Here we first present various expressions for a simple cubic splines.

(i) In terms of the values \( y_j \) and the moments (or the second derivations) \( M_j \) taken at the nodal points \( x_j \).

and

(ii) in terms of the values \( y_j \) and the slopes (or the first derivatives) \( m_j \) taken at the nodal points \( x_j \).

2.1 Various expressions for simple cubic splines.

Theorem 2.1.

Let \( S_\Delta(x) \) be a simple cubic spline on a mesh \( \Delta: a = x_0 < x_1 < \ldots < x_N = b \). If \( y_j = S_\Delta(x_j) \) and \( M_j = S_\Delta''(x_j); j = 0,1,2,\ldots,N \) and if \( h_j \) is the width of \( j^{th} \) subinterval, then on the subinterval \([x_{j-1}, x_j]\).
Moreover, $M_0, M_1, \ldots M_N$ satisfy the following system of equations.

$$\lambda_j M_{j-1} + 2M_j + \mu_j M_{j+1} = d_j; \quad (j = 1, 2, \ldots N-1) \quad (2.2)$$

where $\lambda_j = \frac{h_{j+1}}{h_j + h_{j+1}}; \quad \mu_j = 1 - \lambda_j$ and

$$d_j = \frac{6}{h_j + h_{j+1}} \left\{ \frac{y_{j+1} - y_j}{h_{j+1}} - \frac{y_j - y_{j-1}}{h_j} \right\}$$

Proof:

Fix $j$, consider the $j^{th}$ subinterval $[x_{j-1}, x_j]$. On this subinterval, $S_\Delta(x)$ is a cubic polynomial. Hence $S''_\Delta(x)$ is linear in this subinterval. Also

$$S''_\Delta(x_{j-1}) = M_{j-1} \quad \text{and} \quad S''_\Delta(x_j) = M_j$$

Hence for $x$ in $[x_{j-1}, x_j]$

$$S''_\Delta(x) = M_{j-1} \frac{x - x_{j-1}}{h_j} + M_j \frac{x - x_j}{h_j}$$

Integrating, we get, on $[x_{j-1}, x_j]$. 

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for some constants \( C_1 \) and \( C_2 \). Also

\[
y_j = S_\Delta(x_j) = M_{j-1} \frac{h_j^2}{6j} + C_1 x_j + C_2
\]

and

\[
y_{j-1} = S_\Delta(x_{j-1}) = M_{j-1} \frac{h_j^2}{6j} + C_1 x_{j-1} + C_2
\]

Solving, we get,

\[
C_1 = \frac{y_j - y_{j-1}}{h_j} - \frac{M_j - M_{j-1}}{6j} h_j
\]

and

\[
C_2 = \left( M_j x_{j-1} - M_{j-1} x_j \right) h_j
\]

Substituting for \( C_1 \) and \( C_2 \) in \( C_1x + C_2 \) and simplifying, we get

\[
C_1 x + C_2 = \left( \frac{x_j - x}{h_j} \right) \left[ y_{j-1} - \frac{M_{j-1} h_j^2}{6j} \right] + \frac{(x-x_{j-1})}{h_j} \left[ y_j - \frac{M_j h_j^2}{6j} \right]
\]

Substituting this value of \( C_1 \) and \( C_2 \) in (2.3) we get the required expression for \( S_\Delta(x) \) on \( [x_{j-1}, x_j] \).

From expression (2.1) on differentiating we get on \( [x_{j-1}, x_j] \).
\[ S_\Delta'(x) = -M_{j-1} \frac{(x_j - x)^2}{2h_j} + M_j \frac{(x - x_{j-1})^2}{2h_j} + \frac{y_j - y_{j-1}}{h_j} - \frac{1}{6} \left[ M_j - M_{j-1} \right] h_j \]

This gives

\[ S_\Delta'(x) = M_j \frac{h_j}{2} + \frac{y_j - y_{j-1}}{h_j} - \left( M_j - M_{j-1} \right) \frac{h_j}{6} \]

Similarly, from the expression for \( S_\Delta(x) \) on \((x_j, x_{j+1})\), we get

\[ S_\Delta'(x) = -M_j \frac{h_{j+1}}{2} + \frac{y_{j+1} - y_j}{h_{j+1}} - \left( M_{j+1} - M_j \right) \frac{h_{j+1}}{6} \]

Since \( S_\Delta'(x) \) is continuous \( S_\Delta'(x-) = S_\Delta'(x+) \)

Hence, for \( j = 1, 2, \ldots, N - 1 \)

\[ M_j \frac{h_j}{2} + \frac{y_j - y_{j-1}}{h_j} - \left( M_j - M_{j-1} \right) \frac{h_j}{6} = -M_j \frac{h_{j+1}}{2} + \frac{y_{j+1} - y_j}{h_{j+1}} - \left( M_{j+1} - M_j \right) \frac{h_{j+1}}{6} \]

That is, for \( j = 1, 2, \ldots, N - 1 \),

\[ \frac{h_j}{6} M_{j-1} + \frac{h_j + h_{j+1}}{3} M_j + \frac{h_{j+1}}{6} M_{j+1} = \frac{y_{j+1} - y_j}{h_{j+1}} - \frac{y_j - y_{j-1}}{h_j} \]

Substituting for \( \lambda_j, \mu_j \) and \( d_j \), we arrive at the system of equation (2.2)
Remark 2.1

If $S_\Delta(x)$ is a periodic cubic spline then $M_0 = M_N$ and in this case $M_1, M_2, \ldots, M_N$ satisfy the equation (2.2) for $j = N$ also, where $M_{N+1} = M_1, y_0 = y_N, y_{N+1} = y_1$. Thus the $N$ equations in $M_1, M_2, \ldots, M_N$ completely determine the values $M_1, M_2, \ldots, M_N$. In matrix form these equations can be written as follows.

$$
\begin{bmatrix}
2 & \lambda_1 & 0 & 0 & \cdots & \mu_1 \\
\mu_2 & 2 & \lambda_2 & 0 & \cdots & 0 \\
0 & \mu_3 & 2 & \lambda_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mu_{N-1} & 2 & \lambda_{N-1} \\
\lambda_N & 0 & \cdots & 0 & 2 & \mu_N
\end{bmatrix}
\begin{bmatrix}
M_1 \\
M_2 \\
\vdots \\
M_{N-1} \\
M_N
\end{bmatrix}
= 
\begin{bmatrix}
d_1 \\
d_2 \\
\vdots \\
d_{N-1} \\
d_N
\end{bmatrix}
$$

where $M_0 = M_N$ and $\lambda_N = \frac{h_1}{h_N + h_1}, \mu_N = 1 - \lambda_N$

Remark 2.2.

If $S_\Delta(x)$ is not periodic, to determine $M_0, \ldots, M_N$ completely we require two more conditions apart from the system of $N - 1$ equations (2.2). Then two end conditions are also specified.

$$M_0 + \lambda_0 M_1 = d_0$$
and

\[ \mu_N M_{N-1} + M_N = d_N \]

Together with these end conditions, the system of \( N+1 \) equations for the non-periodic spline can be written in the matrix form as follows.

\[
\begin{bmatrix}
2 & \lambda_0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\mu_1 & 2 & \lambda_1 & 0 & \ldots & 0 & 0 & 0 \\
0 & \mu_2 & 2 & \lambda_2 & \ldots & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & \mu_{N-1} & 2 & \hat{\lambda}_{N-1} \\
0 & 0 & 0 & \ldots & 0 & \mu_N & 2 \\
\end{bmatrix}
\begin{bmatrix}
M_0 \\
M_1 \\
M_2 \\
\vdots \\
\vdots \\
M_{N-1} \\
M_N \\
\end{bmatrix} =
\begin{bmatrix}
d_0 \\
d_1 \\
d_2 \\
\vdots \\
\vdots \\
d_{N-1} \\
D_N \\
\end{bmatrix}
\]  

\[ \text{(2.5)} \]

Note:

If \( \Delta : a = x_0 < x_1 < \ldots < x_N = b \) is a uniform mesh of points and if each subinterval of length \( 'h' \), then the equation (2.2) reduces to the following simple form.

For \( j=1,2,\ldots N-1 \)

\[ M_{j-1} + 4M_j + M_{j+1} = \frac{6}{h^2} [y_{j+1} - 2y_j + y_{j-1}] \]  

\[ \text{(2.6)} \]
For many applications, it is more convenient to work with the slopes \( m_j = S'_\Delta(x_j) \) rather than the moments \( M_j = S''(x_j) \).

In terms of the slopes \( m_j \) and the values \( y_j \) at \( x_j \), we have the following expressions for the simple cubic splines \( S_\Delta(x) \).

**Theorem 2.2.**

*If \( S_\Delta(x) \) is a simple cubic spline on a mesh \( \Delta: a = x_0 < x_1 < \ldots < x_N = b \). Taking on the values \( y_j \) at \( x_j \) and having slopes \( m_j \) at these points and if \( h_j \) is the width of the \( j^{th} \) subinterval \( [x_{j-1}, x_j] \), then on the subinterval, \( S_\Delta(x) \) is given by the following expression.*

\[
S_\Delta(x) = m_{j-1} \frac{(x_j - x)^2(x - x_{j-1})}{h_j^2} - m_j \frac{(x - x_{j-1})^2(x_j - x)}{h_j^2} + y_{j-1}(x_j - x)^2 \left[ \frac{2(x - x_{j-1}) + h_j}{h_j^3} \right] + y_j \frac{(x - x_{j-1})^2}{h_j^3} \left[ 2(x_j - x) + h_j \right] \quad \text{....(2.7)}
\]

Further \( m_0, m_1, \ldots, m_N \) satisfy the following equations for \( j = 1, 2, \ldots, N - 1 \)

\[
\lambda_j m_{j-1} + 2m_j + \mu_j m_{j+1} = c_j \quad \text{....................(2.8)}
\]

where

\[
c_j = 3\lambda_j \cdot \frac{y_j - y_{j-1}}{h_j} + 3\mu_j \frac{y_{j+1} - y_j}{h_{j+1}}
\]

\[
\lambda_j = \frac{h_j}{h_j + h_{j+1}}, \quad \mu_j = 1 - \lambda_j
\]
Proof:

We know that any polynomial of degree ‘3’ on an interval \([x_1, x_2]\)
can be expressed as

\[
y(x) = y(x_1)A_1(x) + y(x_2)A_2(x) + y'(x_1)B_1(x) + y'(x_2)B_2(x)
\]

where \(A_1, A_2, B_1\) and \(B_1, B_2\) are cubic polynomials satisfying the following conditions.

\[
A_i(x_j) = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{otherwise}
\end{cases}; \quad A'_i(x_j) = 0 \text{ for } j = 1, 2
\]

And

\[
B_i(x_j) = 0 \text{ for } j = 1, 2; \quad B'_i(x_j) = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{otherwise}
\end{cases}
\]

The explicit expression for \(A_1, A_2, B_1\) and \(B_2\) are given by

\[
A_1(x) = (x - x_2)^2 \left( \frac{2(x_1 - x) + (x_1 - x_2)}{(x_1 - x_2)^3} \right);
\]

\[
A_2(x) = (x - x_1)^2 \left( \frac{2(x_2 - x) + (x_2 - x_1)}{(x_1 - x_2)^3} \right);
\]

\[
B_1(x) = \frac{(x - x_1)(x - x_2)}{(x_1 - x_2)^2}; \quad B_2(x) = \frac{(x - x_2)(x - x_1)}{(x_1 - x_2)^2}
\]

From this, we get if \(x \in [x_{j-1}, x_j]\)
\[ S_\Delta (x) = m_{j-1} \frac{(x_j - x)^2(x - x_{j-1})}{h_j^2} - m_j \frac{(x - x_{j-1})^2(x_j - x)}{h_j^2} + y_{j-1}(x_j - x)^2 \frac{2(x - x_{j-1}) + h_j}{h_j^3} + y_j(x - x_{j-1})^2 \frac{(x_j - x) + h_j}{h_j^3} \]

Hence on \([x_{j-1}, x_j]\)

\[ \therefore S'_\Delta (x) = \frac{m_{j-1}}{h_j^2} [x_j - x] [2x_{j-1} + x_j - 3x] - \frac{m_j}{h_j^2} (x - x_{j-1}) (2x_j + x_{j-1} - 3x) \]

\[ + \frac{(y_j - y_{j-1})}{h_j^3} 6(x_j - x)(x - x_{j-1}) \]

And on \([x_{j-1}, x_j]\)

\[ S''_\Delta (x) = \frac{m_{j-1}}{h_j^2} [6x - 4x_j - 2x_{j-1}] - \frac{m_j}{h_j^2} [-6x + 4x_{j-1} + 2x_j] \]

\[ + \frac{6|y_j - y_{j-1}| [x_j + x_{j-1} - 2x]}{h_j^3} \]

\[ S''_\Delta (x_j-) = \frac{m_{j-2}}{h_j} + \frac{4m_j}{h_j} - \frac{6(y_j - y_{j-1})}{h_j^2} \]

\[ S''_\Delta (x_{j+}) = \frac{-4m_j}{h_{j+1}} - \frac{2m_{j+1}}{h_{j+1}} + \frac{6}{h_{j+1}^2} (y_{j+1} - y_j) \]

Since \(S''_\Delta\) is continuous
\[ S''_\Delta (x_j) = S''_\Delta (x^+_{j+1}) \quad j = 1, 2, \ldots, N - 1 \]

i.e.,
\[ \frac{m_{j-1}}{h_j} + 2m_j \left[ \frac{1}{h_j} + \frac{1}{h_{j+1}} \right] + \frac{m_{j+1}}{h_j} = \frac{3(y_j - y_{j-1})}{h_j^2} + \frac{3(y_{j+1} - y_j)}{h_j^2 + 1} \]

Substituting for \( \lambda_j, \mu_j \) and \( c_j \) we arrive at the system of equation (2.8)

\begin{align*}
\text{Remark 2.3.} \\
\text{If } S_\Delta(x) \text{ is periodic cubic simple splines then } m_0 = m_N \text{ and in this case } m_1, m_2, \ldots, m_N \text{ satisfy the equation (2.8) for } j = N \text{ also where } m_{N+1} = m_1, y_0 = y_N, y_{N+1} = y_1. \text{ Thus the } N \text{ equations in } m_1, m_2, \ldots, m_N \text{ completely determine the values } m_1, m_2, \ldots, m_N. \text{ In matrix form these equations can be written as follows.}
\end{align*}

In matrix form these equations can be written as follows.

\[ \begin{bmatrix}
2 & \lambda_1 & 0 & 0 & \ldots & \mu_1 \\
\mu_2 & 2 & \lambda_2 & 0 & \ldots & 0 \\
0 & \mu_3 & 2 & \lambda_3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mu_{N-1} & \vdots & \vdots & \vdots & \ddots & 2 \\
\lambda_N & 0 & \ldots & 0 & \mu_N & 2
\end{bmatrix}
\begin{bmatrix}
m_1 \\
m_2 \\
m_3 \\
\vdots \\
m_{N-1} \\
m_N
\end{bmatrix}
= 
\begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
\vdots \\
c_{N-1} \\
c_N
\end{bmatrix}
\]

\[ \text{..(2.9)} \]
where \( m_0 = m_N \) and \( \lambda_N = \frac{h_1}{h_N + h_1}, \mu_N = 1 - \lambda_N \)

**Remark 2.4.**

If \( S_\Delta(x) \) is periodic, to determine \( m_0 \ldots m_N \) completely we require two more conditions apart from the system of \( N-1 \) equations (2.8.) Then two end conditions are also specified.

\[
m_0 + \lambda_0 m_1 = c_0
\]

and

\[
\mu_N m_{N-1} + m_N = c_N
\]

Together with these end conditions the system of \( N+1 \) equations for the non-periodic splines can be written in the matrix form as follows.

\[
\begin{bmatrix}
2 & \mu_0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\mu_1 & 2 & \lambda_1 & 0 & \ldots & 0 & 0 & 0 \\
0 & \mu_2 & 2 & \lambda_2 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \mu_{N-1} & 2 & \lambda_{N-1} & \mu_N \\
0 & 0 & 0 & \ldots & 0 & \mu_N & 2
\end{bmatrix}
\begin{bmatrix}
m_0 \\
m_1 \\
m_2 \\
\vdots \\
m_{N-1} \\
m_N
\end{bmatrix}
= \begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
\vdots \\
c_{N-1} \\
c_N
\end{bmatrix}
\]

**Note:**

If \( \Delta : a = x_0 < x_i \ldots < x_N = b \) is a uniform mesh of points and if each subinterval is of length ‘\( h \)’, then the equation (2.5) reduces to
the following simple form.

\[ m_{j-1} + 4m_j + m_{j+1} = \frac{6}{h} \left[ y_{j+1} - 2y_j + y_{j-1} \right]; \quad j = 1, \ldots, N - 1 \ldots \ldots \ldots \ldots \ldots (2.11) \]

Note: The end conditions \( m_0 = y'_0 \) and \( m_N = y'_N \) are equivalent to the end conditions.

\[
2M_0 + M_1 = \frac{6}{h} \left[ \frac{y_1 - y_0}{h} - y'_0 \right]
\]
\[
M_{N-1} + 2M_N = \frac{6}{h_N} \left[ \frac{y'_N - y_N - y_{N-1}}{h_N} \right]
\].................................(2.12)

Similarly the end conditions \( M_0 = y''_0 \) and \( M_N = y''_N \) are equivalent to end conditions

\[
2m_0 + m_1 = 3 \frac{y_1 - y_0}{h} - \frac{h}{2} y''_0
\]
\[
m_{N-1} + 2m_N = 3 \frac{y_N - y_{N-1}}{h_N} + \frac{h_N}{2} y''_N
\].................................(2.13)

From the expression for \( S''_{\Delta}(x_j) \) derived in the proof of Theorem 2.2 we get

\[
M_0 = S''_{\Delta}(x_0) = -\frac{4m_0 + 2m_1}{h_1} + 6 \frac{y_1 - y_0}{h_1^2}
\]

and \( M_1 = S''_{\Delta}(x_1) = \frac{2m_0 + 4m_1}{h_1} - 6 \frac{y_1 - y_0}{h_1^2} \).

Eliminating \( m_1 \) from these equations we get the first equation (2.12). Second equation in (2.12) is obtained in a similar manner. Similarly from the expression for \( S''_{\Delta}(x_j) \) derived in the proof of Theorem 2.1 we get equation (2.13).
Remark 2.5

From the expression for \( S''_\Delta(x_j) \) in the proof of Theorem 2.2. we get,

\[
M_0 = S''_\Delta(x_0) = -\frac{4m_0 + 2m_1}{h_1} + 6 \frac{(y_1 - y_0)}{h_1^2}
\]

and

\[
M_1 = S''_\Delta(x_i) = \frac{2m_0 + 4m_1}{h} - 6 \frac{(y_1 - y_0)}{h_1^2}
\]

Hence \( 2M_0 + M_1 = -\frac{6m_0}{h_1} + 6 \left( \frac{y_1 - y_0}{h_1^2} \right) \)

i.e., \( 2M_0 + M_1 = \frac{6}{h_1} \left( \frac{y_1 - y_0}{h_1^2} - y'_0 \right) \quad \text{where} \quad m_0 = y_0 \quad \text{...........}(2.12) \)

Similarly from the expression for \( S'_\Delta(x_j) \) in the proof Theorem 2.1, we get

\[
2m_0 + m_i = 3 \frac{y_i - y_0}{h_1} - \frac{h_1}{2} y''_0 \quad \text{..................} \quad (2.13)
\]

where \( y''_0 = M_0 = S''_\Delta(x_0) \)

2.2 Existence of Simple Cubic Splines:

We have in chapter I obtained some existence theorems for general simple splines. In the case of simple cubic splines we have the following additional results.

Theorem 2.3

Let \( \Delta: a = x_0 < x_1 < \ldots < x_N = b \) be a partition of \([a,b]\).

(i) If \( y_0, y_1, \ldots, y_N \) are given values with \( y_0 = y_N \) there exists a unique
simple periodic cubic spline on \( \Delta \) assuming the values \( y_0, y_1, ..., y_N \) respectively at \( x_0, x_1, ..., x_N \).

(ii) If values \( y_0, y_1, ..., y_N, \lambda_0, d_0, \mu_N, d_N \) with \( |\lambda_0| < 2 \) and \( |\mu_N| < 2 \) are given, then there exists a unique simple cubic spline \( S_\Delta \) assuming the values \( y_0, y_1, ..., y_N \) respectively at \( x_0, x_1, ..., x_N \) and satisfying the end conditions,

\[
2M_0 + \lambda_0 M_1 = d_0; \quad \mu_N M_{N-1} + 2M_N = d_N,
\]

where \( M_i \) denotes the value of the second derivative of \( S_\Delta \) at \( x_i \) \( (i = 0, 1, 2, ..., N) \).

(iii) If values \( y_0, y_1, ..., y_N, \mu_0, c_0, \lambda_N, c_N \) with \( |\mu_0| < 2 \) and \( |\lambda_N| < 2 \) are given, then there exists a unique simple cubic spline \( S_\Delta \) assuming the values \( y_0, y_1, ..., y_N \) respectively at \( x_0, x_1, ..., x_N \) and satisfying the end conditions.

\[
2m_0 + \mu_0 m_1 = c_0; \quad \lambda_N m_{N-1} + 2m_N = c_N,
\]

where \( m_i \) denotes the value of the first derivative of \( S_\Delta \) at \( x_i \) \( (i = 0, 1, 2, ..., N) \).

For proving Theorem 2.3, Gershgorin’s Theorem is made use of \( \{ \text{cf. [24]} \} \). This may be stated as follows.

**Gershgorin’s Theorem:**

Let \( (a_{ij}) \) be any \( n \times n \) matrix. The eigen values of this matrix lie in the union of the circles.

\[
|x - a_{ii}| = \sum_{j \neq i} |a_{ij}| \quad (1 \leq i \leq n)
\]

in the complex plane.
By an application of Gershgorin’s theorem we have the following.

**Corollary:**

*A square matrix \((a_{ij})\) with dominant main diagonal is invertible.*

**Proof of the Corollary**

If \((a_{ij})\) is a square matrix with dominant main diagonal, then

\[
\sum_{j \neq i} |a_{ij}| < |a_{ii}| \quad (1 \leq i \leq n)
\]

Hence, applying Gershgorin’s theorem, we see that the eigenvalues of \((a_{ij})\) lies in the union of the circles

\[
|x - a_{ii}| < |a_{ii}| \quad (1 \leq i \leq n)
\]

Clearly zero cannot be inside any of these circles. Consequently it cannot be an eigenvalues of the matrix \((a_{ij})\).

It follows that \((a_{ij})\) is non singular and hence invertible.

**Proof of Theorem 2.3**

(1) Consider the matrix equation (2.4). This can be written as

\[
A^* \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_N \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{bmatrix}
\]
where

\[
A^* = \begin{bmatrix}
2 & \lambda_1 & 0 & \cdots & 0 & 0 & \mu_1 \\
\mu_2 & 2 & \lambda_2 & \cdots & 0 & 0 & 0 \\
0 & \mu_3 & 2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \mu_{N-1} & 2 & \lambda_{N-1} \\
\lambda_N & 0 & 0 & \cdots & 0 & \mu_N & 2
\end{bmatrix}
\]

To obtain periodic spline, we take

\[
h_{N+1} = h_i; \quad y_{N+1} = y_i; \quad \lambda_N = \frac{h_{N+1}}{h_N + h_{N+1}} = \frac{h_i}{h_N + h_i} \quad \text{and} \quad \mu_N = 1 - \lambda_N.
\]

In this case, \( \lambda_i, \mu_i > 0 \) (i = 1, ..., N) and \( \lambda_i + \mu_i = 1 \). Hence the matrix \( A^* \) has dominant main diagonal. By the corollary to Gershgorin’s theorem it follows that \( A^* \) is invertible. Consequently there exists unique set of values \( M_1, M_2, \ldots, M_N \) satisfying (2.4). It follows that there exists a unique simple periodic cubic spline assuming the value \( y_i \) at \( x_i \); (i = 0, 1, 2, ..., N).

The expression for these spline is given by (2.1) with \( M_0 = M_N \).

(ii) Here consider the matrix equation (2.5) given by

\[
\begin{bmatrix}
M_0 \\
M_1 \\
\vdots \\
M_N
\end{bmatrix}
= \begin{bmatrix}
d_0 \\
d_1 \\
\vdots \\
d_N
\end{bmatrix}
\]

Here again, since \( |\lambda_i| < 2 \) and \( |\mu_i| < 2 \), Corollary to Gershgorin’s theorem is extended. Consequently there exists unique solution of the values \( y_i \) at \( x_i \), and \( \lambda_i, \mu_i < 4 \) and \( \lambda_i, \mu_i < 4 \) respectively. Theorem extended in this case since \( \lambda_i, \mu_i < 4 \) and \( \lambda_i, \mu_i < 4 \) where the partition \( \Delta x \) is uniform.
\[
A = \begin{bmatrix}
2 \mu_0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\mu_1 & 2 \lambda_1 & 0 & \ldots & 0 & 0 & 0 \\
0 & \mu_2 & 2 \lambda_2 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \mu_{N-1} & 2 & \lambda_{N-1} \\
0 & 0 & 0 & \ldots & 0 & \mu_N & 2
\end{bmatrix}
\]

Here again, since \(|\lambda_0| < 2; |\mu_N| < 2\), Corollary to Gershgorin's theorem can be applied. It follows that \(A\) is invertible. Consequently there exists a unique simple cubic spline \(S_\Delta(x)\) assuming the values \(y_i\) at \(x_i\) \((i = 0,1,2,\ldots,N)\) and satisfying the end conditions.

\[
2M_0 + \lambda_0 M_1 = d_0; \quad \mu_N M_{N-1} + 2M_N = d_N
\]

(iii) This follows from the matrix equation (2.10) by proceeding as in the above two cases.

The statements (ii) and (iii) of Theorem 2.3 has been extended to the cases of \(\lambda_0 < 4; \mu_N < 4\) and \(\mu_0 < 4; \lambda_N < 4\) respectively [cf. [1] Theorem 2.9].

In the following theorem we obtain a further extension of this to the cases \(\lambda_0 < 6; \mu_N < 6\) and \(\mu_0 < 6, \lambda_N < 6\) where the partition \(\Delta\) is uniform.
Theorem 2.4:

Let $N \geq 3$ and let $\Delta: a = x_0 < x_1 < \ldots < x_N = b$ be a uniform partition of $[a,b]$.

(i) Given values $y_0, y_1, \ldots, y_N; \mu_0, \mu_N$ with $\lambda_0 < 6; \mu_N < 6$ there exists a unique simple cubic spline $S_\Delta$ assuming the values $y_0, y_1, \ldots, y_N$ respectively at $x_0, x_1, \ldots, x_N$ and satisfying the end conditions.

$$2M_0 + \lambda_0 M_1 = d_0; \mu_N M_{N-1} + 2M_N = d_N$$ where $M_i$ denotes the values of the $S_\Delta$ at $x_i$, ($i = 0,1,2,\ldots,N$).

(ii) Given values $y_0, y_1, \ldots, y_N; \mu_0, \mu_N, \lambda_0, \lambda_N$ with $\mu_0 < 6, \lambda_N < 6_1$ there exists a unique simple cubic spline $S_\Delta$ assuming the values $y_0, y_1, \ldots, y_N$ respectively at $x_0, x_1, \ldots, x_N$ and satisfying the end conditions.

$$2m_0 + \mu_0 m_1 = C_0; \lambda_N m_{N-1} + 2m_N = C_N$$

where $m_i$ denotes the values of the first derivative of $S_\Delta$ at $x_i$ ($i = 0,1,2,\ldots,N$).

Proof:

(i) From Theorem 2.1 and its proof, we see that the existence of a unique simple cubic spline $S_\Delta$ satisfying the given conditions is equivalent to the existence of a unique solution to the following system of equations.

The condition $\lambda_0 < 6$ implies $\lambda_0 < 3$ substituting this value of $\lambda_0$ in the equation.
\[ 2M_0 + \lambda_0 M_1 = d_0 \\
2M_0 + \lambda_0 M_1 = d_0 \mu_j M_{j-1} + 2M_j + \lambda_j M_{j+1} = d_j; j = 1, 2, \ldots, N - 1 \\
\mu_N M_{N-1} + 2M_N = d_N \]
\hspace{1cm} \text{(2.14)}

Since the partition is uniform.
\[ \lambda_j = \mu_j = \frac{1}{2}; (j = 1, 2, \ldots, N - 1) \]
and
\[ d_j = \frac{1}{h^2} \left[ y_{j+1} - 2y_j + y_{j-1} \right]; (j = 1, 2, \ldots, N - 1) \]
where \( h \) is the uniform width of the subintervals of the partition.

The system of equations (2.14) has a unique solution if and only if the corresponding system of homogeneous equations,
\[ 2M_0 + \lambda_0 M_1 = 0 \\
\mu_j M_{j-1} + 2M_j + \lambda_j M_{j+1} = 0; j = 1, 2, \ldots, N - 1 \\
\mu_N M_{N-1} + 2M_N = 0 \]
\hspace{1cm} \text{(2.15)}

has only the trivial solution.

Since \( \mu_j = \lambda_j = \frac{1}{2} \) \( (j = 1, 2, \ldots, N - 1) \), the system of equations (2.15) can be written as follows.
\[ 2M_0 + \lambda_0 M_1 = 0 \\
M_{j-1} + 4M_j + M_{j+1} = 0; j = 1, 2, \ldots, N - 1 \\
\mu_N M_{N-1} + 2M_N = 0 \]
\hspace{1cm} \text{(2.16)}

To show that the system of equations (2.16) has only a trivial solution, we proceed as follows.
\[ 2M_0 + \lambda_0 M_1 = 0 \Rightarrow M_0 = \frac{-\lambda_0}{2} M_1 = -\alpha_1 M_1, \text{ say} \]

The condition \( \lambda_0 < 6 \) implies \( \alpha_1 < 3 \) substituting this value of \( M_0 \) in the equation
we get

\[(4 - \alpha_1)M_1 + M_2 = 0\]

This implies that \(M_1 = -\alpha_2 M_2\), where \(\alpha_2 = \frac{1}{4 - \alpha_1}\)

Also

\[\alpha_1 < 3 \Rightarrow \alpha_2 < 1\]

Similarly substituting this value of \(M_1\) in the equation,

\[M_1 + 4M_2 + M_3 = 0\]

we get

\[M_2 = -\alpha_3 M_3\]

where

\[\alpha_3 = \frac{1}{4 - \alpha_2} < \frac{1}{3}; \text{ since } \alpha_2 < 1\]

Repeating these steps, we get

\[M_{i-1} = -\alpha_i M_i \quad (i = 1, 2, \ldots, N)\]

where \(\alpha_i\)'s are given by the recurrence relation

\[\alpha_i = \frac{1}{4 - \alpha_{i-1}} \quad (i = 2, 3, \ldots, N - 1)\]

and \(\alpha_1 = \frac{\lambda_0}{2} < 3\)

From this we see that \(\alpha_i\)'s form a decreasing sequence and since \(\alpha_3 < \frac{1}{3}\),

\[\alpha_i < \frac{1}{3} \quad \text{for } i \geq 3.\]

In particular, \(\alpha_N < \frac{1}{3}\). This gives

\[M_{N-1} = -\alpha_N M_N\]

where \(\alpha_N < \frac{1}{3}\)
This together with the end condition
\[ \mu_N M_{N-1} + 2M_N = 0, \]
gives
\[ M_{N-1} = -\alpha_N x - \left( \frac{\mu_N}{2} \right) M_{N-1} = \frac{\alpha_N \mu_N}{2} M_{N-1} \]
where \( \frac{\alpha_N \mu_N}{2} < 1 \) since \( \alpha_N < \frac{1}{3} \) and \( \mu_N < 6 \).

This is possible only if \( M_{N-1} = 0 \)
\[ M_{N-1} = 0 \Rightarrow M_N = \frac{-\mu_N}{2} M_{N-1} = 0 \]
Now \( M_{N-1} = M_N = 0, \) gives \( M_{N-2} = 0 \) from the relation
\[ M_{N-2} + 4M_{N-1} + M_N = 0. \]

Proceeding in this manner we see that each \( M_i = 0 \) (\( i = 0,1,2,\ldots,N \)). That is, the system of equation (2.16) has only a trivial solution.

Hence the system of equation (2.14) has a unique solution. This guarantees the existence of a unique simple cubic spline \( S_\Delta \) satisfying the conditions.
\[ S_\Delta (x_i) = y_i \quad (i = 0,1,\ldots,N) \]
\[ 2M_0 + \lambda_0 M_1 = d_0; \quad \mu_N M_{N-1} + 2M_N = d_N, \]
where \( M_i = S''_\Delta (x_i) \); (\( i = 0,1,2,\ldots,N \))

(ii) Proof of (ii) is exactly similar to the proof of (i). In this case we use Theorem 2.2, which gives the expression of the simple cubic spline \( S_\Delta \) in terms of slopes \( m_i \); (\( i = 0,1,\ldots,N \)) and the values \( y_i \) at \( x_i \); (\( i = 0,1,2,\ldots,N \)).
Note:

If $|\lambda_0| < 2$ and $|\mu_N| < 2$, then $A$, the coefficient matrix in the matrix equations (2.5) has dominant main diagonal and hence is invertible. Moreover for each $x = (x_0, \ldots, x_N)$.

$$\|Ax\| = \max_i \left| \sum_{j=0}^{N} a_{i,j} x_j \right|$$

Now,

$$\|x\| = \max_i |x_i| = |x_k|, \text{ say}$$

Then

$$\|Ax\| \geq \left| \sum_{j=0}^{k} a_{k,j} x_k \right|$$

$$\geq |a_{kk}| |x_k| - \sum_{j \neq k} |a_{kj}| \|x\|$$

$$= \left( |a_{kk}| - \sum_{j=0}^{N} |a_{kj}| \right) \|x\|$$

$$\geq \min_i \left( |a_{ii}| - \sum_{j \neq i} |a_{ij}| \right) \|x\|$$

Using the expression for $A$, we see that

$$\|Ax\| \geq \min(2 - \lambda_0, 2 - \mu_N, l) \cdot \|x\|$$

Hence if $y = Ax$, we have

$$\|y\| \geq \min(2 - \lambda_0, 2 - \mu_N, l) \cdot \|A^{-1}y\|.$$
It follows that
\[
\|A^{-1}\| = \sup_{y \neq 0} \left( \frac{\|A^{-1}y\|}{\|y\|} \right) \leq \max\{(2 - \lambda_0)^{-1}, (2 - \mu_N)^{-1}\} \tag{2.18}
\]

\[\blacksquare\]

2.3 Convergence Theorem for Cubic Splines

Approximation by splines have striking convergence properties. As an illustration, we now prove a convergence theorem for \( f \) in \( C^2[a,b] \).

**Theorem 2.5.**

Let \( f \) belongs to \( C^2[a,b] \). Let \( \{\Delta_k\} \) be a sequence of partitions of the interval \([a,b]\) with \( \lim_{k \to \infty} \|\Delta_k\| = 0 \). For each \( k, (k = 1,2,3, \ldots) \), let \( S_{\Delta_k}(x) \) be the unique simple cubic spline, interpolating to \( f(x) \) on \( \Delta_k \) and satisfying the end conditions.

\[ S'_{\Delta_k}(a) = f'(a) \quad \text{and} \quad S'_{\Delta_k}(b) = f'(b) \]

Then \( S_{\Delta_k}^{(p)}(x) - f^{(p)}(x) = o\|\Delta_k\|^{2-p} \) \( (p = 0,1,2) \)

uniformly with respect to \( x \) in \([a,b]\).

**Proof:**

Let \( \Delta : a = x_0 < x_1 < \ldots < x_N = b \) be a partition of \([a,b]\) and

\( S_{\Delta} \) be the unique simple cubic spline interpolating to \( f \) at \( x_0, \ldots, x_N \) and satisfying the end conditions.
\[ S'_{\Delta}(a) = f'(a) \quad \text{and} \quad S'_{\Delta}(b) = f'(b) \]

For \( i = 0, 1, \ldots, N \), let \( f_i, f_i', f_i'' \) denote respectively the values \( f(x_i), f'(x_i), f''(x_i) \). Then

\[ S_{\Delta}(x_i) = f_i \quad (i = 0, 1, \ldots, N); \]
\[ S'_{\Delta}(x_0) = f'_0 \quad \text{and} \quad S'_{\Delta}(x_N) = f'_N. \]

By (2.12) these end conditions are equivalent to the conditions

\[ 2M_0 + M_1 = d_0 \quad \text{and} \quad M_{N-1} + 2M_N = d_N, \]

where \( d_0 = \frac{6}{h_1} \left( \frac{f_1 - f_0}{h_1} - f'_0 \right) \)

and \( d_N = \frac{6}{h_N} \left( \frac{f'_N - f_{N-1}}{h_N} \right) \)

where \( h_i \) is the length of the \( i^{th} \) sub interval \((x_{i-1}, x_i)\) and \( M_i = S''_{\Delta}(x_i) \) \( (i = 0, 1, \ldots, N) \)

Then \( M_0, M_1, \ldots, M_N \) satisfies the matrix equation (2.5) where \( d_1, \lambda_i, \mu_i \) \( (i = 1, \ldots, N-1) \) are as in Theorem 2.12

That is

\[
\begin{bmatrix}
M_0 \\
\vdots \\
M_{n-1} \\
M_n
\end{bmatrix}
\begin{bmatrix}
d_0 \\
\vdots \\
d_{n-1} \\
d_n
\end{bmatrix}
\]

Now for \( i = 1, \ldots, N - 1 \)
\[
d_i = -\frac{6}{h_i + h_{i+1}} \left[ \frac{(f_{i+1} - f_i)}{h_{i+1}} - \frac{(f_i - f_{i-1})}{h_i} \right]
\]

\[
= 6f[x_{i-1}, x_i, x_{i+1}]
\]

where \( f[x_{i-1}, x_i, x_{i+1}] \) is the divided difference of \( f \) taken at the points \( x_{i-1}, x_i, x_{i+1} \). By the property of divided differences,

\[
f[x_{i-1}, x_i, x_{i+1}] = \frac{1}{2} f''(\xi_i) \text{ for some } \xi_i \text{ in } (x_{i-1}, x_{i+1})
\]

Hence for \( i = 1, \ldots, N - 1 \),

\[
d_i = 3f''(\xi_i) \text{ for some } \xi_i \text{ in } (x_{i-1}, x_{i+1}).
\]

For \( i = 0 \) and \( i = N \), by Mean Value Theorem, we get

\[
d_0 = 3f''(\xi_0) \text{ for some } \xi_0 \text{ in } (x_0, x_i)
\]

and

\[
d_N = 3f''(\xi_N) \text{, for some } \xi_N \text{ in } (x_{N-1}, x_N)
\]

Hence

\[
\begin{bmatrix}
  d_0 \\
  \cdot \\
  \cdot \\
  \cdot \\
  d_N
\end{bmatrix}
= 3
\begin{bmatrix}
  f''(\xi_0) \\
  \cdot \\
  \cdot \\
  \cdot \\
  f''(\xi_N)
\end{bmatrix}
\]

\[\text{..........................(2.19)}\]

If \( M \) denotes the matrix \((M_0, \ldots, M_N)^T\) and \( d \) denotes the matrix \([d_0, \ldots, d_N]^T\), then

\[A \times (M - \frac{1}{3}d) = d - \frac{1}{3}A \times d\]
Since $A$ is given by the matrix

$$
\begin{bmatrix}
2 & 1 & 0 & \ldots & 0 & 0 & 0 \\
\mu_1 & 2 & \lambda_1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \mu_{N-1} & 2 & \lambda_{N-1} \\
0 & 0 & 0 & \ldots & 0 & 1 & 2
\end{bmatrix}
$$

(2.21)

From (2.20) and (2.21) on rearranging, we get

$$
A \times d =
\begin{bmatrix}
2d_0 + d_1 \\
\mu_1 d_0 + 2d_1 + \lambda_1 d_2 \\
\vdots \\
\vdots \\
\mu_{N-1} d_{N-2} + 2d_{N-1} + \lambda_{N-1} d_N \\
d_{N-1} + 2d_N
\end{bmatrix}
$$

Hence, for $i = 0, \ldots, N$

$$
A \times (\mu - \gamma s) = \frac{1}{3}.
$$

(2.22)

Since $d_i = 3f''(\xi_i)$  $(i = 0, \ldots, N)$
and since $|\xi_i - \xi_{i-1}| \leq 3 \| \Delta \| (i = 1, \ldots, N)$

$$|d_i - d_{i-1}| \leq \frac{1}{3} |f''(\xi_i) - f''(\xi_{i-1})|$$

$$\leq 3\omega \left( f''; 3\| \Delta \| \right); \quad (i = 1, \ldots, N)$$

where $\omega$ denotes the modulus of continuity.

Also $\lambda_i + \mu_i = 1$; $(i = 1, \ldots, N - 1)$.

Thus $\| A(\mu - \frac{1}{2} d) \| \leq \omega \left( f''; 3\| \Delta \| \right)$

$$\leq 3\omega \left( f''; \| \Delta \| \right)$$ by the property of modulus of continuity

Hence

$$\| M - \frac{1}{2} d \| \leq \| A^{-1} \| \cdot \| A(\mu - \frac{1}{2} d) \|$$

$$\leq 3\omega \left( f''; \| \Delta \| \right)$$

Since by (2.18)

$$\| A^{-1} \| \leq \max\{1, \lambda_0, \mu_N\} = 1,$$ in this case.

Hence, for $i = 0, \ldots, N$

$$|M_i - \frac{1}{2} d_i| \leq 3\omega \left( f''; \| \Delta \| \right)$$

It follows that, for $i = 0, \ldots, N$

$$|M_i - f''(\xi_i)| \leq 3\omega \left( f''; \| \Delta \| \right)$$

Also,

$$|f''(\xi_i) - f''| = |f''(\xi_i) - f''(x_i)|$$
\[ \leq \omega \left( f''; \|\Delta\| \right) ; \text{ for } i = 0, \ldots, N \]

Hence \[ \| M_i - f'' \| \leq 4 \omega \left( f''; \|\Delta\| \right) ; \quad (i = 0, \ldots, N) \]

That is \[ |S''_\Delta (x_i) - f''(x_i)| \leq \omega \left( f''; \|\Delta\| \right) ; \quad (i = 0, \ldots, N) \]

Now let \( x \in [a, b] \). Then for since \( i,i = 1, \ldots, N \), \( x \in [x_{i-1}, x_i] \).

Hence

\[ x = (1-t)x_{i-1} + tx_i \quad \text{for some } t, 0 \leq t \leq 1 \]

Since \( S''_\Delta (x) \) is linear in \([x_{i-1}, x_i]\)

\[ S''_\Delta (x) = (1-t)S''_\Delta (x_{i-1}) + tS''_\Delta (x_i) \]

\[ = (1-t)M_{i-1} + tM_i \]

Hence

\[ |S''_\Delta (x) - f''(x)| \leq (1-t)|M_{i-1} - f''(x)| + t|M_i - f''(x)| \]

Also

\[ |M_i - f''(x)| \leq |S''_\Delta (x_i) - f''(x_i)| + |f''(x_i) - f''(x)| \]

\[ \leq 4 \omega \left( f''; \|\Delta\| \right) + \omega \left( f''; \|\Delta\| \right) \]

\[ \leq 5 \omega \left( f''; \|\Delta\| \right) \]

Similarly,

\[ |M_{i-1} - f''(x)| \leq 5 \omega \left( f''; \|\Delta\| \right) \]

From (2.23), (2.24) and (2.25),

\[ |S''_\Delta (x) - f''(x)| \leq 5 \omega \left( f''; \|\Delta\| \right) \]

(2.26)
Now for each $i$ ($i = 1, \ldots, N$)

$$f(x) - S_\Delta(x) \text{ vanishes at both end points of } [x_{i-1}, x_i]$$

Consequently for some point $\xi_i$ in $(x_{i-1}, x_i)$,

$$f'(\xi_i) - S'_\Delta(\xi_i) = 0,$$

by Rolle's theorem.

Thus for $x$ in $(x_{i-1}, x_i)$

$$\left| f''(x) - S''_\Delta(x) \right| = \left| \int_{\xi_i}^{x} (f''(x) - S''_\Delta(x)) \, dx \right|$$

$$\leq 5\omega \left( f''; \|\Delta\| \right) |x - \xi_i|$$

$$\leq 5\|\Delta\| \omega \left( f''; \|\Delta\| \right)$$

Hence for all $x$ in $[a, b]$.

$$\left| f'(x) - S'_\Delta(x) \right| \leq 5\|\Delta\| \omega \left( f''; \|\Delta\| \right). \tag{2.27}$$

One more integration, yields,

$$\left| f(x) - S_\Delta(x) \right| \leq \frac{5}{2} \|\Delta\| \omega \left( f''; \|\Delta\| \right). \tag{2.28}$$

for all $x \in [a, b]$

From (2.26), (2.27) and (2.28) it follows that if $\{\Delta_k\}$ is a sequence of partitions in $[a, b]$ with $\|\Delta_k\| \to 0$, then

$$f^{(b)}(x) - S^{(b)}_\Delta(x) = o \left( \left| \|\Delta_k\| \right|^{2-p} \right) \quad (p = 0, 1, 2) \text{ uniformly for } x \text{ in } [a, b].$$

$\blacksquare$
For \( f \) in \( C^2[a,b] \), we now consider the question of convergence when the splines \( S_{\Delta_k}(x) \) are the natural cubic splines interpolating to \( f(x) \) at the nodes of \( \Delta_k \). In this case in the next theorem, we obtain the convergence of \( S_{\Delta_k}^{(p)}(x) \) to \( f^{(p)}(x) \) \( (P=0,1,2) \) as \( \|\Delta_k\| \to 0 \), uniformly on compact subintervals of \((a,b)\).

**Theorem 2.6**

Let \( f(x) \) be of class \( C^2[a,b] \). Let \( \{\Delta_k\} \) be a sequences of meshes in \([a,b]\) with \( \lim_{k \to \infty} \|\Delta_k\| = 0 \). For each \( k \), let \( S_{\Delta_k}(x) \) be the unique natural cubic spline interpolating to \( f(x) \) on \( \Delta_k \).

Then

\[
f^{(p)}(x) - S_{\Delta_k}^{(p)}(x) = o\left(\|\Delta_k\|^{2-p}\right); \quad (p=0,1,2),
\]

uniformly on each closed subinterval \([a,b]\) of \((a,b)\).

**Proof:**

Let \( \Delta: a = x_0 < x_1 < \ldots < x_N = b \) be any partition of \([a,b]\). For each \( i \), let \( f_i = f(x_i), \ M_i = S''(x_i) \) and \( h_i = x_i - x_{i-1} \).

Then \( S_{\Delta}(x) \) satisfies the following conditions.

\[
S_{\Delta}(x_i) = f_i; \quad (i = 0,1,\ldots,N)
\]
\[
M_0 = S''(x_0) = 0; \quad M_N = S''(x_N) = 0
\]

Then the values \( M_0, M_1, \ldots, M_N \) are given by the matrix equation
where $A$ is the coefficient matrix of the matrix equation (2.5) with $\lambda_0 = \mu_N = 0$

That is

\[
A = \begin{bmatrix}
2 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\lambda_1 & 2 & \mu_1 & \ldots & 0 & 0 & 0 \\
0 & \lambda_2 & 2 & \ldots & 0 & 0 & 0 \\
& & & \ddots & & & \\
0 & 0 & 0 & \ldots & 2 & \mu_{N-2} & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 2
\end{bmatrix}
\]

Also $d_0 = d_N = 0$ and for $i = 1, 2, \ldots, N - 1$

\[
d_i = \frac{6}{h_i + h_{i+1}} \left[ \frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i} \right]
\]

As in the previous Theorem, it can be seen that for $i = 1, 2, \ldots, N - 1$

\[
d_i = 3 f''(\xi_i) \text{ for some } \xi_i \in (x_{i-1}, x_{i+1}).
\]
Further,

\[
\begin{bmatrix}
0 \\
M_1 - f''(\xi_1) \\
M_2 - f''(\xi_2) \\
\vdots \\
M_{N-1} - f''(\xi_{N-1}) \\
0
\end{bmatrix}
= \begin{bmatrix}
0 \\
M_1 - \frac{1}{3} d_1 \\
M_2 - \frac{1}{3} d_2 \\
\vdots \\
M_{N-1} - \frac{1}{3} d_{N-1} \\
0
\end{bmatrix}
\]

Here

Now let 

\[ M = (0, M_1, \ldots, M_{N-1}, 0)^T \]

\[ d = (0, d_1, \ldots, d_{N-1}, 0)^T \]

Then

\[ A \times [M - \frac{1}{3} d] = d - \frac{1}{3} A \times d = [I - \frac{1}{3} A] d \]

\[
\begin{bmatrix}
0 \\
\mu_i (d_i - d_{i-1}) - \lambda_0 (d_2 - d_1) \\
\vdots \\
\mu_{N-1} (d_{N-1} - d_N) - \lambda_{N-1} (d_N - d_{N-1}) \\
0
\end{bmatrix}
\]

Also

\[ (I - \frac{1}{3} A) \times d = \frac{1}{3} \]

with

\[ d_0 = d_N = 0 \]

Here, for

\[ i = 2, 3, \ldots, N - 2 \]

\[
\left| \mu_i (d_i - d_{i-1}) - \lambda_i (d_{i+1} - d_i) \right|
\]

\[
= 3 \left\{ \mu_i \left[ f''(\xi_i) - f''(\xi_{i-1}) \right] - \lambda_i \left[ f''(\xi_{i+1}) - f''(\xi_i) \right] \right\}
\]

\[ \leq 3 \omega \left( f''; \|A\| \right) \]

Hence proceeding as in the previous theorem
we get for \( i = 2, 3, \ldots, N - 2 \)

\[
| M_i - \frac{1}{2} d | \leq 3 \omega \left[ f''; \| \Delta \| \right]
\]

Further,

\[
| f_i'' - \frac{1}{2} d | = | f_i'' - f(\xi_i) | \leq \omega \left( f''; \| \Delta \| \right);
\]

Hence

\[
| M_i - f_i'' | \leq 4 \omega \left( f''; \| \Delta \| \right); \quad i = 2, \ldots, N - 2
\]

Hence if \( x \in [x_2, x_{N-2}] \), proceeding as in the previous theorem, we get

\[
| f''(x) - S''_\Delta (x) | \leq 5 \omega \left( f''; \| \Delta \| \right);
\]

Now if \([a, b_1] \) is any closed subinterval of \((a, b)\), then \([a, b_1] \subseteq [x_2, x_{N-1}] \)

if \( \| \Delta \| \) is sufficiently small. Consequently,

\[
| f''(x) - S''_\Delta (x) | \leq 5 \omega \left( f''; \| \Delta \| \right)
\]

for all \( x \) in \([a, b_1] \) if \( \| \Delta \| \) is sufficiently small.

It follows that

\[
f''(x) - S''_\Delta (x) = o(1) \quad \text{as} \quad \| \Delta \| \to 0,
\]

uniformly on \([a, b]\). In a similar manner, we have

\[
f^{(p)}(x) - S^{(p)}_\Delta (x) = o \left( \| \Delta \|^{2-p} \right); \quad p = 0, 1, 2,
\]

uniformly on \([a, b_1]\).

Hence if \( \{ \Delta_k \} \) is any sequence of meshes in \([a, b]\), with \( \| \Delta_k \| \to 0 \)

it follows that

\[
f^{(p)}(x) - S^{(p)}_\Delta (x) = o \left( \| \Delta_k \|^{2-p} \right) \quad (p = 0, 1, 2)
\]

uniformly on \([a_1, b_1]\).