Shannon’s inequalities are well known in the field of information theory. Researchers have found many inequalities by using well known Holders’s inequality, Jensen’s inequality etc and have found many applications for Shannon’s entropy, Renyi’s entropy etc. Dragomir [46] have found many information inequalities for the logarithmic mapping and convex mappings by using Jensen’s inequality. In this chapter, several generalized information inequalities have been obtained by introducing a independent variable ‘s’ and applied for Shannon’s entropy, Renyi’s entropy and mutual information. The information inequalities obtained here are not only new but also generalizes some established inequalities in information theory given by Dragomir [46].

Also some upper bounds for the relative arithmetic geometric divergence measure have been obtained by using some classical inequalities like Kantorovic inequality, Diaz-Metcalf inequality and other inequality for logarithmic function.

5.1. Introduction

The following converse of Jensen’s discrete inequality for convex mappings of a real variable was proved by Dragomir and Ionescu [40].

**Theorem (5.1.1).** Let $f : I \subseteq R \rightarrow R$ be a differentiable convex function on the interval $I$, $x_i \in \mathring{I}$ ($\mathring{I}$ is the interior of $I$), $p_i \geq 0 \ (i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} p_i = 1$. Then we have the inequality

$$0 \leq \sum_{i=1}^{n} p_i f(x_i) - f \left( \sum_{i=1}^{n} p_i x_i \right) \leq \sum_{i=1}^{n} p_i x_i f' (x_i) - \sum_{i=1}^{n} p_i x_i \sum_{i=1}^{n} p_i f' (x_i)$$

They also pointed out some natural applications of (5.1.1) in connection to the arithmetic geometric mean inequality, the generalized triangular inequality etc. A generalization of (5.1.1) for differentiable convex mappings of several variables was obtained by Dragomir and Goh [39]. They also consider the following analytical inequality for the logarithmic mapping.

**Theorem (5.1.2).** Let $x_i, p_i > 0, \ (i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} p_i = 1$ and $b > 1$. Then

$$0 \leq \log_b \left( \sum_{i=1}^{n} p_i x_i \right) - \sum_{i=1}^{n} p_i \log_b x_i$$
\[
\leq \frac{1}{\ln b} \left[ \sum_{i=1}^{n} \frac{p_i}{x_i} \sum_{i=1}^{n} p_i x_i - 1 \right]
\]

\[
= \frac{1}{\ln b} \sum_{1 \leq i < j \leq n} p_i p_j \frac{(x_i - x_j)^2}{x_i x_j}
\]
equality holds in (5.1.2) iff \(x_1 = x_2 = \ldots = x_n\).

They applied inequality (5.1.2) in information theory for the entropy mapping, conditional entropy, mutual information etc. An integral version of (5.1.2) was employed by Dragomir and Goh [39] to obtain different bounds for the entropy, conditional entropy and mutual information for continuous random variables.

In the next section, some generalized inequalities have been obtained and applications for Shannon’s entropy, Renyi’s entropy and mutual information are also given. The results obtained in this section generalizes some results of Dragomir [46]. This work is published in International Journal of pure and applied Mathematics, Vol 31 (2) PP 253-263 (2006) (Baig and Rayees [11]).

5.2 Some generalized inequalities for convex functions

**Theorem** (5.2.1). Let \(f : [a, b] \rightarrow \mathbb{R}\) be twice differentiable on \((a, b)\), continuous in \([a, b]\) and \(m \leq x^{2-s} f''(x) \leq M \quad \forall x \in (a, b)\) and \(s \in \mathbb{R}\). If \(x_i \in [a, b]\), \(i = 1, 2, \ldots, n\) and \(p = p_i, i = 1, 2, \ldots, n\) is a probability distribution, then

\[
m \left[ \sum_{i=1}^{n} p_i \phi_s(x_i) - \phi_s \left( \sum_{i=1}^{n} p_i x_i \right) \right]
\]

\[
\leq n \left[ \sum_{i=1}^{n} p_i f(x_i) - f \left( \sum_{i=1}^{n} p_i x_i \right) \right]
\]

\[
\leq M \left[ \sum_{i=1}^{n} p_i \phi_s(x_i) - \phi_s \left( \sum_{i=1}^{n} p_i x_i \right) \right]
\]

**Proof:** Let us consider the function

\[
\phi_s(x) = \begin{cases} 
    [s (s - 1)]^{-1} [x^s - 1 - s (x - 1)] & s \neq 0, 1 \\
    x - 1 - \ln x & s = 0 \\
    1 - x + \ln x & s = 1
\end{cases}
\]

Then

\[
\phi_s'(x) = \begin{cases} 
    (s - 1)^{-1} [x^{s-1} - 1] & s \neq 0, 1 \\
    1 - x^{-1} & s = 0 \\
    \ln x & s = 1
\end{cases}
\]

and
\[ \phi_s''(x) = \begin{cases} 
  x^{s-2} ; & s \neq 0, 1 \\
  x^{-2} ; & s = 0 \\
  x^{-1} ; & s = 1 
\end{cases} \]

Here \( \phi_s''(x) > 0, \ \forall \ x > 0, \) here \( \phi_s(x) \) is strictly convex for all \( x > 0 \) and \( s \in \mathbb{R} \).

Let us consider the function \( g : [a, b] \rightarrow \mathbb{R} \)
\[ g(x) = f(x) - m\phi_s(x), \ x \in (a, b), \ s \in \mathbb{R} \]
\[ g''(x) = f''(x) - m\phi_s''(x) = x^{s-2}(x^{2-s}f''(x) - m) \geq 0 \]
which shows that the mapping \( g(x) \) is convex on \([a, b] \).

Applying Jensen’s discrete inequality for the convex mapping \( g(x) \), i.e.
\[ g\left(\sum_{i=1}^{n} p_i x_i \right) \leq \sum_{i=1}^{n} p_i g(x_i) \]
Therefore,
\[ f\left(\sum_{i=1}^{n} p_i x_i \right) - m\phi_s\left(\sum_{i=1}^{n} p_i x_i \right) \leq \sum_{i=1}^{n} p_i \left[f(x_i) - m\phi_s(x_i)\right] \]
\[ \sum_{i=1}^{n} p_i f(x_i) - \left(\sum_{i=1}^{n} p_i x_i \right) \geq m\left[\sum_{i=1}^{n} p_i \phi_s(x_i) - \phi_s\left(\sum_{i=1}^{n} p_i x_i \right)\right] \]
The first inequality in (5.2.1) is proved.

The proof of the second inequality goes likewise for the mapping \( h : [a, b] \rightarrow \mathbb{R}, h(x) = M\phi_s(x) - f(x) \) which is convex on \([a, b] \).

**Corollary (5.2.1)**. Let \( x_i, w_i > 0 \) \( (i = 1, 2, \ldots, n) \) and put \( W_n = \sum_{i=1}^{n} w_i \). Also consider Arithmetic mean \( A_n(w, a) = \frac{1}{n} \sum_{i=1}^{n} w_i a_i \). If \( x_i \in [m, M] \subset (0, \infty) , \ i = 1, 2, \ldots, n \) and \( s \in \mathbb{R} \), then we have the inequalities
\[ \exp\left[\frac{1}{nW_n} \left\{ \sum_{i=1}^{n} w_i \phi_s(x_i) - \phi_s\left(\sum_{i=1}^{n} w_i x_i \right) \right\}\right] \]
\[ \leq \exp\left[\frac{1}{n} \frac{\prod_{i=1}^{n} w_i x_i}{A_n(w, a)^{\phi_s}}\right] \]
\[ \leq \exp\left[\frac{1}{mW_n} \left\{ \sum_{i=1}^{n} w_i \phi_s(x_i) - \phi_s\left(\sum_{i=1}^{n} w_i x_i \right) \right\}\right] \]

**Proof:** Consider the mapping \( f(x) = x \ln x, \ x > 0. \) Then
\[ f'(x) = \ln x + 1, \ x \in (0, \infty) \]
\[ f''(x) = \frac{1}{x}, \ x \in (0, \infty) \]
which shows that \( f \) is strictly convex on the interval \((0, \infty)\).

\[
\ln f \left( x \right) = \frac{1}{M} \quad \text{for } x \in [m, M]
\]

\[
\sup_{x \in [m, M]} f''(x) = \frac{1}{m}
\]

Applying theorem (5.2.1) for this mapping and \( p_i = \frac{m}{\rho_n} \), \( i = 1, 2, \ldots, n \). We deduce (5.2.5).

The case of equality follows by the strict convexity of the mapping

\[
g(x) = x \ln x - \frac{1}{M} \phi_s(x), \quad h(x) = \frac{1}{m} \phi_s(x) - x \ln x \text{ on } (m, M).
\]

**Theorem (5.2.2).** Let \( f : [a, b] \to \mathbb{R}_+ \) be twice differentiable on \((a, b)\), continuous in \([a, b]\) and \( m \leq x^{2-s} f''(x) \leq M \quad \forall x \in [a, b] \) and \( s \in \mathbb{R} \). If \( x_i \in [a, b], \ i = 1, 2, \ldots, n \) and \( p = p_i \ (i = 1, 2, \ldots, n) \) is a probability distribution. Then we have the inequalities

\[
\frac{1}{2} \sum_{i,j} p_i p_j (x_i - x_j) \left( f'(x_i) - f'(x_j) \right)
\]

\[
+ M \left[ \sum_{i=1}^n p_i \phi_s(x_i) - \phi_s \left( \sum_{i=1}^n p_i x_i \right) - \frac{1}{2} \sum_{i,j} p_i p_j (x_i - x_j) \left( \phi'_s(x_i) - \phi'_s(x_j) \right) \right]
\]

\[
\leq \sum_{i=1}^n p_i f(x_i) - f \left( \sum_{i=1}^n p_i x_i \right)
\]

\[
\leq \frac{1}{2} \sum_{i,j} p_i p_j (x_i - x_j) \left( f'(x_i) - f'(x_j) \right)
\]

\[
+ m \left[ \sum_{i=1}^n p_i \phi_s(x_i) - \phi_s \left( \sum_{i=1}^n p_i x_i \right) - \frac{1}{2} \sum_{i,j} p_i p_j (x_i - x_j) \left( \phi'_s(x_i) - \phi'_s(x_j) \right) \right]
\]

**Proof:** Consider the mapping

\[
g : [a, b] \to \mathbb{R}, \quad g(x) = f(x) - m \phi_s(x), \quad s \in \mathbb{R}, \quad x \in (a, b)
\]

where \( \phi_s(x) \) is given in (5.2.2).

Then \( g \) is twice differentiable on \((a, b)\) and

\[
g''(x) = f''(x) - m \phi''_s(x) = f''(x) - mx^{s-2}
\]

\[
g''(x) = x^{s-2} [x^{2-s} f''(x) - m] \geq 0 \quad \forall \ x \in (a, b), \ s \in \mathbb{R}
\]

which shows that the mapping is convex on \([a, b], s \in \mathbb{R}\). Also \( \phi''_s(x) \) is given by (5.2.4). We apply inequality (5.1.1) for the convex mapping \( g \), i.e.

\[
0 \leq \sum_{i=1}^n p_i g(x_i) - g \left( \sum_{i=1}^n p_i x_i \right) \leq \frac{1}{2} \sum_{i,j} p_i p_j (x_i - x_j) \left( g'(x_i) - g'(x_j) \right)
\]

to obtain

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\[ 0 \leq \sum_{i=1}^{n} p_i \left[ f(x_i) - m\phi_s(x_i) \right] - \left[ f \left( \sum_{i=1}^{n} p_i x_i \right) - m\phi_s \left( \sum_{i=1}^{n} p_i x_i \right) \right] \]
\[ \leq \frac{1}{2} \sum_{ij} p_i p_j (x_i - x_j) \left[ f'(x_i) - f'(x_j) - m\phi'_s(x_i) + m\phi'_s(x_j) \right] \]
\[ = \frac{1}{2} \sum_{ij} p_i p_j (x_i - x_j) \left( f'(x_i) - f'(x_j) \right) - m\frac{1}{2} \sum_{ij} p_i p_j (x_i - x_j) \left( \phi'_s(x_i) - \phi'_s(x_j) \right) \]

Thus
\[ \sum_{i=1}^{n} p_i f(x_i) - f \left( \sum_{i=1}^{n} p_i x_i \right) \]
\[ \leq \frac{1}{2} \sum_{ij} p_i p_j (x_i - x_j) \left( f'(x_i) - f'(x_j) \right) \]
\[ + m \left[ \sum_{i=1}^{n} p_i \phi_s(x_i) - \phi_s \left( \sum_{i=1}^{n} p_i x_i \right) - \frac{1}{2} \sum_{ij} p_i p_j (x_i - x_j) \left( \phi'_s(x_i) - \phi'_s(x_j) \right) \right] \]

and the second inequality is proved.

The proof of the first inequality goes likewise for the mapping \( h : [a, b] \to \mathbb{R}, h(x) = M\phi_s(x) - f(x) \).

**Corollary (5.2.2).** Let \( x_i \in [m, M] \subset (0, \infty) \) and \( p_i > 0 \) \( (i = 1, \ldots, n) \) with \( \sum_{i=1}^{n} p_i = 1 \). Then we have the inequality

\[ \frac{1}{2} \sum_{ij} p_i p_j \frac{(x_i - x_j)^2}{x_ix_j} \]
\[ + \frac{1}{m^2} \left[ \sum_{i=1}^{n} p_i \phi_s(x_i) - \phi_s \left( \sum_{i=1}^{n} p_i x_i \right) - \frac{1}{2} \sum_{ij} p_i p_j (x_i - x_j) \left( \phi'_s(x_i) - \phi'_s(x_j) \right) \right] \]
\[ \leq \ln \left( \sum_{i=1}^{n} p_i x_i \right) - \sum_{i=1}^{n} p_i \ln x_i \]
\[ \leq \frac{1}{2} \sum_{ij} p_i p_j \frac{(x_i - x_j)^2}{x_ix_j} \]
\[ + \frac{1}{m^2} \left[ \sum_{i=1}^{n} p_i \phi_s(x_i) - \phi_s \left( \sum_{i=1}^{n} p_i x_i \right) - \frac{1}{2} \sum_{ij} p_i p_j (x_i - x_j) \left( \phi'_s(x_i) - \phi'_s(x_j) \right) \right] \]

**Proof:** Consider the mapping \( f : [m, M] \subset (0, \infty) \) given by \( f(x) = -\ln x \)

Then
\[ f'(x) = -\frac{1}{x}, \quad f''(x) = \frac{1}{x^2} \]

Also
\[ \inf_{x \in [m,M]} f'(x) = \frac{1}{M^2}, \quad \sup_{x \in [m,M]} f'(x) = \frac{1}{m^2} \]

Applying inequality (5.2.6) for this mapping, we can write

\[
\frac{1}{2} \sum_{ij} p_i p_j \frac{(x_i - x_j)^2}{x_i x_j} + \frac{1}{m^2} \left[ \sum_{i=1}^n p_i \phi_s(x_i) - \phi_s \left( \sum_{i=1}^n p_i x_i \right) \right] - \frac{1}{2} \sum_{i=1}^n p_i \left( x_i - x_j \right) \left( \phi'_s(x_i) - \phi'_s(x_j) \right) \frac{1}{x_i x_j} \leq \ln \left( \sum_{i=1}^n p_i x_i \right) - \sum_{i=1}^n p_i \ln x_i \]

\[
\leq \frac{1}{2} \sum_{ij} p_i p_j \frac{(x_i - x_j)^2}{x_i x_j} + \frac{1}{m^2} \left[ \sum_{i=1}^n p_i \phi_s(x_i) - \phi_s \left( \sum_{i=1}^n p_i x_i \right) \right] - \frac{1}{2} \sum_{i=1}^n p_i \left( x_i - x_j \right) \left( \phi'_s(x_i) - \phi'_s(x_j) \right) \frac{1}{x_i x_j} \]

which is equivalent to (5.2.7). The case of equality follows by the strict convexity of the mapping \( g(x) = -\ln x - \frac{1}{m^2} \phi_s(x), \quad h(x) = \frac{1}{m^2} \phi_s(x) + \ln x \) on the interval \([m, M] \).

**Corollary (5.2.3).** Let \( x_i \in [m, M] \subset (0, \infty) \), also \( s \in \mathbb{R}, \ p_i > 0 \ (i = 1, 2, \ldots, n) \) with \( \sum_{i=1}^n p_i = 1 \). Then we have the inequality

\[
\exp \left[ \frac{1}{2} \sum_{ij} p_i p_j \frac{(x_i - x_j)^2}{x_i x_j} + \frac{1}{m^2} \left[ \sum_{i=1}^n p_i \phi_s(x_i) - \phi_s \left( \sum_{i=1}^n p_i x_i \right) \right] \right] \]

\[
\leq A_n(p, x) \quad \exp \left[ \frac{1}{2} \sum_{ij} p_i p_j \frac{(x_i - x_j)^2}{x_i x_j} + \frac{1}{m^2} \left[ \sum_{i=1}^n p_i \phi_s(x_i) - \phi_s \left( \sum_{i=1}^n p_i x_i \right) \right] \right] \]

The equality holds in (5.2.8) iff \( x_1 = x_2 = \ldots = x_n \).

The proof is obvious by (5.2.7). Also, \( A_n(p, x) = \sum_{i=1}^n p_i x_i (A.M), \quad G_n(p, x) = \prod_{i=1}^n x_i^{p_i} (G.M) \).

If in (5.2.8), we put \( x \) instead of \( \frac{1}{2} \), we obtain the following corollary.
**Corollary (5.2.4).** Let $x_i, p_i, i = 1, 2, ..., n$ be as in corollary (5.2.3). Then we have the inequality

\[(5.2.9) \quad \exp \left[ \frac{1}{M} \left( \frac{1}{2} \sum_{ij} p_i p_j \frac{(x_i - x_j)^2}{x_i x_j} \right) \right] \leq \frac{G_n(p,x)}{H_n(p,x)} \]

\[\leq \exp \left[ \frac{1}{M^2} \left( \frac{1}{2} \sum_{ij} p_i p_j \frac{(x_i - x_j)^2}{x_i x_j} \right) \right] \]

The equality holds in (5.2.9) iff $x_1 = x_2 = ... = x_n$.

Also, $H_n(p,x) = \frac{1}{\sum_{i=1}^n 1/p_i}$ (Harmonic mean).

**5.3 Application in Shannon’s entropy**

The following inequalities for the logarithmic mapping holds.

**Lemma (5.3.1).** Let $\xi_i \in [m, M] \subset (0, \infty), p_i > 0, i = 1, 2, ..., n$ with $\sum p_i = 1$ and $s \in \mathbb{R}$. Then we have the inequality

\[(5.3.1) \quad \frac{1}{M} \left[ \sum_{i=1}^n p_i \phi_s (\xi_i) - \phi_s \left( \sum_{i=1}^n p_i \xi_i \right) \right] \leq \sum_{i=1}^n p_i \xi_i \ln \xi_i - \sum_{i=1}^n p_i \xi_i \ln \left( \sum_{i=1}^n p_i \xi_i \right) \]

\[\leq \frac{1}{M} \left[ \sum_{i=1}^n p_i \phi_s (\xi_i) - \phi_s \left( \sum_{i=1}^n p_i \xi_i \right) \right] \]

The case of equality holds iff $\xi_1 = \xi_2 = ... = \xi_n$.

The proof is obvious by theorem (5.2.1) for the convex mapping

\[f : [0, \infty) \to \mathbb{R}, \quad f(x) = x \ln x\]

**Corollary (5.3.1).** Under the assumption for $\xi_i (i = 1, 2, ..., n)$, we have
\[(5.3.2) \quad 0 \leq \frac{1}{M} \left[ \sum_{i=1}^{n} \phi_s (\xi_i) - \phi_s \left( \sum_{i=1}^{n} \xi_i \right) \right] \leq \sum_{i=1}^{n} \xi_i \ln \xi_i - \sum_{i=1}^{n} \xi_i \ln \left( \frac{\xi_i}{n} \right) \]
\[ \leq \frac{1}{m} \left[ \sum_{i=1}^{n} \phi_s (\xi_i) - \phi_s \left( \sum_{i=1}^{n} \xi_i \right) \right] \]

equality holds iff \( \xi_1 = \xi_2 = \ldots = \xi_n \).

The proof is obvious by lemma \((5.3.1)\), choosing \( p_i = \frac{1}{n} \) (i = 1, 2, ..., n).

**Theorem (5.3.1).** Let \( X \) be a random variable with probability distribution \( p_i \) (i = 1, 2, ..., n).

Assume that \( p = \min \{ p_i/i = 1, 2, ..., n \} > 0 \) and \( P = \max \{ p_i/i = 1, 2, ..., n \} < 1. \)

Then
\[(5.3.3) \quad \frac{1}{2} \sum_{ij} p_i p_j (p_j - p_i)^2 \]
\[ + p^2 \left[ \sum_{i=1}^{n} p_i \phi_s (\frac{1}{p_i}) - \phi_s (n) - \frac{1}{2} \sum_{ij} (p_j - p_i) \left( \phi_s' \left( \frac{1}{p_j} \right) - \phi_s' \left( \frac{1}{p_i} \right) \right) \right] \]
\[ \leq \ln (x) - H (X) \]
\[ \leq \frac{1}{2} \sum_{ij} p_i p_j (p_j - p_i)^2 \]
\[ + p^2 \left[ \sum_{i=1}^{n} p_i \phi_s (\frac{1}{p_i}) - \phi_s (n) - \frac{1}{2} \sum_{ij} (p_j - p_i) \left( \phi_s' \left( \frac{1}{p_j} \right) - \phi_s' \left( \frac{1}{p_i} \right) \right) \right] \]

**Proof:** If we choose \( x_i = \frac{1}{p^*} \in \left[ \frac{1}{p^*}, \frac{1}{p} \right] \) in \((5.2.7)\). We can deduce that \((5.3.3)\) with \((m = \frac{1}{p^*}, M = \frac{1}{p})\).

### 5.4 Application in Renyi’s entropy

\( (1) \) If \( x_i = p_i^{\alpha-1} \) (i = 1, 2, ..., n); \( \alpha \in (0, 1) \), then \( P^{\alpha-1} \leq x_i \leq p^{\alpha-1} \), then by \((5.2.7)\), we deduce that
\[ \frac{1}{2} \sum_{ij} p_i p_j \frac{(p_i^{\alpha-1} - p_j^{\alpha-1})}{p_i^{\alpha-1} - p_j^{\alpha-1}} \]
\[ + \frac{1}{Pa^{(1-\alpha)}} \left[ \sum_{i=1}^{n} p_i \phi_s (p_i^{\alpha-1}) - \phi_s \left( \sum_{i=1}^{n} p_i^{\alpha-1} \right) - \frac{1}{2} \sum_{ij} p_i p_j \left( p_i^{\alpha-1} - p_j^{\alpha-1} \right) \left( \phi_s' (p_i^{\alpha-1}) - \phi_s' (p_j^{\alpha-1}) \right) \right] \]
\[ \leq \ln \left( \sum_{i=1}^{n} p_i^{\alpha} \right) - \sum_{i=1}^{n} p_i \ln p_i^{\alpha-1} \]
\[ \leq \frac{1}{2} \sum_{ij} p_i p_j \frac{(p_i^{\alpha-1} - p_j^{\alpha-1})}{p_i^{\alpha-1} - p_j^{\alpha-1}} \]
\[
+ \frac{1}{p^{(\alpha - 1)/2}} \left[ \sum_{i=1}^{n} p_i \phi_s \left( p_i^{\alpha - 1} \right) - \phi_s \left( \sum_{i=1}^{n} p_i^{\alpha} \right) - \frac{1}{2} \sum_{ij} p_i p_j \left( p_i^{\alpha - 1} - p_j^{\alpha - 1} \right) \left( \phi_s \left( p_i^{\alpha - 1} \right) - \phi_s \left( p_j^{\alpha - 1} \right) \right) \right]
\]

\[
\frac{1}{2} \sum_{ij} p_i p_j \left( p_i^{\alpha - 1} - p_j^{\alpha - 1} \right)
\]

\[
+ \frac{1}{p^{(\alpha - 1)/2}} \left[ \sum_{i=1}^{n} p_i \phi_s \left( p_i^{\alpha - 1} \right) - \phi_s \left( \sum_{i=1}^{n} p_i^{\alpha} \right) - \frac{1}{2} \sum_{ij} p_i p_j \left( p_i^{\alpha - 1} - p_j^{\alpha - 1} \right) \left( \phi_s \left( p_i^{\alpha - 1} \right) - \phi_s \left( p_j^{\alpha - 1} \right) \right) \right]
\]

\[
\leq (1 - \alpha) |H_\alpha (X) - H (X)|
\]

\[
\frac{1}{2} \sum_{ij} p_i p_j \left( p_i^{\alpha - 1} - p_j^{\alpha - 1} \right)
\]

\[
+ \frac{1}{p^{(\alpha - 1)/2}} \left[ \sum_{i=1}^{n} p_i \phi_s \left( p_i^{\alpha - 1} \right) - \phi_s \left( \sum_{i=1}^{n} p_i^{\alpha} \right) - \frac{1}{2} \sum_{ij} p_i p_j \left( p_i^{\alpha - 1} - p_j^{\alpha - 1} \right) \left( \phi_s \left( p_i^{\alpha - 1} \right) - \phi_s \left( p_j^{\alpha - 1} \right) \right) \right]
\]

\[
(2) \text{ If } x_i = p_i^{\alpha - 1} (i = 1, 2, ..., n); \alpha \in (1, \infty), \text{ Then } p^{\alpha - 1} \leq x_i \leq P^{\alpha - 1}, \text{ then by (5.2.7), we deduce that}
\]

\[
\frac{1}{2} \sum_{ij} p_i p_j \left( p_i^{\alpha - 1} - p_j^{\alpha - 1} \right)
\]

\[
+ \frac{1}{p^{(\alpha - 1)/2}} \left[ \sum_{i=1}^{n} p_i \phi_s \left( p_i^{\alpha - 1} \right) - \phi_s \left( \sum_{i=1}^{n} p_i^{\alpha} \right) - \frac{1}{2} \sum_{ij} p_i p_j \left( p_i^{\alpha - 1} - p_j^{\alpha - 1} \right) \left( \phi_s \left( p_i^{\alpha - 1} \right) - \phi_s \left( p_j^{\alpha - 1} \right) \right) \right]
\]

\[
\leq (1 - \alpha) |H_\alpha (X) - H (X)|
\]

\[
\frac{1}{2} \sum_{ij} p_i p_j \left( p_i^{\alpha - 1} - p_j^{\alpha - 1} \right)
\]

\[
+ \frac{1}{p^{(\alpha - 1)/2}} \left[ \sum_{i=1}^{n} p_i \phi_s \left( p_i^{\alpha - 1} \right) - \phi_s \left( \sum_{i=1}^{n} p_i^{\alpha} \right) - \frac{1}{2} \sum_{ij} p_i p_j \left( p_i^{\alpha - 1} - p_j^{\alpha - 1} \right) \left( \phi_s \left( p_i^{\alpha - 1} \right) - \phi_s \left( p_j^{\alpha - 1} \right) \right) \right]
\]

\[
5.5 \text{ Application in mutual information}
\]

\textbf{Theorem (5.5.1).} Let X and Y be two random variables with a joint probability mass function \( r (x, y) \) and marginal probability mass function \( p (x) \) and \( q (y) \) respectively, also \( 0 < m \leq \frac{r(x,y)}{p(x)q(y)} \leq M < \infty, \forall (x, y) \in X \times Y. \) Then we have the inequalities

\[
(5.5.1) \quad \frac{1}{M} \left[ \sum_{(x,y) \in X \times Y} p (x) q (y) \phi_s \left( \frac{r(x,y)}{p(x)q(y)} \right) \right] \leq I (X; Y)
\]
\[ \leq \frac{1}{m} \left\{ \sum_{(x,y) \in X \times Y} p(x) q(y) \phi_s \left( \frac{r(x,y)}{p(x)q(y)} \right) \right\} \]

**Proof:** Choosing \( p_i = p(x) q(y) \), \( \xi_i = \frac{r(x,y)}{p(x)q(y)} \) and \( ((x,y) \in X \times Y) \) in lemma (5.3.1) and taking

\[ \sum_{(x,y) \in X \times Y} p(x) q(y) \frac{r(x,y)}{p(x)q(y)} \ln \frac{r(x,y)}{p(x)q(y)} = I(X;Y), \]

Also, \( \phi_s \left( \sum_{(x,y) \in X \times Y} p(x) q(y) \frac{r(x,y)}{p(x)q(y)} \right) = \phi_s(1) = 0. \) Then we have the desired inequality.

Dragomir et al [38] has obtained many interesting bounds for relative entropy and he has shown applications of these bounds in the field of information theory. In the next section, some upper bounds have been obtained for the relative arithmetic geometric divergence measure with the help of some well known inequalities.

**5.6 Upper bounds for the relative arithmetic geometric divergence measure.**

Let \( p(x), q(x), x \in \chi, card(\chi) < \infty \), be two probability mass functions. Taneja [125] defined the relative arithmetic geometric divergence measure as

(5.6.1) \[ G(p//q) = \sum_{x \in \chi} \frac{p(x) + q(x)}{2} \log \frac{p(x) + q(x)}{2p(x)} \]

Also, the \( \chi^2 \)- distance measure is given by Pearson [94]

(5.6.2) \[ D_{\chi^2} (q//p) = \sum_{x \in \chi} \frac{q(x)^2}{p(x)} - 1 \]

**Theorem (5.6.1).** Let \( p(x), q(x), x \in \chi \) be two probability mass functions. Then

(5.6.3) \[ G(p//q) \geq 0 \]

with equality iff \( p(x) = q(x) \ \forall x \in \chi. \)

**Proof:** Let \( A = \{ x : p(x) > 0 \} \) be the support of \( p(x) \). Then

\[ -G(p//q) = -\sum_{x \in A} \frac{p(x) + q(x)}{2} \log \frac{p(x) + q(x)}{2p(x)} \]

\[ = \sum_{x \in A} \frac{p(x) + q(x)}{2} \log \frac{2p(x)}{p(x) + q(x)} \]

\[ \leq \log \left( \sum_{x \in A} \frac{p(x) + q(x)}{2} \cdot \frac{2p(x)}{p(x) + q(x)} \right) \]

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\[= \log \left( \sum_{x \in \chi} p(x) \right) \leq \log \left( \sum_{x \in \chi} q(x) \right) = \log(1) = 0\]

Thus,

\[G(p/q) \geq 0\]

Where the first inequality follows from Jensen’s inequality. Since \( \log \) is strictly concave, we have equality above iff \( \frac{q(x)}{p(x)} = 1 \) everywhere. Hence we have \( G(p/q) = 0 \) iff \( p(x) = q(x) \ \forall x \in \chi \).

**Theorem (5.6.2).** Let \( p(x), q(x), x \in \chi \) be two probability mass functions. Then

\[(5.6.4) \quad G(p/q) \leq \frac{1}{4} \left( \sum_{x \in \chi} \frac{q^2(x)}{p(x)} - 1 \right)\]

with equality iff \( p(x) = q(x) \ \forall x \in \chi \).

**Proof:** We know that for every differentiable real valued strictly convex function \( f \) defined on an interval \( I \) of the real line, we have the inequality

\[(5.6.5) \quad f'(b)(b-a) \geq f(b) - f(a) \quad \forall a, b \in I\]

The equality holds iff \( a = b \).

Now, apply (5.6.5) to \( f(x) = -\log(x) \) and \( I = (0, \infty) \) to get

\[(5.6.6) \quad \frac{1}{b} (a-b) \geq \log a - \log b \quad \forall a, b > 0\]

choose \( a = p(x) + q(x), b = 2p(x), x \in \chi \)

Then by (5.6.6), we get

\[\frac{1}{2p(x)} (q(x) - p(x)) \geq \log \frac{p(x)+q(x)}{2p(x)}, \quad x \in \chi\]

Multiplying by \( \frac{p(x)+q(x)}{2p(x)} > 0 \), we get

\[\frac{(p(x)+q(x))(q(x)-p(x))}{2p(x)} \geq \frac{p(x)+q(x)}{2} \log \frac{p(x)+q(x)}{2p(x)}, \quad \forall x \in \chi\]

Summing over \( x \in \chi \), we get

\[G(p/q) = \sum_{x \in \chi} \frac{p(x)+q(x)}{2p(x)} \log \frac{p(x)+q(x)}{2p(x)} \leq \sum_{x \in \chi} \frac{q^2(x)}{p(x)} - \frac{p^2(x)}{4p(x)}\]

\[G(p/q) \leq \frac{1}{4} \left( \sum_{x \in \chi} \frac{q^2(x)}{p(x)} - 1 \right)\]

The case of equality follows by the strict convexity of \(-\log(.)\).
Theorem (5.6.3). Let \( p(x), q(x) \geq 0, x \in \chi \) be two probability mass functions. Then we have the inequality
\[
0 \leq G(p//q) \leq \log \left[ \frac{1}{4} D_{\chi^2}(q//p) + 1 \right] \leq \frac{1}{4} D_{\chi^2}(q//p)
\]
equality holds iff \( p(x) = q(x) \), \( \forall x \in \chi \).

Proof: We use Jensen’s discrete inequality
\[
f \left( \sum_{x \in \chi} \frac{p(x)+q(x)}{2} t(x) \right) \leq \sum_{x \in \chi} \frac{p(x)+q(x)}{2} f(t(x))
\]
provided that \( f \) is convex on a given interval \( I, t(x) \in I \forall x \in \chi \) and \( p(x), q(x) > 0 \) are probability mass function on \( \chi \).

choose \( f(x) = -\log x, x > 0 \), we obtain from (5.6.8)
\[
-\log \left( \sum_{x \in \chi} \frac{p(x)+q(x)}{2} t(x) \right) \leq - \sum_{x \in \chi} \frac{p(x)+q(x)}{2} \log t(x)
\]
\[
\log \left( \sum_{x \in \chi} \frac{p(x)+q(x)}{2} t(x) \right) \geq \sum_{x \in \chi} \frac{p(x)+q(x)}{2} \log \frac{p(x)+q(x)}{2p(x)}
\]
\[
G(p//q) \leq \log \left( \sum_{x \in \chi} \frac{(p(x)+q(x))^2}{4p(x)} \right)
\]
\[
G(p//q) \leq \log \left( \frac{1}{4} D_{\chi^2}(q//p) + 1 \right)
\]
We use elementary inequality \( \log(u+1) \leq u, u \geq 0 \) with equality iff \( u = 0 \).
\[
\log \left( \frac{1}{4} D_{\chi^2}(q//p) + 1 \right) \leq \frac{1}{4} D_{\chi^2}(q//p)
\]
Thus we can write
\[
G(p//q) \leq \log \left( \frac{1}{4} D_{\chi^2}(q//p) + 1 \right) \leq \frac{1}{4} D_{\chi^2}(q//p)
\]

Lemma (5.6.1). Let \( p(x), q(x) > 0, x \in \chi \) be two probability mass functions. Define \( r(x) = \frac{q(x)}{p(x)}, x \in \chi \) and assume that
\[
0 < r \leq r(x) \leq R < \infty, \forall x \in \chi.
\]
Then we have the inequality
\[
0 \leq G(p//q) \leq \frac{(R-x)^2}{16rR}
\]

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equality holds in (5.6.10) iff \( p(x) = q(x), \ \forall x \in \chi \).

**Proof:** Using the Kantorovitch inequality [89]

\[
\sum_{k=1}^{n} r_k u_k^2 \sum_{k=1}^{n} \frac{1}{n} u_k^2 \leq \frac{1}{4} \left( \sqrt{\frac{M}{m}} + \sqrt{\frac{M}{r}} \right)^2 \left( \sum_{k=1}^{n} u_k^2 \right)^2
\]

where \( 0 < m \leq r_k \leq M < \infty \) for \( k = 1, 2, \ldots, n \)

Put \( u_k = \sqrt{q(x)}, \ r_k = r(x) \) in (5.6.11), we get

\[
\sum_{x \in \chi} r(x) q(x) \sum_{x \in \chi} \frac{1}{p(x)} q(x) \leq \frac{1}{4} \left( \sqrt{\frac{M}{r}} + \sqrt{\frac{M}{r}} \right)^2 \left( \sum_{x \in \chi} q(x) \right)^2
\]

which is equivalent to

\[
\sum_{x \in \chi} \frac{q(x)}{p(x)} \leq \frac{1}{4} \left( \frac{R}{r} + \sqrt{\frac{R}{r}} \right)^2
\]

or

\[
D_{\chi^2}(q/p) \leq \frac{1}{4} \left( \frac{R}{r} + \sqrt{\frac{R}{r}} \right)^2 - 1
\]

\[
- \frac{1}{4} \left( \frac{R}{r} - \sqrt{\frac{R}{r}} \right)^2
\]

(5.6.12) \( D_{\chi^2}(q/p) \leq \frac{(r-r)^2}{4rR} \)

Also, proved in theorem (5.6.2)

\[
G(p/q) \leq \frac{1}{4} D_{\chi^2}(q/p)
\]

Thus,

\[
4G(p/q) \leq D_{\chi^2}(q/p) \leq \frac{(r-r)^2}{4rR}
\]

or

\[
G(p/q) \leq \frac{(r-r)^2}{16rR}
\]

**Theorem (5.6.4).** Let \( p(x) , q(x) > 0, x \in \chi \) be two probability mass functions.

Define \( r(x) = \frac{q(x)}{p(x)}, x \in \chi \) and assume that

\[
0 < r \leq r(x) \leq R < \infty, \ \forall x \in \chi.
\]

Then we have the inequality

(5.6.13) \( G(p/q) \leq \log \left( \frac{(r-r)^2}{16rR} + 1 \right) \leq \frac{(r-r)^2}{16rR} \)

equality holds iff \( p(x) = q(x) \) \( \forall x \in \chi \).

**Proof:** Using the inequality (5.6.7) and (5.6.12), we have

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\[ G(p/q) \leq \log \left( \frac{1}{4}D_{\chi^2}(q/p) + 1 \right) \leq \log \left( \frac{(R-r)^2}{10rR} + 1 \right) \]

or

\[ G(p/q) \leq \log \left( \frac{(R-r)^2}{10rR} + 1 \right) \leq \frac{(R-r)^2}{10rR} \]

The last inequality in (5.6.14) follows by the elementary inequality \( \log (u + 1) \leq u, u \geq 0 \) with equality iff \( u=0 \).

**Lemma (5.6.2).** Let \( p(x), q(x) > 0, x \in \chi \) be two probability mass functions satisfying the condition

\[ 0 < r \leq \frac{p(x)}{q(x)} \leq R < \infty , \ \forall \ x \in \chi. \]

Then we have the inequality

(5.6.15) \[ G(p/q) \leq \frac{1}{4} (1-r) (R-1) \leq \frac{1}{10} (R-r)^2 \]

equality holds in (5.6.15) iff \( p(x) = q(x) \), \( \forall \ x \in \chi \).

**Proof:** Using the Diaz Metcalf inequality for real numbers [89]

(5.6.16) \[ \sum_{k=1}^{n} q_k b_k^2 + m M \sum_{k=1}^{n} q_k a_k^2 \leq (m + M) \sum_{k=1}^{n} q_k a_k b_k \]

provided that

(5.6.17) \[ m \leq \frac{b_k}{a_k} \leq M \text{ for } k = 1, 2, \ldots, n \text{ and } q_k > 0 \text{ with } \sum_{k=1}^{n} q_k = 1 \]

The equality holds in (5.6.16) if either \( b_k = m a_k \) or \( b_k = M a_k \) for \( k = 1, 2, \ldots, n \)

Define

\[ b(x) = \sqrt{\frac{p(x)}{q(x)}}, \quad a(x) = \sqrt{\frac{q(x)}{p(x)}}, \quad x \in \chi. \]

Then

\[ \frac{b(x)}{a(x)} = \frac{p(x)}{q(x)} \in [r, R] \subset (0, \infty), \ \forall \ x \in \chi. \]

From (5.6.16), we get

\[ \sum_{x \in \chi} q(x) \left( \sqrt{\frac{p(x)}{q(x)}} \right)^2 + r R \sum_{x \in \chi} q(x) \left( \sqrt{\frac{q(x)}{p(x)}} \right)^2 \]

\[ \leq (R + r) \sum_{x \in \chi} q(x) \sqrt{\frac{p(x)}{q(x)}} \sqrt{\frac{q(x)}{p(x)}} \]

i.e

\[ \sum_{x \in \chi} \frac{p(x)}{p(x)} + r R \sum_{x \in \chi} p(x) \leq (R + r) \sum_{x \in \chi} q(x) \]
In addition as \( \sum_{x \in \chi} p(x) = \sum_{x \in \chi} q(x) = 1 \), we obtain
\[
\sum_{x \in \chi} \frac{q(x)}{p(x)} + rR \leq (R + r)
\]
\[
\sum_{x \in \chi} \frac{q(x)}{p(x)} \leq R + r - rR
\]
or
\[
\sum_{x \in \chi} \frac{q(x)}{p(x)} - 1 \leq (1 - r)(R - 1)
\]
or
\[
D_{\chi^2} \left( \frac{q}{p} \right) \leq (1 - r)(R - 1)
\]
We use here elementary inequality
\[
ab \leq \frac{1}{4} (a + b)^2, \quad a, b \in \mathbb{R}.
\]
Thus
\[
(1 - r)(R - 1) \leq \frac{1}{4} (R - r)^2
\]
Thus we can write
\[
(5.6.18) \quad D_{\chi^2} \left( \frac{q}{p} \right) \leq (1 - r)(R - 1) \leq \frac{1}{4} (R - r)^2
\]
Using the inequality (5.6.4), we can write
\[
G \left( \frac{p}{q} \right) \leq \frac{1}{4} (1 - r)(R - 1) \leq \frac{1}{16} (R - r)^2
\]
**Theorem (5.6.5).** Let \( p(x), q(x) > 0, x \in \chi \) be two probability mass functions.
Define \( r(x) = \frac{q(x)}{p(x)}, x \in \chi \) and assume that
\[
(5.6.19) \quad 0 < r \leq r(x) \leq R < \infty, \quad \forall \ x \in \chi.
\]
Then we have the inequality
\[
(5.6.20) \quad G \left( \frac{p}{q} \right) \leq \log \left( \frac{(1-r)(R-1)}{4} + 1 \right) \leq \log \left( \frac{(R-r)^2}{16} + 1 \right)
\]
equality holds iff \( p(x) = q(x), \quad \forall \ x \in \chi. \)

**Proof:** Using theorem (5.6.3) and inequality (5.6.8), we have
\[
G \left( \frac{p}{q} \right) \leq \log \left( \frac{1}{4} D_{\chi^2} \left( \frac{q}{p} \right) + 1 \right) \leq \log \left( \frac{(1-r)(R-1)}{4} + 1 \right) \leq \log \left( \frac{(R-r)^2}{16} + 1 \right)
\]
or
\[
G \left( \frac{p}{q} \right) \leq \log \left( \frac{(1-r)(R-1)}{4} + 1 \right) \leq \log \left( \frac{(R-r)^2}{16} + 1 \right)
\]

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