Chapter 4

INVERSE EQUITABLE DOMINATION IN GRAPHS

4.1 Introduction

The concept of inverse domination in graphs introduced by V.R. Kulli and S.C. Sigarkanti [21]. Let $G = (V, E)$ be a graph with no isolated vertex. A classical observation in domination theory is that if $D$ is a minimum dominating set of $G$, then $V - D$ is also a dominating set of $G$. A set $D'$ is an inverse dominating set of $G$ if $D'$ is a dominating set of $G$ and $D' \subseteq V - D$ or some minimum dominating set $D$ of $G$. The inverse domination number of $G$ is the minimum cardinality among all inverse dominating sets of $G$. In this chapter, we introduce the inverse equitable domination in a graph and begin an investigation of this concept, some of the properties and interesting results of this new parameter are obtained.
**Definition 4.1.1.** Let $G = (V, E)$ be a graph with no equitable isolated vertices. If $D$ is a minimum equitable dominating set of $G$, then $V - D$ is also equitable dominating set of $G$. A set $D'$ is an inverse equitable dominating set of $G$ if $D'$ is an equitable dominating set of $G$ and $D' \subseteq V - D$ for some minimum equitable dominating set $D$ of $G$. The inverse equitable domination number of $G$ is the minimum cardinality among all inverse equitable dominating sets of $G$ and is denoted by $\gamma_{e^{-1}}(G)$.

**Definition 4.1.2.** Let $G = (V, E)$ be a graph with no equitable isolated vertices. If $D$ is a minimum equitable dominating set and $D'$ is an inverse equitable dominating set with respect to $D$. Then $D'$ is called minimal inverse equitable dominating set if no proper subset of $D'$ is an equitable dominating set of $G$.

**Example 4.1.3.** From the Figure 4.1 $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$. There is only two minimum dominating set $\{v_4, v_5\}$, $\{v_2, v_7\}$ of graph $G$ and obviously the minimum inverse dominating set corresponding to $\{v_4, v_5\}$ is $\{v_2, v_7\}$ and vise versa. Therefore $\gamma(G) = 2$ and $\gamma'(G) = 2$.

There is only minimum equitable dominating set $\{v_2, v_7\}$ and there are two corresponding minimum inverse equitable dominating set $\{v_1, v_3, v_4, v_6, v_8\}$ and $\{v_1, v_3, v_5, v_6, v_8\}$. Thus $\gamma_e(G) = 2$ and $\gamma_{e^{-1}}(G) = 5$. 

43
Note that every graph without equitable isolated vertices contains an inverse
equitable dominating set, since if $D$ is any minimal dominating set then $V - D$
is also equitable dominating set. So here by graph we mean graph without any
isolated vertices.

First we obtain the exact values of inverse equitable domination of some
standard graphs. The proof of the following propositions is straightforward.

Proposition 4.1.4. For any cycle $C_p$ with $p$ vertices, $\gamma_e^{-1}(C_p) = \lceil \frac{p}{3} \rceil$.

Proposition 4.1.5. For any path $P_p$ with $p$ vertices,

$$\gamma_e^{-1}(P_p) = \begin{cases} 
\lceil \frac{p}{3} \rceil + 1 & \text{if } p \equiv 0 \pmod{3} \\
\lceil \frac{p}{3} \rceil & \text{otherwise}.
\end{cases}$$

Proposition 4.1.6. For any complete graph $G$, $\gamma_e^{-1}(K_p) = 1$.

Proposition 4.1.7. For any complete bipartite graph $K_{m,n}$ without any equitable
isolated vertices, $\gamma_e^{-1}(K_{m,n}) = 2$.

Proposition 4.1.8. For any graph $G$ without any equitable isolated vertices,
$\gamma_e(G) \leq \gamma_e^{-1}(G)$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure41.png}
\caption{$G$}
\end{figure}
Proposition 4.1.9. For any graph $G$ without any equitable isolated vertices, $\gamma_e(G) + \gamma_e^{-1}(G) \leq p$.

Observation 4.1.10. Any minimum equitable dominating set is also minimal but the converse is not true.

The following theorem we get the sufficient and necessary condition for the inverse equitable dominating set to be minimal.

Theorem 4.1.11. An inverse equitable dominating set $D'$ of a graph $G$ is minimal if and only if for every vertex $u \in D'$ one of the following conditions holds.

1. There exists a vertex $v \in V - D'$ such that $N_e(v) \cap D' = \{u\}$.

2. $N_e(u) \cap D' = \phi$.

Proof. Suppose that $D'$ is an inverse equitable dominating set of $G$ and let conditions (1) and (2) not hold. Then for some vertex $u \in D'$ there exists $v \in N_e(u) \cap D'$. Therefore $D' - \{u\}$ is an equitable dominating set of $G$, a contradiction with the minimality of $D'$.

Conversely, let for every $u \in D'$ one of the conditions (1) or (2) holds. Suppose that $D'$ is not minimal. Then there exists $u \in D'$ such that $D' - \{u\}$ is an equitable dominating set of $G$. That means there exists $v \in D' - \{u\}$ which is an equitable adjacent to $u$. Hence (2) does not satisfy.
Theorem 4.1.12. For any graph $G$ without any equitable isolated vertices, 
\[ \gamma(G) \leq \min\{\gamma^{-1}(G), \gamma_e(G), \gamma_e^{-1}(G)\} \].

Proof. Since every inverse equitable dominating set is an inverse dominating set of $G$ and every inverse dominating set is dominating set, similarly every equitable dominating set is dominating set. Hence $\gamma(G) \leq \min\{\gamma^{-1}(G), \gamma_e(G), \gamma_e^{-1}(G)\}$.

In the following result gives a sufficient condition for a graph $G$ with no equitable isolated vertices to have $\gamma_e(G) = \gamma_e^{-1}(G)$.

Theorem 4.1.13. Let $G$ be a graph without any equitable isolated vertices and let $D$ be a minimum equitable dominating set. If for every $v \in D$ the induced subgraph $\langle N_e[v] \rangle$ is a complete graph of order at least 2, then $\gamma_e(G) = \gamma_e^{-1}(G)$.

Proof. Let $D = \{v_1, v_2, \ldots, v_p\}$ be a minimum equitable dominating set of $G$ and $\{u_1, u_2, \ldots, u_p\}$ be the vertices which they are equitable adjacent to $\{v_1, v_2, \ldots, v_p\}$ respectively. Since for each vertex $v_i \in D$ the graph $\langle N_e[v_i] \rangle$ is complete. Then $N_e[v_i] \subseteq N_e[u_i]$. Hence $V(G) = \bigcup_{i=1}^{p} N_e[v_i] \subseteq \bigcup_{i=1}^{p} N_e[u_i] = V(G)$. Thus $\{u_1, u_2, \ldots, u_p\} = D'$ is an inverse equitable dominating set of $G$ that means $\gamma^{-1}(G) \leq |D'| = |D| = \gamma_e(G)$. Also by proposition 4.1.8 we have $\gamma_e(G) \leq \gamma_e^{-1}(G)$. Hence $\gamma_e(G) = \gamma_e^{-1}(G)$. 

46
**Theorem 4.1.14.** Let $G$ be a graph with $p$ vertices and let $\tau$ denote the family of minimum equitable dominating set of $G$. If for any $D \in \tau$ we have $V - D$ is an independent set, then $\gamma_e(G) + \gamma_e^{-1}(G) = p$.

**Proof.** Since for any $D \in \tau$ we have $V - D$ is an independent set, then $V - D$ is minimum inverse equitable dominating set. Hence $\gamma_e(G) + \gamma_e^{-1}(G) = p$.

**Theorem 4.1.15.** Let $G$ be a graph a $(p, q)$ graph and has no equitable isolated vertices and $\gamma_e(G) = \gamma_e^{-1}(G)$. Then $\frac{2p-q}{3} \leq \gamma_e^{-1}(G)$.

**Proof.** Let $D$ and $D'$ be the minimum equitable dominating set and the corresponding inverse equitable dominating set of $G$ respectively. Let $A = \{V - (D \cup D')\}$ obviously $|A| = p - 2\gamma_e^{-1}(G)$. Now each vertex in $A$ has at least one edge to $D$ and at least one edge to $D'$. Therefore $q \geq 2(p - 2\gamma_e^{-1}(G)) + \gamma_e^{-1}(G)$. Hence $\frac{2p-q}{3} \leq \gamma_e^{-1}(G)$.

**Corollary 4.1.16.** For any tree $T_p$ without any equitable isolated vertices, $\frac{p+1}{3} \leq \gamma_e^{-1}(G)$. 

47
4.2 Inverse Equitable Edge Domination Number

4.3 Introduction

Anwar Alwardi and N.D. Soner [3] introduced the notion Equitable edge domination in graphs. Let \( G = (V, E) \) be a graph, for any edge \( f \in E \) the degree of an edge \( f = uv \) in \( G \) is defined by \( \text{deg}(f) = \text{deg}(u) + \text{deg}(v) - 2 \). A set \( S \subseteq E \) of edges is an equitable edge dominating set of \( G \) if every edge \( f \) not in \( S \) is adjacent to at least one edge \( f' \in S \) such that \(|\text{deg}(f) - \text{deg}(f')| \leq 1\). The minimum cardinality of such equitable edge dominating set is denoted by \( \gamma'_e(G) \) and is called an equitable edge domination number of \( G \). \( S \) is minimal if for any edge \( f \in S \), \( S - \{f\} \) is not an equitable edge dominating set of \( G \). A subset \( S \) of \( E \) is called an equitable edge independent set, if for any \( f \in S \), \( f \notin N_e(g) \), for all \( g \in S - \{f\} \). If an edge \( f \in E \) be such that \(|\text{deg}(f) - \text{deg}(g)| \geq 2 \) for every \( g \in N(f) \) then \( f \) is in every equitable edge dominating set such edges are called equitable edge isolates. Let \( f \in E \) the equitable edge neighborhood of \( f \) denoted by \( N_e(f) \) is defined as \( N_e(f) = \{g \in E : g \in N(f), |\text{deg}(f) - \text{deg}(g)| \leq 1\} \). The maximum and minimum equitable edge degree of a point in \( G \) are denoted by \( \Delta'_e(G) \) and \( \delta'_e(G) \) that is \( \Delta'_e(G) = \max_{f \in E(G)} |N_e(f)| \) and \( \delta'_e(G) = \min_{f \in E(G)} |N_e(f)| \). The open equitable neighborhood and closed equitable neighborhood of \( g \) are denoted by \( N_e(g) \) and \( N_e[g] = N_e(g) \cup \{g\} \) respectively. If \( S \subseteq E \) then \( N_e(S) = \cup_{g \in S} N_e(g) \). 

48
and \(N[S] = N_e(S) \cup S\). If \(S \subseteq V\) and \(f \in S\) then the private equitable edge neighbor set of \(f\) with respect to \(S\) is defined by \(pne[f, S] = N_e[f] - N_e[S - \{f\}]\).

**Definition 4.3.1.** Let \(F\) be minimum equitable edge dominating set of a graph \(G = (V, E)\). If \(E - F\) contains an equitable edge dominating set \(F'\), then \(F'\) is called an inverse equitable edge dominating set of \(G\) with respect to \(F\). The minimum number of edges in an inverse equitable edge dominating set of \(G\) is called the inverse equitable edge domination number and denoted by \(\gamma_{eed}^{-1}(G)\).

**Example 4.3.2.** From the graph of \(G\) Figure 4.2 the minimum equitable edge dominating set is \(\{e_2, e_4, e_6\}\) and the minimum inverse equitable edge dominating sets are \(\{e_1, e_3, e_5, e_7\}\) and \(\{e_1, e_7, e_8, e_9\}\). Hence \(\gamma_e'(G) = 3\) and \(\gamma_{eed}^{-1}(G) = 4\).

![Figure 4.2: G](image)

Obviously the inverse equitable edge dominating set exist if \(G\) has no equitable isolated edge.

**Theorem 4.3.3.** A graph \(G = (V, E)\) has an inverse equitable edge dominating set if and only if \(G\) has no equitable isolated edge.

**Proof.** If \(G = (V, E)\) has no equitable isolated edge and \(F\) is an equitable edge dominating set, then \(E - F\) is an inverse equitable edge dominating set.
Conversely, Let $G$ has an equitable edge dominating set $F$ and inverse equitable edge dominating set $F'$. Suppose $G$ has an equitable isolated edge $f$, then $f$ must belong to $F$ and $F'$, a contradiction.

**Proposition 4.3.4.** For any graph $G$ without isolated edges, $\gamma'_e(G) \leq \gamma_{eed}^{-1}(G)$, where $\gamma'_e(G)$ is the equitable edge domination number of $G$.

**Proof.** Since each inverse equitable edge dominating set of graph $G$ is an equitable edge dominating set, then the proof is straightforward.

**Theorem 4.3.5.** Let $F$ be minimum equitable edge dominating set of $G$. If for each edge $f \in F$, the induced subgraph $N_e(f)$ is star, then $\gamma'_e(G) = \gamma_{eed}^{-1}(G)$.

**Proof.** Let $F$ be minimum equitable edge dominating set of $G$. Since for each edge $f \in F$, the induced subgraph $N_e(f)$ is star, then $F' = \{f' : f' is an equitable adjacent to $f \in F\}$ is a minimum inverse equitable edge dominating set. Thus $\gamma_{eed}^{-1}(G) = |F'| = |F| = \gamma'_e(G)$.

In the following Proposition we list the inverse equitable edge domination number for some standard graph. The proof of this proposition is straightforward so we omitted the proof.

**Proposition 4.3.6.** For any complete graph $K_p$, path $P_p$, cycle graph $C_p$ and complete bipartite graph $K_{r,s}$, we have.

1. $\gamma_{eed}^{-1}(K_p) = \lfloor \frac{p}{2} \rfloor$, if $p \geq 3$.

2. $\gamma_{eed}^{-1}(P_p) = \lceil \frac{p}{3} \rceil$, if $p \geq 3$.
3. \( \gamma_{eed}(C_p) = \left\lceil \frac{p}{3} \right\rceil \), if \( p \geq 3 \).

4. \( \gamma_{eed}(K_{r,s}) = \min\{r, s\} \), if \( r, s \geq 2 \).

**Theorem 4.3.7.** [3] For any \((p, q)\) graph \( G \), \( \left\lceil \frac{q - \beta_e'}{\Delta_e'(G) + 1} \right\rceil \leq \gamma_e'(G) \leq q - \beta_e' + q_0 \), where \( q_0 \) is the number of equitable isolated edges.

**Theorem 4.3.8.** If \( G \) is a graph without equitable isolated edges and \( p \geq 3 \), then
\[
\gamma_e'(G) + \gamma_{eed}^{-1}(G) \leq q.
\]

**Proof.** Let \( G = (V, E) \) be any graph without any equitable isolated edges and let \( F \) and \( F' \) be the minimum equitable and inverse equitable dominating sets of \( G \). Then \( F \cup F' \subseteq E(G) \), Hence \( \gamma_e'(G) + \gamma_{eed}^{-1}(G) \leq q \).

**Theorem 4.3.9.** If \( G \) is a graph without equitable isolated edges and \( p \geq 3 \), then
\[
\gamma_{eed}^{-1}(G) \leq \left\lceil \frac{q \Delta_e'(G)}{\Delta_e(G) + 1} \right\rceil.
\]
Further the equality holds if \( G = P_3 \) or \( P_4 \).

**Proof.** By Theorem 4.3.7. [3] we have \( \left\lceil \frac{q}{\Delta_e'(G) + 1} \right\rceil \leq \gamma_e'(G) \) and by Theorem 4.3.8, we have \( \gamma_e'(G) + \gamma_{eed}^{-1}(G) \leq q \), therefore \( \gamma_{eed}^{-1}(G) \leq q - \gamma_e'(G) \leq q - \left\lceil \frac{q}{\Delta_e(G) + 1} \right\rceil = \left\lceil \frac{q \Delta_e'(G)}{\Delta_e(G) + 1} \right\rceil. \)