy given by Taneja and Tuteja [119].

2. When $\beta = 1, p_i = q_i \ \forall \ i = 1, 2, ..., n$, the measure (3.2.1) reduces to the measure given by Autar and Khan [6] as ‘useful’ information measure.

3. When $\alpha \to 1, \beta = 1$ and $p_i = q_i \ \forall \ i = 1, 2, ..., n$, the measure (3.2.1) reduces to the measure of ‘useful’ information for incomplete probability distribution given by Belis and Guiasu [17]. Further, when utility aspect of the scheme is ignored, the measure reduces to Shannon’s [107] entropy.

4. When the probability distribution is complete and the utility aspect of the scheme is ignored as well as $\alpha \to 1, \beta = 1$. The measure (3.2.1) becomes the Kerridge’s [73] measure of inaccuracy. We call (3.2.1) as generalized ‘useful’ inaccuracy measure of order $\alpha$ and type $\beta$ for incomplete probability distribution.

Further, consider a generalized ‘useful’ codeword mean length credited with utilities and probabilities as
(3.2.2) \[ L_\alpha^\beta (U) = \frac{1}{D^{\frac{\alpha}{\alpha - 1}}} \left[ 1 - \sum_{i=1}^{n} p_i ^\beta \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i ^\beta} \right)^{\frac{1}{\alpha}} D^{-l_i \left( \frac{\alpha - 1}{\alpha} \right)} \right] \]

where \( \alpha > 0 \ (\neq 1) \), \( \beta > 0 \), \( p_i \geq 0 \), \( \sum_{i=1}^{n} p_i \leq 1 \), \( i = 1, 2, \ldots, n \). D is the size of the code alphabet.

**Remark (3.2.2)**

(1) When \( \alpha \to 1 \), \( \beta = 1 \) the measure (3.2.2) reduces to ‘useful’ mean length \( L(U) \) of the code, given by Guisau and Picard [51].

(2) When the utility aspect of the scheme is ignored by taking \( u_i = 1 \ \forall \ i = 1, 2, \ldots, n \) also \( \sum_{i=1}^{n} p_i = 1 \) and \( \alpha \to 1 \), \( \beta = 1 \), the mean length of the code (3.2.2) becomes optimal code length identical to Shannon [107].

Now, we find the bounds for \( L_\alpha^\beta (U) \) in terms of \( I_\alpha^\beta (P, Q; U) \) under the condition

(3.2.3) \[ \sum_{i=1}^{n} p_i ^\beta q_i^{-1} D^{-l_i} \leq 1 \]

where D is the size of the code alphabet. It is easy to see that for \( \beta = 1 \) and \( p_i = q_i \ \forall \ i = 1, 2, \ldots, n \). Inequality (3.2.3) reduces to Kraft [80] inequality.

**Theorem (3.2.1).** For all integers \( D (D > 1) \). Let \( l_i \) satisfies the condition (3.2.3), then the generalized ‘useful’ codeword mean length satisfies

(3.2.4) \[ L_\alpha^\beta (U) \geq I_\alpha^\beta (P, Q; U) \]

equality holds iff

(3.2.5) \[ l_i = - \log \left( \frac{u_i q_i^\alpha \left( \sum_{i=1}^{n} u_i p_i ^\beta q_i ^{\alpha - 1} \right)}{\sum_{i=1}^{n} u_i p_i ^\beta q_i ^{\alpha - 1}} \right) \]

**Proof:** By Holder’s inequality [116]

(3.2.6) \[ \sum_{i=1}^{n} x_i y_i \geq \left( \sum_{i=1}^{n} x_i ^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} y_i ^q \right)^{\frac{1}{q}} \]

for all \( x_i, y_i > 0 \), \( i = 1, 2, \ldots, n \) and \( \frac{1}{p} + \frac{1}{q} = 1 \), \( p < 1 \ (\neq 0) \), \( q < 0 \) or \( q < 1 \ (\neq 0) \), \( p < 0 \). We see the equality holds iff there exists a positive constant c such that

(3.2.7) \[ x_i ^p = cy_i ^q \]

Making the substitution
\[ x_i = p_i^{\alpha - 1} \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i^\alpha} \right)^{\frac{1}{\alpha - 1}} D^{-l_i}, \quad y_i = p_i^{\alpha - n} \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i^\alpha} \right)^{\frac{1}{\alpha - n}} q_i^{-1} \]

In (3.2.6), we get
\[ \sum_{i=1}^{n} p_i^\beta q_i^{-1} D^{-l_i} \geq \left[ \sum_{i=1}^{n} p_i^\beta \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i^\beta} \right)^{\frac{1}{\alpha}} D^{-l_i \left( \frac{\alpha - 1}{\alpha} \right)} \right]^{\frac{1}{1 - \alpha}} \]

Using the inequality (3.2.3), we get
\[ \left( \sum_{i=1}^{n} p_i^\beta \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i^\beta} \right)^{\frac{1}{\alpha}} D^{-l_i \left( \frac{\alpha - 1}{\alpha} \right)} \right)^{\frac{1}{1 - \alpha}} \geq \left[ \sum_{i=1}^{n} p_i^\beta \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i^\beta} \right) q_i^{-1} \right] \]

Let \( 0 < \alpha < 1 \), raising both sides of (3.2.8) to the power \( \frac{1 - \alpha}{\alpha} \), we get
\[ \left[ \sum_{i=1}^{n} p_i^\beta \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i^\beta} \right)^{\frac{1}{\alpha}} D^{-l_i \left( \frac{\alpha - 1}{\alpha} \right)} \right]^{\frac{1}{1 - \alpha}} \geq \left[ \sum_{i=1}^{n} p_i^\beta \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i^\beta} \right) q_i^{-1} \right] \]

After making suitable operations we get (3.2.4) for \( D_{\alpha}^{\frac{\alpha - 1}{\alpha}} = 0 \) when \( \alpha \neq 1 \). For \( \alpha > 1 \), the proof follows along the similar lines.

**Theorem (3.2.2):** For every code with lengths \( l_1, l_2, \ldots, l_n \) satisfies the condition (3.2.3), \( L_{\alpha}^\beta (U) \) can be made to satisfy the inequality
\[ L_{\alpha}^\beta (U) < I_{\alpha}^\beta (P, Q; U) D_{\alpha}^{\frac{\alpha - 1}{\alpha}} + \frac{1 - D_{\alpha}^{\frac{\alpha - 1}{\alpha}}}{D_{\alpha} - 1} \]

**Proof:** Let \( l_i \) be the positive integer satisfying the inequality
\[ -\log \left( \frac{u_i q_i^\alpha}{\sum_{i=1}^{n} u_i p_i^\beta q_i^{-1}} \right) \leq l_i < -\log \left( \frac{u_i q_i^\alpha}{\sum_{i=1}^{n} u_i p_i^\beta q_i^{-1}} \right) + 1 \]

Consider the interval
\[ \delta_i = \left[ -\log \left( \frac{u_i q_i^\alpha}{\sum_{i=1}^{n} u_i p_i^\beta q_i^{-1}} \right), -\log \left( \frac{u_i q_i^\alpha}{\sum_{i=1}^{n} u_i p_i^\beta q_i^{-1}} \right) + 1 \right] \]
of length 1. In every \( \delta_i \), there lies exactly one positive integer \( l_i \) such that
(3.2.12) \[ 0 < -\log \left( \frac{u_i q_i^\alpha}{\sum_{i=1}^{n} u_i p_i^\alpha q_i^{\alpha-1}} \right) \leq l_i \leq -\log \left( \frac{u_i q_i^\alpha}{\sum_{i=1}^{n} u_i p_i^\alpha q_i^{\alpha-1}} \right) + 1 \]

We will first show that the sequence \( l_1, l_2, \ldots, l_n \) thus defined satisfies (3.2.3). From (3.2.12), we have

\[ -\log \left( \frac{u_i q_i^\alpha}{\sum_{i=1}^{n} u_i p_i^\alpha q_i^{\alpha-1}} \right) \leq l_i \]

or

\[ \left( \frac{u_i q_i^\alpha}{\sum_{i=1}^{n} u_i p_i^\alpha q_i^{\alpha-1}} \right) \geq D^{-l_i} \]

Multiply both sides by \( p_i^\beta q_i^{-1} \) and summing over \( i = 1, 2, \ldots, n \) we get (3.2.3). The last inequality of (3.2.12) gives

\[ l_i \leq -\log \left( \frac{u_i q_i^\alpha}{\sum_{i=1}^{n} u_i p_i^\alpha q_i^{\alpha-1}} \right) + 1 \]

or

\[ D^{l_i} < \left( \frac{u_i q_i^\alpha}{\sum_{i=1}^{n} u_i p_i^\alpha q_i^{\alpha-1}} \right)^{-1} D \]

Let \( 0 < \alpha < 1 \), raising both sides to the power \( \left( \frac{1-\alpha}{\alpha} \right) \), we get

\[ D^{-l_i} \left( \frac{\alpha}{1-\alpha} \right) < \left( \frac{u_i q_i^\alpha}{\sum_{i=1}^{n} u_i p_i^\alpha q_i^{\alpha-1}} \right)^{\frac{\alpha}{1-\alpha}} D \frac{\alpha}{1-\alpha} \]

Multiply both sides by \( p_i^\beta \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i^\alpha} \right)^{\frac{1}{\alpha}} \) and summing over \( i = 1, 2, \ldots, n \) and after suitable operations, we get

(3.2.13) \[ I_0^\beta (U) < I_0^\beta (P, Q; U) D^{1-\alpha} + \frac{1-D^{1-\alpha}}{D^{1-\alpha}-1} \]

Again, bounds have been obtained by considering a more generalized inaccuracy measure of order \( \alpha \) and type \( \beta \). The main aim of studying this new function is that it generalizes some information measures already existing in the literature. This new function can be used for more complex distributions other than exponential.

Consider a function
\[ I_\alpha^\beta (P, Q; U) = \frac{1}{1-\alpha} \log \frac{\sum_{i=1}^{n} u_i^\alpha p_i^\beta q_i^{-1}}{\sum_{i=1}^{n} u_i^\alpha p_i^\beta}, \quad \alpha > 0 \neq 1, \beta > 0 \]

**Remark (3.2.3)**

(1) When \( \beta = 1 \), (3.2.14) reduces to the measure given by Bhatia [23].

(2) When \( \beta = 1 \), \( p_i = q_i \) \( \forall \, i = 1, 2, \ldots, n \). (3.2.14) reduces to measure given by Taneja, Hooda and Bhaker [120].

(3) When \( p_i = q_i \) \( \forall \, i = 1, 2, \ldots, n \). (3.2.14) reduces to the measure given by Hooda and Bhaker [59], further it reduces to Renyi’s [104] entropy when \( \beta = 1 \), \( u_i = 1 \) \( \forall \, i = 1, 2, \ldots, n \).

Further, we define a parametric codeword mean length credited with utilities and probabilities as

\[ L_\alpha^\beta (t) = \frac{1}{t} \log \frac{\sum_{i=1}^{n} u_i^\alpha p_i^\beta t^v_i}{\sum_{i=1}^{n} u_i^\alpha p_i^\beta}, \quad -1 < t < \infty, \, t \neq 0, \beta > 0 \]

**Remark (3.2.4)**

(1) When \( \beta = 1 \), \( t \to 0 \), (3.2.15) reduces to \( L (U) \) given in (3.1.6).

Now, we establish a result that in a way, gives a characterization of \( I_\alpha^\beta (P, Q; U) \), under the condition

\[ \sum_{i=1}^{n} u_i^\alpha p_i^\beta t^{-l_i} D^{-l_i} \leq \sum_{i=1}^{n} u_i^\alpha p_i^\beta \]

which is generalization of Kraft [80] inequality. Also D is the size of the code alphabet.

**Theorem (3.2.3):** For all integers \( D (D > 1) \). Let \( l_i \) satisfies the condition (3.2.16), then the generalized ‘useful’ codeword mean length satisfies

\[ L_\alpha^\beta (t) \geq I_\alpha^\beta (P, Q; U) \]

where \( \alpha = \frac{1}{1+t} \), equality holds iff

\[ l_i = -\log q_i^\alpha + \log \frac{\sum_{i=1}^{n} u_i^\alpha p_i^\beta q_i^{-1}}{\sum_{i=1}^{n} u_i^\alpha p_i^\beta} \]

**Proof:** By Holder’s inequality [116]
(3.2.19) \[ \sum_{i=1}^{n} x_i y_i \geq \left( \frac{\sum_{i=1}^{n} x_i^p}{\sum_{i=1}^{n} y_i^q} \right)^{\frac{1}{p}} \left( \frac{\sum_{i=1}^{n} y_i^q}{\sum_{i=1}^{n} y_i^q} \right)^{\frac{1}{q}} \]

for all \( x_i, y_i > 0, \ i = 1, 2, \ldots, n \) and \( \frac{1}{p} + \frac{1}{q} = 1, p < 1 (\neq 0), q < 0 \) or \( q < 1 (\neq 0), p < 0 \). We see the equality holds if there exists a positive constant \( c \) such that

(3.2.20) \[ x_i^p = cy_i^q \]

Making the substitution

\[ x_i = \left[ \frac{u_i^p q_i^l}{\sum_{i=1}^{n} u_i^p q_i^l} \right]^{\frac{1}{l}}, \ y_i = \left[ \frac{u_i^q p_i^t q_i^{n-1}}{\sum_{i=1}^{n} u_i^q p_i^t} \right] \]

in (3.2.19), we get

\[ \sum_{i=1}^{n} x_i y_i \geq \left[ \frac{\sum_{i=1}^{n} u_i^p q_i^l D_s^{l-t}}{\sum_{i=1}^{n} u_i^p q_i^l} \right] \leq \left[ \frac{\sum_{i=1}^{n} u_i^q p_i^t q_i^{n-1}}{\sum_{i=1}^{n} u_i^q p_i^t} \right] \]

using the inequality (3.2.16), we get

\[ \left[ \frac{\sum_{i=1}^{n} u_i^p q_i^l D_s^{l-t}}{\sum_{i=1}^{n} u_i^p q_i^l} \right] \leq \left[ \frac{\sum_{i=1}^{n} u_i^q p_i^t q_i^{n-1}}{\sum_{i=1}^{n} u_i^q p_i^t} \right] \]

Taking logarithms to both sides with base \( D \), we obtain (3.2.17).

**Theorem (3.2.4).** For every code with lengths \( l_1, l_2, \ldots, l_n \) satisfies the condition (3.2.16), \( L_{\alpha}^t \) can be made to satisfy the inequality

(3.2.21) \[ L_{\alpha}^t < I_{\alpha}^t (P, Q; U) + 1 \]

**Proof:** Let \( l_i \) be the positive integer satisfying the inequality

(3.2.22) \[ -\log q_i^a + \log \frac{\sum_{i=1}^{n} u_i^p q_i^l q_i^{n-1}}{\sum_{i=1}^{n} u_i^p q_i^l} \leq l_i < -\log q_i^a + \log \frac{\sum_{i=1}^{n} u_i^q p_i^t q_i^{n-1}}{\sum_{i=1}^{n} u_i^q p_i^t} + 1 \]

Consider the interval

(3.2.23) \[ \delta_i = \left[ -\log q_i^a + \log \frac{\sum_{i=1}^{n} u_i^p q_i^l q_i^{n-1}}{\sum_{i=1}^{n} u_i^p q_i^l}, \ -\log q_i^a + \log \frac{\sum_{i=1}^{n} u_i^q p_i^t q_i^{n-1}}{\sum_{i=1}^{n} u_i^q p_i^t} + 1 \right] \]

of length 1. In every \( \delta_i \), there lies exactly one positive integer \( l_i \) such that
(3.2.24) \[ 0 < -\log q_i^\alpha + \log \frac{\sum_{i=1}^n u_i^\beta p_i^\beta q_i^{\alpha - 1}}{\sum_{i=1}^n u_i^\beta p_i^\beta} \leq l_i < -\log q_i^\alpha + \log \frac{\sum_{i=1}^n u_i^\beta p_i^\beta q_i^{\alpha - 1}}{\sum_{i=1}^n u_i^\beta p_i^\beta} + 1 \]

We will first show that the sequence \( l_1, l_2, \ldots, l_n \) thus defined satisfy (3.2.16). From (3.2.24) we have

\[ -\log q_i^\alpha + \log \frac{\sum_{i=1}^n u_i^\beta p_i^\beta q_i^{\alpha - 1}}{\sum_{i=1}^n u_i^\beta p_i^\beta} \leq l_i \]

or

\[ q_i^{-\alpha} \frac{\sum_{i=1}^n u_i^\beta p_i^\beta q_i^{\alpha - 1}}{\sum_{i=1}^n u_i^\beta p_i^\beta} \leq D_i \]

\[ q_i^\alpha \frac{\sum_{i=1}^n u_i^\beta p_i^\beta}{\sum_{i=1}^n u_i^\beta p_i^\beta} \geq D^{-l_i} \]

Multiply both sides by \( u_i^\beta p_i^\beta q_i^{\alpha - 1} \) and summing over \( i = 1, 2, \ldots, n \) we get (3.2.16). The last inequality of (3.2.24) gives

\[ l_i < -\log q_i^\alpha + \log \frac{\sum_{i=1}^n u_i^\beta p_i^\beta q_i^{\alpha - 1}}{\sum_{i=1}^n u_i^\beta p_i^\beta} + 1 \]

or

\[ D^{l_i} < q_i^{-\alpha} \frac{\sum_{i=1}^n u_i^\beta p_i^\beta q_i^{\alpha - 1}}{\sum_{i=1}^n u_i^\beta p_i^\beta} D \]

Raising both sides to the power \( t \) and multiplying both sides by \( \frac{\sum_{i=1}^n u_i^\beta p_i^\beta}{\sum_{i=1}^n u_i^\beta p_i^\beta} \) and also summing over \( i = 1, 2, \ldots, n \), and simplifying, we get (3.2.21).

In the next section, bounds have been obtained by considering another type of inaccuracy measure of order \( \alpha \) and type \( \beta \). This work has been published in “Sarajevo Journal of Mathematics”, Vol 3(1), PP 137-143 (2007)(Baig and Rayees [12]). The function considered here is also suitable for the distributions other than exponential.

### 3.3. Noiseless coding theorems of inaccuracy measure of order \( \alpha \) and type \( \beta \).

Consider a function

\[ I_\alpha^\beta (P; Q; U) = \frac{1}{1-\alpha} \log \left[ \frac{\sum_{i=1}^n u_i^\beta p_i^\beta q_i^{(\alpha-1)}}{\sum_{i=1}^n u_i^\beta p_i^\beta} \right] , \quad \alpha > 0 \neq 1, \beta > 0 \]

**Remark (3.3.1)**
(1) When $\beta = 1$, (3.3.1) reduces to ‘useful’ information measure of order $\alpha$ due to Bhatia [23].

(2) When $\beta = 1, u_i = 1 \ \forall \ i = 1, 2, ..., n$, (3.3.1) reduces to the inaccuracy measure given by Nath [90], further it reduces to Renyi’s [104] entropy by taking $p_i = q_i \ \forall \ i = 1, 2, ..., n$.

(3) When $\beta = 1, u_i = 1 \ \forall \ i = 1, 2, ..., n$ and $\alpha \to 1$. Then (3.3.1) reduces to the measure due to Kerridge [73].

(4) When $u_i = 1 \ \forall \ i = 1, 2, ..., n$ and $p_i = q_i \ \forall \ i = 1, 2, ..., n$. The measure (3.3.1) becomes the entropy for the $\beta$ power distribution derived from P studied by Roy [105]. We call $I_\alpha^\beta (P; Q; U)$ in (3.3.1) the generalized ‘useful’ inaccuracy measure of order $\alpha$ and type $\beta$.

Further, we define a parametric codeword mean length credited with utilities and probabilities as

$$L_\beta^t (U) = \frac{1}{t} \log \left[ \frac{\sum_{i=1}^n u_i^{t+1} p_i^{\beta} D^{t \alpha}}{\left( \sum_{i=1}^n u_i p_i^\beta \right)^{t+1}} \right], \ t > -1, t \neq 0, \beta > 0$$

**Remark (3.3.2).**

(1) When $\beta = 1$, $L_\beta^t (U)$ in (3.3.2) reduces to ‘useful’ mean length $L^t (U)$ of the code given by Bhatia [23].

(2) When $\beta = 1, u_i = 1 \ \forall \ i = 1, 2, ..., n$, $L_\beta^t (U)$ in (3.3.2) reduces to the code mean length given by Campbell [29].

(3) When $\beta = 1, u_i = 1 \ \forall \ i = 1, 2, ..., n$ and $\alpha \to 1$. $L_\beta^t (U)$ in (3.3.2) reduces to the optimal code length identical to Shannon [107].

(4) When $u_i = 1 \ \forall \ i = 1, 2, ..., n$, $L_\beta^t (U)$ in (3.3.2) reduces to the codeword mean length given by Khan and Haseen [74].

Now, we find the bounds for $L_\beta^t (U)$ in terms of $I_\alpha^\beta (P; Q; U)$ under the condition

$$\sum_{i=1}^n p_i^\beta q_i^{-\beta} D^{-t_i} \leq 1$$

where $D$ is the size of the code alphabet. Also (3.3.3) is a generalization of Kraft [80] inequality.

**Theorem (3.3.1).** For every code whose lengths $l_1, l_2, ..., l_n$ satisfies the condition
(3.3.3). Then the code mean length satisfies

(3.3.4) \[ L^t_\beta(U) \geq I^\beta_\alpha(P, Q; U) \]

where \( \alpha = \frac{1}{1+\epsilon} \), equality holds iff

(3.3.5) \[ l_i = -\log \frac{u_i q_i^{\alpha \beta}}{\sum_{i=1}^{n} u_i p_i q_i^{\alpha \beta(\alpha - 1)}} \]

Proof: By Holder’s inequality [116]

(3.3.6) \[ \sum_{i=1}^{n} x_i y_i \geq \left( \sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} y_i^q \right)^{\frac{1}{q}} \]

for all \( x_i, y_i > 0, i = 1, 2, \ldots, n \) and \( \frac{1}{p} + \frac{1}{q} = 1, p < 1 (\neq 0), q < 0 \) or \( q < 1 (\neq 0), p < 0 \). We see the equality holds iff there exists a positive constant \( c \) such that

(3.3.7) \[ x_i^p = cy_i^q \]

Making the substitution

\[ x_i = u_i \left( \frac{1 + t}{1 + t} \right)^{\frac{1}{p}} \left( \frac{1}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{p}} \]

\[ y_i = u_i \left( \frac{1 + t}{1 + t} \right)^{\frac{1}{q}} \left( \frac{1}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{q}} \]

\[ p = -t, \quad q = \frac{t}{1 + t} \]

in (3.3.6) and using (3.3.3), we get

\[ \left[ \frac{\sum_{i=1}^{n} u_i p_i q_i^{\alpha \beta}}{\sum_{i=1}^{n} u_i p_i} \right]^{\frac{1}{t}} \geq \left[ \frac{\sum_{i=1}^{n} u_i p_i q_i^{\alpha \beta(\alpha - 1)}}{\sum_{i=1}^{n} u_i p_i} \right]^{\frac{1}{t}} \]

Taking logarithms to both sides with base \( D \), we obtain (3.3.4).

**Theorem (3.3.2)**. For every code with lengths \( l_1, l_2, \ldots, l_n \) satisfies the condition (3.3.3), \( I^t_\beta(U) \) can be made to satisfy the inequality

(3.3.8) \[ I^t_\beta(U) < I^\beta_\alpha(P, Q; U) + 1 \]

Proof: Let \( l_i \) be the positive integer satisfying

(3.3.9) \[ -\log \frac{u_i q_i^{\alpha \beta}}{\sum_{i=1}^{n} u_i p_i q_i^{\alpha \beta(\alpha - 1)}} \leq l_i < -\log \frac{u_i q_i^{\alpha \beta}}{\sum_{i=1}^{n} u_i p_i q_i^{\alpha \beta(\alpha - 1)}} + 1 \]

Consider the interval

54
\[(3.3.10) \quad \delta_i = \left[ -\log \frac{u_i q_i^{\alpha \beta}}{\sum_{i=1}^{n} u_i p_i \beta q_i^{\beta (\alpha - 1)}}, \quad -\log \frac{u_i q_i^{\alpha \beta}}{\sum_{i=1}^{n} u_i p_i \beta q_i^{(\alpha - 1)}} + 1 \right] \]

of length 1. In every \( \delta_i \), there lies exactly one positive integer \( l_i \) such that

\[(3.3.11) \quad 0 < -\log \frac{u_i q_i^{\alpha \beta}}{\sum_{i=1}^{n} u_i p_i \beta q_i^{\beta (\alpha - 1)}} \leq l_i < -\log \frac{u_i q_i^{\alpha \beta}}{\sum_{i=1}^{n} u_i p_i \beta q_i^{(\alpha - 1)}} + 1 \]

We will first show that the sequence \( l_1, l_2, \ldots, l_n \) thus defined satisfies (3.3.3). From

(3.3.11) we have

\[-\log \frac{u_i q_i^{\alpha \beta}}{\sum_{i=1}^{n} u_i p_i \beta q_i^{\beta (\alpha - 1)}} \leq l_i\]

or

\[\frac{u_i q_i^{\alpha \beta}}{\sum_{i=1}^{n} u_i p_i \beta q_i^{\beta (\alpha - 1)}} \geq D^{-l_i}\]

Multiply both sides by \( p_i^{\beta} q_i^{-\beta} \) and summing over \( i = 1, 2, \ldots, n \), we get (3.3.3). The last inequality in (3.3.11) gives

\[l_i < -\log \frac{u_i q_i^{\alpha \beta}}{\sum_{i=1}^{n} u_i p_i \beta q_i^{\beta (\alpha - 1)}} + 1\]

or

\[D^{l_i} < \left( \frac{\sum_{i=1}^{n} u_i q_i^{\alpha \beta}}{\sum_{i=1}^{n} u_i p_i \beta q_i^{\beta (\alpha - 1)}} \right)^{-l_i} D^l\]

Multiplying both sides by \( \frac{\sum_{i=1}^{n} u_i^{l+1} p_i^{\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta}} \) and summing over \( i = 1, 2, \ldots, n \). we get

\[\frac{\sum_{i=1}^{n} u_i^{l+1} p_i^{\beta} D^{l_i}}{\left( \sum_{i=1}^{n} u_i p_i^{\beta} \right)^{l+1}} < \left[ \frac{\sum_{i=1}^{n} u_i p_i^{\beta} q_i^{\beta (\alpha - 1)}}{\sum_{i=1}^{n} u_i p_i^{\beta}} \right]^{l+1} D^l\]

Taking logarithms to both sides with base \( D \) and then dividing both sides by \( t \), we obtain (3.3.8).

### 3.4. Some results on weighted parametric information measures.

In this section, two generalized measures of information are considered and their bounds are obtained. The results obtained by considering first measure has been presented in the 2nd J&K science congress held in University of Kashmir, Srinagar (2006) (Baig and Rayees [9]).

Consider a ‘useful’ inaccuracy measure of order \( \beta \) given by Om Parkash [93]
\[ I_n^\beta (P, Q; U) = \frac{1}{2-\beta-1} \left[ \sum_{i=1}^{n} u_i p_i \left( q_i^\beta - 1 \right) \right], \beta \neq 0 \]

**Remark (3.4.1):**

1. When \( \beta \to 0 \), \( I_n^\beta (P, Q; U) \) reduces to ‘useful’ inaccuracy measure given by Taneja and Tuteja [119].

2. When \( \beta \to 0 \) and \( u_i = 1 \ \forall \ i = 1, 2, \ldots, n, \) \( I_n^\beta (P, Q; U) \) reduces to measure of inaccuracy given by Kerridge [73].

Further, consider a parametric codeword mean length

\[ L^\beta (U) = \frac{1}{2-\beta-1} \left[ \left( \frac{\sum_{i=1}^{n} u_i p_i D^{-\beta} \left( \frac{1}{q_i^\beta} \right) }{\sum_{i=1}^{n} u_i p_i} \right)^{\beta+1} \right]^{\frac{1}{\beta+1}}, \beta \neq 0 \]

**Remark (3.4.2):**

1. When \( \beta \to 0 \), \( L^\beta (U) \) reduces to the ‘useful’ codeword mean length \( L (U) \) given by Guiasu and Picard [51].

2. When \( \beta \to 0 \) and \( u_i = 1 \ \forall \ i = 1, 2, \ldots, n, \) \( L^\beta (U) \) becomes the optimal code length defined by Shannon [107].

In the following theorem, we obtain lower bound for \( L^\beta (U) \) in terms of \( I_n^\beta (P, Q; U) \) under the condition

\[ \sum_{i=1}^{n} u_i p_i q_i^{-1} D^{-l_i} \leq \sum_{i=1}^{n} u_i p_i \]

**Theorem (3.4.1):** If \( l_1, l_2, \ldots, l_n \) be the lengths of the code satisfying the inequality (3.4.3). Then the mean codeword length satisfies

\[ L^\beta (U) \geq \frac{I_n^\beta (P, Q; U)}{\overline{U}}, \beta \neq 0 \]

where \( \overline{U} = \sum_{i=1}^{n} u_i p_i \), equality holds iff

\[ l_i = -\log q_i^\beta + 1 + \log \frac{\sum_{i=1}^{n} u_i p_i q_i^\beta}{\sum_{i=1}^{n} u_i p_i} \]

**Proof:** By Holder’s inequality [116]

\[ \sum_{i=1}^{n} x_i y_i \geq \left( \sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} y_i^q \right)^{\frac{1}{q}} \]

56
for all \( x_i, y_i > 0, i = 1, 2, \ldots, n \) and \( \frac{1}{p} + \frac{1}{q} = 1, p < 1(\neq 0), q < 0 \) or \( q < 1(\neq 0), p < 0 \). We see the equality holds iff there exists a positive constant \( c \) such that

\[(3.4.7) \quad x_i^p = cy_i^q\]

Making the substitution

\[
x_i = \left( \frac{u_i p_i}{\sum_{i=1}^n u_i p_i} \right)^{\frac{1}{p-1}} D^{-l_i}, \quad y_i = \left( \frac{u_i p_i}{\sum_{i=1}^n u_i p_i} \right)^{\frac{1}{q-1}} q_i^{-1}
\]

\[
p = \frac{\beta}{\beta+1}, \quad q = -\beta
\]

in (3.4.6), we get

\[
\frac{\sum_{i=1}^n u_i p_i q_i^{-1} D^{-l_i}}{\sum_{i=1}^n u_i p_i} \geq \left[ \frac{\sum_{i=1}^n u_i p_i D^{-l_i} \left( \frac{1}{\beta+1} \right)}{\sum_{i=1}^n u_i p_i} \right]^{\frac{1}{\beta}} \left[ \frac{\sum_{i=1}^n u_i p_i q_i}{\sum_{i=1}^n u_i p_i} \right]^{-\frac{1}{\beta}}
\]

using the inequality (3.4.3), we get

\[(3.4.8) \quad \left[ \frac{\sum_{i=1}^n u_i p_i D^{-l_i} \left( \frac{1}{\beta+1} \right)}{\sum_{i=1}^n u_i p_i} \right]^{\frac{1}{\beta+1}} \leq \left[ \frac{\sum_{i=1}^n u_i p_i q_i}{\sum_{i=1}^n u_i p_i} \right]^{-\frac{1}{\beta}}
\]

For \( \beta > 0 \), raising both sides of (3.4.8) to the power \( (-\beta) \), we get

\[(3.4.9) \quad \left[ \frac{\sum_{i=1}^n u_i p_i D^{-l_i} \left( \frac{1}{\beta+1} \right)}{\sum_{i=1}^n u_i p_i} \right]^{\beta+1} \leq \left[ \frac{\sum_{i=1}^n u_i p_i q_i}{\sum_{i=1}^n u_i p_i} \right]^{-\beta}
\]

Since \( 2^{-\beta} - 1 < 0 \) for \( \beta > 0 \), a simple manipulation proves (3.4.4) for \( \beta > 0 \). The proof for \( \beta < 0 \) follows on the same lines.

**Theorem (3.4.2).** For every code with lengths \( l_1, l_2, \ldots, l_n \) satisfies the condition (3.4.3), \( L^\beta (U) \) can be made to satisfy the inequality

\[(3.4.10) \quad L^\beta (U) < \frac{\sum_{i=1}^n u_i p_i D^{-l_i}}{\beta} + \frac{\beta-1}{2}, \quad \beta \neq 0
\]

**Proof:** Let \( l_i \) be the positive integer satisfying the inequality

\[(3.4.11) \quad -\log q_i^{\beta+1} + \log \frac{\sum_{i=1}^n u_i p_i q_i^\beta}{\sum_{i=1}^n u_i p_i} \leq l_i \leq -\log q_i^{\beta-1} + \log \frac{\sum_{i=1}^n u_i p_i q_i^\beta}{\sum_{i=1}^n u_i p_i} + 1
\]

Consider the interval
\[
\delta_i = \left[ -\log q_i^{\beta+1} + \log \frac{\sum_{i=1}^{n} u_i p_i q_i^\beta}{\sum_{i=1}^{n} u_i p_i}, -\log q_i^{\beta+1} + \log \frac{\sum_{i=1}^{n} u_i p_i q_i^\beta}{\sum_{i=1}^{n} u_i p_i} + 1 \right]
\]
of length 1. In every \( \delta_i \), there lies exactly one positive integer \( l_i \) such that

\[
0 < -\log q_i^{\beta+1} + \log \frac{\sum_{i=1}^{n} u_i p_i q_i^\beta}{\sum_{i=1}^{n} u_i p_i} \leq l_i < -\log q_i^{\beta+1} + \log \frac{\sum_{i=1}^{n} u_i p_i q_i^\beta}{\sum_{i=1}^{n} u_i p_i} + 1
\]

We will first show that sequence \( \{l_1, l_2, \ldots, l_n\} \), thus defined satisfies (3.4.3), we have

\[
-\log q_i^{\beta+1} + \log \frac{\sum_{i=1}^{n} u_i p_i q_i^\beta}{\sum_{i=1}^{n} u_i p_i} \leq l_i
\]
or

\[
-\log q_i^{\beta+1} + \log \frac{\sum_{i=1}^{n} u_i p_i q_i^\beta}{\sum_{i=1}^{n} u_i p_i} \leq -\log D^{-l_i}
\]

(3.4.14)

\[
\frac{q_i^{\beta+1}}{\left( \frac{\sum_{i=1}^{n} u_i p_i q_i^\beta}{\sum_{i=1}^{n} u_i p_i} \right)^{l_i}} \geq D^{-l_i}
\]

Multiplying both sides of (3.4.14) by \( u_i p_i q_i^{-1} \) and summing over \( i = 1, 2, \ldots, n \), we get (3.4.3). The last inequality of (3.4.13) gives

\[
l_i < -\log q_i^{\beta+1} + \log \frac{\sum_{i=1}^{n} u_i p_i q_i^\beta}{\sum_{i=1}^{n} u_i p_i} + 1
\]

(3.4.15)

\[
D^{l_i} < q_i^{-(\beta+1)} \frac{\sum_{i=1}^{n} u_i p_i q_i^\beta}{\sum_{i=1}^{n} u_i p_i} D
\]

For \( \beta > 0 \), raising both sides of (3.4.15) to the power \( -\left( \frac{\beta}{\beta+1} \right) \), we get

\[
D^{-l_i} \left( \frac{\sum_{i=1}^{n} u_i p_i q_i^\beta}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{-\beta}{\beta+1}} > q_i^{\beta} \left( \frac{\sum_{i=1}^{n} u_i p_i q_i^\beta}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{-\beta}{\beta+1}} D^{-\frac{\beta}{\beta+1}}
\]

(3.4.16)

Multiplying both sides of (3.4.16) by \( \frac{u_i p_i}{\sum_{i=1}^{n} u_i p_i} \) and summing over \( i = 1, 2, \ldots, n \), then raising both sides to the power \( (\beta + 1) \), we get

(3.4.17)

\[
\left[ \frac{\sum_{i=1}^{n} u_i p_i D^{-l_i} \left( \frac{\sum_{i=1}^{n} u_i p_i q_i^\beta}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{-\beta}{\beta+1}}}{\sum_{i=1}^{n} u_i p_i} \right]^{\beta+1} > \frac{\sum_{i=1}^{n} u_i p_i q_i^\beta}{\sum_{i=1}^{n} u_i p_i} D^{-\beta}
\]
Since $2^{-\beta} - 1 < 0$ for $\beta > 0$, a simple manipulation in (3.4.17) gives

\[ L^\beta (U) < \frac{I^\beta (P, Q; U)}{\beta} + \frac{D_{\beta \rightarrow 1}^{-\beta} - 1}{2^{\beta - 1}}, \quad \beta \neq 0 \]

For $\beta < 0$, the proof follows on the same lines.

Now, we consider weighted parametric measure involving utilities and the bounds have been obtained.

Consider a function

\[ I^\beta_{\alpha} (P, Q; U) = \frac{1}{1 - D(1 - \alpha, \beta)} \left[ 1 - \left( \frac{\sum_{i=1}^{n} u_i q_i^{-1}}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{\alpha}} \right], \quad \alpha > 0 (\neq 1), \beta \neq 0 \]

**Remark (3.4.3).**

(1) When $\beta \to 0$, (3.4.18) reduces to the measure given by Bhatia [23].

(2) When $\beta = \frac{1}{\alpha}$, (3.4.18) reduces to the measure given by Bhatia [22].

(3) When $\beta = \frac{1}{\alpha}$, $p_i = q_i \forall i = 1, 2, ..., n$, (3.4.18) reduces to the measure given by Autar and Khan [6], which can be further reduced to the entropy given by Shannon [107] by taking $\alpha \to 1$ and $u_i = 1 \forall i = 1, 2, ..., n$.

Further, consider a parametric ‘useful’ codeword mean length

\[ L^\beta_{\alpha} (U) = \frac{1}{1 - D(1 - \alpha, \beta)} \left[ 1 - \left\{ \sum_{i=1}^{n} p_i \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{\alpha}} D^{-i_{\alpha}} \left( \frac{\alpha - 1}{\alpha} \right)^{\alpha^\beta} \right\} \right] \]

where $\alpha > 0 (\neq 1), \beta \neq 0$.

**Remark (3.4.4).**

(1) When $\beta \to 0$, (3.4.19) reduces to the codeword mean length given by Gurdial and Pessoa [53].

(2) When $\beta = \frac{1}{\alpha}$, (3.4.19) reduces to the codeword mean length given by Autar and Khan [6], which can be further reduced to ordinary codeword mean length given by Shannon [107] by taking $\alpha \to 1$, $u_i = 1 \forall i = 1, 2, ..., n$.

Now we find the bounds of $L^\beta_{\alpha} (U)$ in terms of $I^\beta_{\alpha} (P, Q; U)$ under the condition

\[ \sum_{i=1}^{n} p_i q_i^{-1} D^{-i_{\alpha}} \leq 1 \]

59
where $D$ is the size of the code alphabet.

**Theorem (3.4.3).** For every code with lengths $l_1, l_2, \ldots, l_n$ satisfies the condition (3.4.20). Then generalized codeword mean length satisfies

\begin{equation}
I_{\alpha}^{*}(U) \geq I_{\alpha}^{*}(P;Q;U)
\end{equation}

equality holds iff

\begin{equation}
l_i = -\log \frac{u_iq_{i}^{\alpha}}{\sum_{i=1}^{n} u_i p_i q_{i}^{\alpha - 1}}
\end{equation}

**Proof:** By Holder’s inequality [116]

\begin{equation}
\sum_{i=1}^{n} x_i y_i \geq \left( \sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} y_i^q \right)^{\frac{1}{q}}
\end{equation}

for all $x_i, y_i > 0$, $i = 1, 2, \ldots, n$ and $\frac{1}{p} + \frac{1}{q} = 1, p < 1 \neq 0, q < 0$ or $q < 1 \neq 0, p < 0$. We see the equality holds iff there exists a positive constant $c$ such that

\begin{equation}
x_i^p = cy_i^q
\end{equation}

Making the substitution

\[p = \frac{\alpha}{\alpha - 1}, \quad q = 1 - \alpha\]

\[x_i = p_i^{\frac{\alpha}{\alpha - 1}} \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{\alpha - 1}} D^{-l_i}, \quad y_i = p_i^{\frac{1}{\alpha - 1}} \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{\alpha - 1}} q_i^{-1}
\]

in (3.4.23) and using (3.4.20), we get

\begin{equation}
\sum_{i=1}^{n} p_i \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{\alpha - 1}} D^{-l_i} \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{\alpha - 1}{\alpha}} \geq \left[ \sum_{i=1}^{n} u_i p_i q_i^{\alpha - 1} \right]^{\frac{1}{\alpha}}
\end{equation}

**Case 1:** For $\alpha > 1, \beta > 0$, equation (3.4.25) becomes

\begin{equation}
\left[ \sum_{i=1}^{n} u_i p_i q_i^{\alpha - 1} \right]^{\beta} \geq \left[ \sum_{i=1}^{n} p_i \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{\alpha}} D^{-l_i} \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{\alpha - 1}{\alpha}} \right]^{\alpha \beta}
\end{equation}

Since $1 - D^{(1-\alpha)\beta} > 0$, we have

\[I_{\alpha}^{*}(U) \geq I_{\alpha}^{*}(P;Q;U)\]

**Case 2:** For $\alpha > 1, \beta < 0$, equation (3.4.25) becomes

60
\[
\left[ \sum_{i=1}^{n} \frac{u_i p_i q_i^{\alpha-1}}{\sum_{i=1}^{n} u_i p_i} \right]^\beta \leq \left[ \sum_{i=1}^{n} p_i \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i} \right) \right] \frac{1}{\alpha} D^{-l_i \left( \frac{n-1}{\alpha} \right)} \alpha \beta
\]

Since \(1 - D^{(1-\alpha)\beta} < 0\), we have
\[
I_\alpha^\beta (U) \geq I_\alpha^\beta (P; Q; U)
\]

**Case 3:** For \(\alpha < 1, \beta > 0\), equation (3.4.25) becomes
\[
\left[ \sum_{i=1}^{n} \frac{u_i p_i q_i^{\alpha-1}}{\sum_{i=1}^{n} u_i p_i} \right]^\beta \leq \left[ \sum_{i=1}^{n} p_i \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i} \right) \right] \frac{1}{\alpha} D^{-l_i \left( \frac{n-1}{\alpha} \right)} \alpha \beta
\]

Since \(1 - D^{(1-\alpha)\beta} < 0\), we have
\[
I_\alpha^\beta (U) \geq I_\alpha^\beta (P; Q; U)
\]

**Case 4:** For \(\alpha < 1, \beta < 0\), equation (3.4.25) becomes
\[
\left[ \sum_{i=1}^{n} \frac{u_i p_i q_i^{\alpha-1}}{\sum_{i=1}^{n} u_i p_i} \right]^\beta \geq \left[ \sum_{i=1}^{n} p_i \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i} \right) \right] \frac{1}{\alpha} D^{-l_i \left( \frac{n-1}{\alpha} \right)} \alpha \beta
\]

Since \(1 - D^{(1-\alpha)\beta} > 0\), we have
\[
I_\alpha^\beta (U) \geq I_\alpha^\beta (P; Q; U)
\]

**Theorem (3.4.4):** For every code with lengths \(l_1, l_2, \ldots, l_n\) satisfies the condition (3.4.20), \(I_\alpha^\beta (U)\) can be made to satisfy
\[
I_\alpha^\beta (P; Q; U) \leq I_\alpha^\beta (U) < D^{(1-\alpha)\beta} I_\alpha^\beta (P; Q; U) + 1
\]

**Proof:** In general we can not hope to construct an absolute optimal code for a given set of probabilities \(p_1, p_2, \ldots, p_n\). Since if we choose \(l_i\) satisfy (3.4.22) then \(l_i\) may not be an integer. However, can do the next thing and select the integer \(\overline{l}_i\) such that
\[
l_i \leq \overline{l}_i < l_i + 1
\]
\[
- \log \frac{u_i q_i^{\alpha}}{\sum_{i=1}^{n} u_i p_i q_i^{\alpha-1}} \leq \overline{l}_i < - \log \frac{u_i q_i^{\alpha}}{\sum_{i=1}^{n} u_i p_i q_i^{\alpha-1}} + 1
\]

We claim that prefix code can be constructed with word lengths \(\overline{l}_1, \overline{l}_2, \ldots, \overline{l}_n\). To prove this we must show that sequences \(l_1, \overline{l}_2, \ldots, \overline{l}_n\) satisfies (3.4.20). From left hand inequality of (3.4.28), it follows that
\[
\frac{u_i q_i^{\alpha}}{\sum_{i=1}^{n} u_i p_i q_i^{\alpha-1}} \geq D^{-\overline{l}_i}
\]
Multiplying both sides by $p_i q_i^{-1}$ and summing over $i = 1, 2, \ldots, n$, we get (3.4.20). Considering $L_\alpha^\beta (U)$ as a function of $l_1, l_2, \ldots, l_n$ only and using differentiable technique it can be easily proved that $L_\alpha^\beta (U)$ is an increasing function of $l_1, l_2, \ldots, l_n$.

From (3.4.27), we have

$$\frac{1}{1-D^{(1-\alpha)\beta}} \left[ 1 - \left( \sum_{i=1}^{n} p_i \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i} \right) \right) \right]$$

$$\leq \frac{1}{1-D^{(1-\alpha)\beta}} \left[ 1 - \left( \sum_{i=1}^{n} p_i \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i} \right) \right) \right]$$

$$\leq \frac{1}{1-D^{(1-\alpha)\beta}} \left[ 1 - D^{(1-\alpha)\beta} \left( \sum_{i=1}^{n} p_i \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i} \right) \right) \right]$$

Clearly,

$$\left[ \sum_{i=1}^{n} p_i \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i} \right) \right]^{\alpha \beta} = \left[ \sum_{i=1}^{n} u_i p_i q_i^{\alpha-1} \right]$$

We get

$$\frac{1}{1-D^{(1-\alpha)\beta}} \left[ 1 - \left( \sum_{i=1}^{n} u_i p_i q_i^{\alpha-1} \right) \right]$$

$$\leq L_\alpha^\beta (U) \leq \frac{1}{1-D^{(1-\alpha)\beta}} \left[ 1 - D^{(1-\alpha)\beta} \left( \sum_{i=1}^{n} u_i p_i q_i^{\alpha-1} \right) \right]$$

Implies that

$$I_\alpha^\beta (P, Q; U) \leq L_\alpha^\beta (U) < D^{(1-\alpha)\beta} I_\alpha^\beta (P, Q; U) + 1$$

### 3.5. Bounds on generalized inaccuracy measures with two and three parameters

In this section, bounds have been obtained on generalized inaccuracy measures.
with two and three parameters given by Tuteja and Bhaker [129].

Consider a function of inaccuracy measure given by Tuteja and Bhaker [129]

\[
I_\alpha^\beta (P, Q; U) = \frac{\sum_{i=1}^{n} (u_i p_i)^\alpha (q_i^{\alpha-1} - 1)}{(2^{\alpha-1}) \sum_{i=1}^{n} p_i^\alpha}
\]

where \(\alpha > 0 \neq 1\), \(\beta > 0\), \(p_i \geq 0\), \(\sum_{i=1}^{n} p_i \leq 1\).

**Remark (3.5.1)**

1. When \(\beta = 1\), (3.5.1) reduces to the measure given by Sharma, Mittal and Mohan [113].

2. When \(\alpha \to 1, \beta = 1\), (3.5.1) reduces to the non-additive ‘useful’ inaccuracy measure for generalized probability distribution characterized by Taneja and Tuteja [119].

3. When \(\beta = 1, p_i = q_i \ \forall \ i = 1, 2, \ldots, n\), (3.5.1) reduces to the measure given by Jain and Tuteja [63].

4. When \(\alpha \to 1, \beta = 1, u_i = 1 \ \forall \ i = 1, 2, \ldots, n\), and probability distribution is complete then (3.5.1) reduces to the measure given by Kerridge [73].

Consider a Parametric ‘useful’ generalized codeword mean length

\[
L_\alpha^\beta (U) = \frac{1}{\prod_{i=1}^{n} (u_i p_i)^\alpha} \left[ \left( \frac{\sum_{i=1}^{n} (u_i p_i)^\alpha D_{1} \left( \frac{1}{\sum_{i=1}^{n} p_i} \right)^{\alpha}}{\sum_{i=1}^{n} p_i^\alpha} \right)^{\alpha} - 1 \right]
\]

where \(\alpha > 0 \neq 1\), \(\beta > 0\), \(p_i \geq 0\), \(\sum_{i=1}^{n} p_i \leq 1\).

**Remark (3.5.2)**

1. When \(\beta = 1\) and distribution is complete, then (3.5.2) reduces to codeword mean length given by Jain and Tuteja [63].

2. When \(\alpha \to 1, \beta = 1\) and distribution is complete, then (3.5.2) reduces to the codeword mean length given by Guisau and Picard [51] and further reduces to ordinary mean length given by Shannon [107] by taking \(u_i = 1 \ \forall \ i = 1, 2, \ldots, n\).

The bounds are obtained here, under the condition

\[
\sum_{i=1}^{n} (u_i p_i)^\beta q_i^{-1} D_i^{\alpha} \leq \sum_{i=1}^{n} (u_i p_i)^\beta
\]

which is a generalization of Kraft [80] inequality.

63
**Theorem (3.5.1):** If \( l_1, l_2, \ldots, l_n \) denote the code lengths satisfying the condition (3.5.3). Then

\[
L_\alpha^\beta (U) \geq \frac{P(U)}{\overline{U}}, \quad \alpha > 0 \neq 1, \beta > 0
\]

where \( \overline{U} = \sum_{i=1}^{n} (u_i p_i)^\beta \), equality holds iff

\[
l_i = -\log q_i^\alpha + \log \left( \frac{\sum_{i=1}^{n} (u_i p_i)^\beta q_i^{\alpha - 1}}{\sum_{i=1}^{n} (u_i p_i)^\beta} \right)
\]

**Proof:** By Holder’s inequality [116]

\[
\sum_{i=1}^{n} x_i y_i \geq \left( \sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} y_i^q \right)^{\frac{1}{q}}
\]

for all \( x_i, y_i > 0, \ i = 1, 2, \ldots, n \) and \( \frac{1}{p} + \frac{1}{q} = 1, p < 1 \neq 0, q < 0 \) or \( q < 1 \neq 0, p < 0 \). We see the equality holds iff there exists a positive constant \( c \) such that

\[
x_i^p = c y_i^q
\]

Making the substitution

\[
x_i = \left[ \frac{(u_i p_i)^\beta}{\sum_{i=1}^{n} (u_i p_i)^\beta} \right]^{\frac{1}{\alpha - 1}} D^{-\frac{1}{\alpha - 1}}, \ y_i = \left[ \frac{(u_i p_i)^\beta}{\sum_{i=1}^{n} (u_i p_i)^\beta} \right]^{-\frac{1}{\alpha - 1}} q_i^{-1}
\]

\( p = \frac{\alpha - 1}{\alpha}, \ q = 1 - \alpha \)

in (3.5.6), we get

\[
\frac{\sum_{i=1}^{n} (u_i p_i)^\beta D^{-\frac{1}{\alpha - 1}} q_i^{\frac{1}{\alpha - 1}}}{\sum_{i=1}^{n} (u_i p_i)^\beta} \geq \left[ \frac{\sum_{i=1}^{n} (u_i p_i)^\beta D^{\frac{1}{\alpha - 1}} (\frac{1}{\alpha - 1})}{\sum_{i=1}^{n} (u_i p_i)^\beta} \right]^{\frac{1}{\alpha - 1}} \left[ \frac{\sum_{i=1}^{n} (u_i p_i)^\beta q_i^{\frac{\alpha - 1}{\alpha - 1}}}{\sum_{i=1}^{n} (u_i p_i)^\beta} \right]^{\frac{1}{\alpha - 1}}
\]

using the inequality (3.5.3), we get

\[
\left[ \frac{\sum_{i=1}^{n} (u_i p_i)^\beta D^{\frac{1}{\alpha - 1}} (\frac{1}{\alpha - 1})}{\sum_{i=1}^{n} (u_i p_i)^\beta} \right]^{\frac{1}{\alpha - 1}} \geq \left[ \frac{\sum_{i=1}^{n} (u_i p_i)^\beta q_i^{\frac{\alpha - 1}{\alpha - 1}}}{\sum_{i=1}^{n} (u_i p_i)^\beta} \right]^{\frac{1}{\alpha - 1}}
\]

(3.5.8)

Let \( 0 < \alpha < 1 \), raising both sides of (3.5.8) to the power \((1 - \alpha)\), we get

\[
\left[ \frac{\sum_{i=1}^{n} (u_i p_i)^\beta D^{\frac{1}{\alpha - 1}} (\frac{1}{\alpha - 1})}{\sum_{i=1}^{n} (u_i p_i)^\beta} \right]^{\alpha} \geq \left[ \frac{\sum_{i=1}^{n} (u_i p_i)^\beta q_i^{\frac{\alpha - 1}{\alpha - 1}}}{\sum_{i=1}^{n} (u_i p_i)^\beta} \right]^{\alpha}
\]

(3.5.9)

Since \( 2^{1-\alpha} - 1 > 0 \), for \( 0 < \alpha < 1 \), a simple manipulation proves (3.5.4) for \( 0 < \alpha < 1 \). The proof for \( 1 < \alpha < \infty \) follows on the same lines.
**Theorem (3.5.2):** For every code with lengths $l_1, l_2, \ldots, l_n$ satisfying the condition (3.5.3), $L^\beta_\alpha (U)$ can be made to satisfy the inequality

\[(3.5.10) \quad L^\beta_\alpha (U) < \frac{\sum_{i=1}^n (u_i p_i)^\beta q_i^{\alpha - 1}}{\sum_{i=1}^n (u_i p_i)^\beta} \leq \frac{2^{1-\alpha} - 1}{(2^{1-\alpha} - 1) \sum_{i=1}^n p_i^\alpha}, \quad \alpha > 0 \neq 1, \beta > 0\]

**Proof:** Let $l_i$ be the positive integer satisfying the inequality

\[(3.5.11) \quad -\log q_i^\alpha + \log \frac{\sum_{i=1}^n (u_i p_i)^\beta q_i^{\alpha - 1}}{\sum_{i=1}^n (u_i p_i)^\beta} \leq l_i \leq -\log q_i^\alpha + \log \frac{\sum_{i=1}^n (u_i p_i)^\beta q_i^{\alpha - 1}}{\sum_{i=1}^n (u_i p_i)^\beta} + 1\]

Consider the interval

\[(3.12) \quad \delta_i = \left[ -\log q_i^\alpha + \log \frac{\sum_{i=1}^n (u_i p_i)^\beta q_i^{\alpha - 1}}{\sum_{i=1}^n (u_i p_i)^\beta}, -\log q_i^\alpha + \log \frac{\sum_{i=1}^n (u_i p_i)^\beta q_i^{\alpha - 1}}{\sum_{i=1}^n (u_i p_i)^\beta} + 1 \right]\]

of length 1. In every $\delta_i$, there lies exactly one positive integer $l_i$ such that

\[(3.13) \quad 0 < -\log q_i^\alpha + \log \frac{\sum_{i=1}^n (u_i p_i)^\beta q_i^{\alpha - 1}}{\sum_{i=1}^n (u_i p_i)^\beta} \leq l_i \leq -\log q_i^\alpha + \log \frac{\sum_{i=1}^n (u_i p_i)^\beta q_i^{\alpha - 1}}{\sum_{i=1}^n (u_i p_i)^\beta} + 1\]

We will first show that the sequence $\{l_1, l_2, \ldots, l_n\}$, thus defined satisfies (3.5.3).

From (3.13) we have

\[-\log q_i^\alpha + \log \frac{\sum_{i=1}^n (u_i p_i)^\beta q_i^{\alpha - 1}}{\sum_{i=1}^n (u_i p_i)^\beta} \leq l_i\]

\[-\log q_i^\alpha + \log \frac{\sum_{i=1}^n (u_i p_i)^\beta q_i^{\alpha - 1}}{\sum_{i=1}^n (u_i p_i)^\beta} \leq -\log D - l_i\]

\[(3.14) \quad \left( \frac{\sum_{i=1}^n (u_i p_i)^\beta q_i^{\alpha - 1}}{\sum_{i=1}^n (u_i p_i)^\beta} \right) \geq D^{-l_i}\]

Multiplying both sides of (3.14) by $(u_i p_i)^\beta q_i^{\alpha - 1}$ and summing over $i = 1, 2, \ldots, n$, we get (3.5.3).

The last inequality of (3.13) gives

\[l_i < -\log q_i^\alpha + \log \frac{\sum_{i=1}^n (u_i p_i)^\beta q_i^{\alpha - 1}}{\sum_{i=1}^n (u_i p_i)^\beta} + 1\]

\[(3.15) \quad D^{l_i} < D_{q_i}^{-\alpha} \frac{\sum_{i=1}^n (u_i p_i)^\beta q_i^{\alpha - 1}}{\sum_{i=1}^n (u_i p_i)^\beta}\]

Let $0 < \alpha < 1$, raising both sides of (3.15) to the power $\frac{1-\alpha}{\alpha}$, we get

65
\begin{equation}
D^{i}(\frac{1}{\alpha}) < D^{\frac{1}{\alpha}} q_{i}^{\alpha - 1} \left[ \frac{\sum_{i=1}^{n} (u_{i} p_{i})^{\beta}}{\sum_{i=1}^{n} (u_{i} p_{i})} \right]^{\frac{1}{\alpha}}
\end{equation}

Multiply both sides of (3.5.16) by \( \frac{(u_{i} p_{i})^{\beta}}{\sum_{i=1}^{n} (u_{i} p_{i})^{\beta}} \), summing over \( i = 1, 2, \ldots, n \) and after then raising both sides to the power \( \alpha \), we get

\begin{equation}
\left[ \frac{\sum_{i=1}^{n} (u_{i} p_{i})^{\beta} D^{i}(\frac{1}{\alpha})}{\sum_{i=1}^{n} (u_{i} p_{i})^{\beta}} \right]^{\alpha} < D^{1/\alpha} \left[ \frac{\sum_{i=1}^{n} (u_{i} p_{i})^{\beta} q_{i}^{\alpha - 1}}{\sum_{i=1}^{n} (u_{i} p_{i})^{\beta}} \right]^{\alpha}
\end{equation}

Since \( 2^{1/\alpha - 1} > 0 \), for \( 0 < \alpha < 1 \), a simple manipulation in (3.5.17) proves (3.5.10) for \( 0 < \alpha < 1 \). Also for \( \alpha > 1 \), the proof follows along the similar lines.

Now, bounds have been obtained here by considering a function studied by Tuteja and Bhaker [129] which involves three parameters. The results obtained here are more generalized than previous results. The function used here is applicable to more complex distributions.

Consider a function of inaccuracy measure given by Tuteja and Bhaker [129]

\begin{equation}
I_{\alpha}^{\beta, \gamma} (P, Q; U) = \frac{\sum_{i=1}^{n} (u_{i} p_{i})^{\beta} (p_{i}^{-1} q_{i}^{\gamma - 1})}{(2^{1/\alpha - 1}) \sum_{i=1}^{n} p_{i}^{\beta}}, \quad \alpha, \beta, \gamma > 0, \alpha \neq 1
\end{equation}

**Remark (3.5.3)**

1. When \( \gamma = 1 \) in (3.5.18), it reduces to measure (3.5.1) given by Tuteja and Bhaker [129].

2. When \( \beta = 1, \gamma = 1 \) and \( \alpha \to 1 \), then (3.5.18) reduces to measure given by Taneja and Tuteja [119].

3. When \( \beta = 1, \gamma = 1 \), \( \alpha \to 1 \) and \( u_{i} = 1 \ \forall \ i = 1, 2, \ldots, n \). Then (3.5.18) reduces to the result given by Kerridge [73]. Further if \( p_{i} = q_{i} \ \forall \ i = 1, 2, \ldots, n \), it reduces to Shannon’s [107] entropy.

4. When \( \beta = 1, \gamma = 1 \) and \( p_{i} = q_{i} \ \forall \ i = 1, 2, \ldots, n \). (3.5.18) reduces to measure given by Jain and Tuteja [63].

Let us consider the three parametric codeword mean length

\begin{equation}
L_{\alpha}^{\beta, \gamma} (U) = \frac{1}{(2^{1/\alpha - 1}) \sum_{i=1}^{n} p_{i}^{\beta}} \left[ \left( \frac{\sum_{i=1}^{n} (u_{i} p_{i})^{\beta} p_{i}^{-1} D^{i}(\frac{1}{\alpha})}{\sum_{i=1}^{n} (u_{i} p_{i})^{\beta}} \right)^{\alpha} - 1 \right]^{\alpha+1-\gamma}
\end{equation}

where \( \alpha, \beta, \gamma > 0, \alpha \neq 1 \).
Remark (3.5.4)

(1) When $\beta = 1, \gamma = 1$ and $\alpha \to 1$, then (3.5.19) reduces to codeword mean length
given by Guiasu and Picard [51].

(2) When $\beta = 1, \gamma = 1, \alpha \to 1$ and $u_i = 1 \forall i = 1, 2, \ldots, n$. Then (3.5.19)
reduces to codeword mean length given by Shannon [107].

(3) When $\beta = 1, \gamma = 1$, then (3.5.19) reduces to the codeword mean length given
by Jain and Tuteja [63].

Now, we obtain the bounds under the condition

$$ \sum_{i=1}^{n} (u_i p_i)^{\beta} p_i^{\gamma - 1} q_i^{-1} D_{-i} \leq \sum_{i=1}^{n} (u_i p_i)^{\beta} $$

which is a generalization of Kraft [80] inequality.

Theorem (3.5.3): If $l_1, l_2, \ldots, l_n$ denote the codeword lengths and satisfying the condi-
tion (3.5.20), then

$$ L_{\alpha, \gamma} (U) \geq \frac{t_{\alpha, \gamma} (U, Q, U)}{U}, \quad \alpha, \beta, \gamma > 0, \alpha \neq 1 $$

where $U = \sum_{i=1}^{n} (u_i p_i)^{\beta}$, equality holds iff

$$ l_i = -\log q_i^{\alpha + 1 - \gamma} + \log \frac{\sum_{i=1}^{n} (u_i p_i)^{\beta} p_i^{\gamma - 1} q_i^{\alpha - \gamma}}{\sum_{i=1}^{n} (u_i p_i)^{\beta}} $$

Proof: By Holder’s inequality [116]

$$ \sum_{i=1}^{n} x_i y_i \geq \left( \sum_{i=1}^{n} x_i^{p} \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} y_i^{q} \right)^{\frac{1}{q}} $$

for all $x_i, y_i > 0, i = 1, 2, \ldots, n$ and $\frac{1}{p} + \frac{1}{q} = 1, p < 1 (\neq 0), q < 0$ or $q < 1 (\neq 0), p < 0$. We see the equality holds iff there exists a positive constant $c$ such that

$$ x_i^{p} = cy_i^{q} $$

For $\alpha > \gamma$,

Making the substitution

$$ x_i = \left[ \frac{(u_i p_i)^{\beta}}{\sum_{i=1}^{n} (u_i p_i)^{\beta}} \right]^{\frac{\gamma - 1}{\gamma - \alpha}} p_i^{\frac{(\gamma - 1)(\gamma - 1 - \alpha)}{\gamma - \alpha}} D_{-i} $$

67
\[
y_i = \left[ \frac{(u_i p_i)^{\alpha}}{\sum_{i=1}^{n} (u_i p_i)^{\alpha}} \right]^{\frac{1}{\gamma - \alpha}} p_i^{\frac{\gamma - 1}{\gamma - \alpha}} q_i^{-1}
\]

\[
p = \frac{\gamma - \alpha}{\gamma - \alpha - 1}, q = \gamma - \alpha
\]
in (3.5.23), we get

\[
\frac{\sum_{i=1}^{n} (u_i p_i)^{\alpha} p_i^{\gamma - 1} q_i^{-1} D^{-i_i}}{\sum_{i=1}^{n} (u_i p_i)^{\alpha}} \geq \left[ \frac{\sum_{i=1}^{n} (u_i p_i)^{\alpha} p_i^{\gamma - 1} D^{-i_i} \left( \frac{\gamma - 1}{\gamma - \alpha} \right) ^{\gamma - \alpha}}{\sum_{i=1}^{n} (u_i p_i)^{\alpha}} \right]^{\frac{\gamma - \alpha - 1}{\gamma - \alpha}} \left[ \frac{\sum_{i=1}^{n} (u_i p_i)^{\alpha} p_i^{\gamma - 1} q_i^{-1}}{\sum_{i=1}^{n} (u_i p_i)^{\alpha}} \right]^{\frac{1}{\gamma - \alpha}}
\]

using the inequality (3.5.20), we get

\[
\left( \frac{\sum_{i=1}^{n} (u_i p_i)^{\alpha} p_i^{\gamma - 1} D^{-i_i} \left( \frac{\gamma - 1}{\gamma - \alpha} \right) ^{\gamma - \alpha}}{\sum_{i=1}^{n} (u_i p_i)^{\alpha}} \right)^{-\frac{\gamma - \alpha - 1}{\gamma - \alpha}} \leq \left[ \frac{\sum_{i=1}^{n} (u_i p_i)^{\alpha} p_i^{\gamma - 1} q_i^{-1}}{\sum_{i=1}^{n} (u_i p_i)^{\alpha}} \right]^{\frac{1}{\gamma - \alpha}}
\]

Raising both sides of (3.5.25) to the power \((\gamma - \alpha)\), we get

\[
\left( \frac{\sum_{i=1}^{n} (u_i p_i)^{\alpha} p_i^{\gamma - 1} D^{-i_i} \left( \frac{\gamma - 1}{\gamma - \alpha} \right) ^{\gamma - \alpha}}{\sum_{i=1}^{n} (u_i p_i)^{\alpha}} \right)^{1 - \gamma} \leq \left[ \frac{\sum_{i=1}^{n} (u_i p_i)^{\alpha} p_i^{\gamma - 1} q_i^{-1}}{\sum_{i=1}^{n} (u_i p_i)^{\alpha}} \right]^{\frac{1}{\gamma - \alpha}}
\]

Since \(2^{\gamma - 1} - 1 < 0\) for \(\alpha > 1\), a simple manipulation in (3.5.26) proves (3.5.21) for \(\alpha > 1\). For other cases, proof follows on the same lines.

**Theorem (3.5.4):** For every code with lengths \(l_1, l_2, \ldots, l_n\) satisfying the condition (3.5.20), \(L_{\alpha, \gamma}^x(U)\) can be made to satisfy the inequality

\[
L_{\alpha, \gamma}^x(U) < \frac{P(\alpha, P(U)) D^{1 - \alpha}}{\gamma - \alpha} + \frac{D^{\gamma - 1}}{(2^{\alpha - 1} - 1) \sum_{i=1}^{n} p_i^{\alpha}} \alpha, \beta, \gamma > 0, \alpha \neq 1.
\]

where \(U = \sum_{i=1}^{n} (u_i p_i)^{\alpha}\)

**Proof:** Let \(l_i\) be the positive integer satisfying the inequality

\[
- \log q_i^{\alpha + 1 - \gamma} + \log \frac{\sum_{i=1}^{n} (u_i p_i)^{\alpha} p_i^{\gamma - 1} q_i^{-1}}{\sum_{i=1}^{n} (u_i p_i)^{\alpha}} \leq l_i < - \log q_i^{\alpha + 1 - \gamma} + \log \frac{\sum_{i=1}^{n} (u_i p_i)^{\alpha} p_i^{\gamma - 1} q_i^{-1}}{\sum_{i=1}^{n} (u_i p_i)^{\alpha}} + 1
\]

Consider the interval

\[
(3.5.29) \delta_i = \left[ - \log q_i^{\alpha + 1 - \gamma} + \log \frac{\sum_{i=1}^{n} (u_i p_i)^{\alpha} p_i^{\gamma - 1} q_i^{-1}}{\sum_{i=1}^{n} (u_i p_i)^{\alpha}}, - \log q_i^{\alpha + 1 - \gamma} + \log \frac{\sum_{i=1}^{n} (u_i p_i)^{\alpha} p_i^{\gamma - 1} q_i^{-1}}{\sum_{i=1}^{n} (u_i p_i)^{\alpha}} + 1 \right]
\]

68
of length 1. In every $\delta_i$, there lies exactly one positive $l_i$ such that

$$0 < -\log q_i^{\alpha+1-\gamma} + \log \frac{\sum_{i=1}^{n} (u_{i,p_i})^d p_i^{\gamma-1} q_i^{\alpha-\gamma}}{\sum_{i=1}^{n} (u_{i,p_i})^d} \leq l_i < -\log q_i^{\alpha+1-\gamma} + \log \frac{\sum_{i=1}^{n} (u_{i,p_i})^d p_i^{\gamma-1} q_i^{\alpha-\gamma}}{\sum_{i=1}^{n} (u_{i,p_i})^d} + 1$$

We will first show that the sequence $\{l_1, l_2, \ldots, l_n\}$, thus defined satisfies (3.5.20). From (3.5.30) we have

$$-\log q_i^{\alpha+1-\gamma} + \log \frac{\sum_{i=1}^{n} (u_{i,p_i})^d p_i^{\gamma-1} q_i^{\alpha-\gamma}}{\sum_{i=1}^{n} (u_{i,p_i})^d} \leq l_i$$

$$-\log q_i^{\alpha+1-\gamma} + \log \frac{\sum_{i=1}^{n} (u_{i,p_i})^d p_i^{\gamma-1} q_i^{\alpha-\gamma}}{\sum_{i=1}^{n} (u_{i,p_i})^d} \leq -\log D D^{-l_i}$$

or

$$\left(\frac{\sum_{i=1}^{n} (u_{i,p_i})^d p_i^{\gamma-1} q_i^{\alpha-\gamma}}{\sum_{i=1}^{n} (u_{i,p_i})^d}\right) \geq D^{-l_i}$$

Multiplying both sides by $\frac{(u_{i,p_i})^d p_i^{\gamma-1} q_i^{\alpha-\gamma}}{\sum_{i=1}^{n} (u_{i,p_i})^d}$ and summing over $i = 1, 2, \ldots, n$ we get (3.5.20).

The last inequality in (3.5.30) gives

$$l_i < \log q_i^{-(\alpha+1-\gamma)} \frac{\sum_{i=1}^{n} (u_{i,p_i})^d p_i^{\gamma-1} q_i^{\alpha-\gamma}}{\sum_{i=1}^{n} (u_{i,p_i})^d} D$$

(3.5.31) \hspace{1cm} D^{l_i} < q_i^{-(\alpha+1-\gamma)} \frac{\sum_{i=1}^{n} (u_{i,p_i})^d p_i^{\gamma-1} q_i^{\alpha-\gamma}}{\sum_{i=1}^{n} (u_{i,p_i})^d} D$$

Let $\alpha > \gamma$, raising both sides of (3.5.31) to the power $-\left(\frac{\gamma-\alpha}{\gamma-\alpha-1}\right)$, we get

$$D^{-l_i} \left(\frac{\gamma-\alpha}{\gamma-\alpha-1}\right) > q_i^{\alpha-\gamma} \left(\frac{\sum_{i=1}^{n} (u_{i,p_i})^d p_i^{\gamma-1} q_i^{\alpha-\gamma}}{\sum_{i=1}^{n} (u_{i,p_i})^d}\right) \left(\frac{\gamma-\alpha}{\gamma-\alpha-1}\right) D$$

Multiply both sides of (3.5.32) by $\frac{\sum_{i=1}^{n} (u_{i,p_i})^d p_i^{\gamma-1}}{\sum_{i=1}^{n} (u_{i,p_i})^d}$ and summing over $i = 1, 2, \ldots, n$ and
after raising both sides to the power \((\alpha + 1 - \gamma)\), we get

\[
(3.5.33) \quad \left[ \frac{\sum_{i=1}^{n} (u_i p_i)^{\beta} p_i^{-1} D^{-1}(\frac{\gamma}{\alpha - 1})}{\sum_{i=1}^{n} (u_i p_i)^{\beta}} \right]^{\alpha + 1 - \gamma} > \frac{\sum_{i=1}^{n} (u_i p_i)^{\beta} p_i^{-1} q_i^{\alpha - \gamma}}{\sum_{i=1}^{n} (u_i p_i)^{\beta}} D^{\gamma - \alpha}
\]

Since for \(\alpha > 1, 2^{1-\alpha} - 1 < 0\), after simple manipulation in (3.5.33) we get for \(\alpha > 1\) and \(\alpha > \gamma\)

\[
L^{\beta, \gamma}_{\alpha} (U) < \frac{f^{\beta, \gamma}_{\alpha}(P, Q, U) D^{\gamma - \alpha}}{U} + \frac{D^{\gamma - \alpha - 1}}{(2^{1-\alpha} - 1) \sum_{i=1}^{n} p_i^{\beta}}
\]