Chapter V

SIMPLE QUINTIC SPLINES

A quintic spline $S_\Delta$ on a mesh $\Delta: a = x_0 < x_1 < \ldots < x_N = b$ is said to be simple if it belongs to $C^4[a,b]$.

As a special case of existence theorem for general simple polynomial splines (cf. Theorem 1.4) we have the following existence theorem for simple quintic splines.

Theorem 5.1

Let $\Delta: a = x_0 < x_1 < \ldots < x_N = b$ be a partition of $[a,b]$. Given values $y_0, y_1, \ldots, y_N, y'_0, y'_1, \ldots, y''_0, y''_1, \ldots, y''_N$, there exists a unique quintic spline $S_\Delta(x)$ on the mesh $\Delta$ satisfying the conditions.

\begin{align*}
S_\Delta(x_i) &= y_i; \quad (i = 0, 1, \ldots, N) \\
S'_\Delta(x_0) &= y'_0; \quad S'_\Delta(x_N) = y'_N \\
S''_\Delta(x_0) &= y''_0; \quad S''_\Delta(x_N) = y''_N
\end{align*}
5.1. Expression for Simple Quintic Splines

Theorem 5.2

Let $S_{\Delta}(x)$ be the simple quintic spline on a uniform mesh $\Delta: a = x_0 < x_1 < \ldots < x_N = b$; taking the values $y_i$ at $x_i$; $(i = 0, 1, 2, \ldots, N)$ and satisfying the end conditions

$$S_{\Delta}'(x_0) = y_0'; \quad S_{\Delta}'(x_N) = y_N'$$

$$S_{\Delta}''(x_0) = y_0''; \quad S_{\Delta}''(x_N) = y_N''$$

If $\Delta$ is the uniform partition of $[a, b]$ of length ‘$h$’ then on $[x_{j-1}, x_j]$, $(j = 1, 2, \ldots, N)$;

$S_{\Delta}(x)$ has the following expression

$$S_{\Delta}(x) = S_{\Delta,j}(x) = y_jT_0(\sigma_j) + y_jT_0(1-\sigma_j) + h\left[m_jT_1(\sigma_j) - m_jT_1(1-\sigma_j)\right] + h^2\left[M_jT_2(\sigma_j) + M_jT_2(1-\sigma_j)\right]$$

(5.1)

where $\sigma_j = \frac{x-x_{j-1}}{h}$

$T_0(\sigma) = 1 - 10\sigma^3 + 15\sigma^4 - 6\sigma^5$

$T_1(\sigma) = \sigma - 6\sigma^3 + 8\sigma^4 - 3\sigma^5$

$T_2(\sigma) = \frac{1}{2}\left[\sigma^2 - 3\sigma^3 + 3\sigma^4 - \sigma^5\right]$ and

$m_j = S_{\Delta}'(x_j); \quad (j = 0, 1, \ldots, N)$
Further \( m_0, m_1, \ldots, m_N, M_0, M_1, \ldots, M_N \) satisfy the following conditions.

\[
m_0 = y'_0; \quad m_N = y'_N; \quad M_0 = y''_0; \quad M_N = y''_N \quad \ldots \ldots (5.2)
\]

For \( j = 1, 2, \ldots, N - 1 \):

\[
\begin{align*}
[M_{j+1} - 6M_j + M_{j+1}] + \frac{8}{h}[m_{j+1} - m_{j-1}] &= \frac{20}{h^2}[y_{j+1} - 2y_j + y_{j-1}] \ldots \ldots (5.3)
\end{align*}
\]

and

\[
\begin{align*}
\frac{1}{h}[7m_{j+1} + 16m_j + 7m_{j-1}] - (M_{j+1} - M_{j-1}) \\
= \frac{15}{h^2}(y_{j+1} - y_{j-1}) \\
\ldots \ldots (5.4)
\end{align*}
\]

**Proof:**

Using the properties of the quintic polynomials (cf. Theorem 4.1) \( T_0, T_1 \) and \( T_2 \) we can easily derive the expression (5.1) for \( S_\Delta(x) \).

Since \( S_\Delta(x) \in C^4[a,b] \); then for \( j = 1, 2, \ldots, N - 1 \)

\[
S''_\Delta(x_j -) = S''_\Delta(x_j +)
\]

and

\[
S^{(iv)}_\Delta(x_j -) = S^{(iv)}_\Delta(x_j +)
\]
Now

\[ S''_\Delta (x_j-) = S''_\Delta (x_j+) \] gives us the equations

\[ \frac{8}{h} (m_{j+1} - m_{j-1}) + (M_{j+1} - 6M_j + M_{j-1}) = \frac{20}{h^2} (y_{j+1} - 2y_j + y_{j-1}); \] [cf. 4.1]

Also

\[ S^{(iv)}_\Delta (x_j-) = S^{(iv)}_\Delta (x_j+) \]

implies

\[ \frac{1}{h^4} \left\{ y_{j-1} T_0^{(iv)} (1) + y_j T_0^{(iv)} (1) \right\} + \frac{1}{h^3} \left\{ m_{j-1} T_1^{(iv)} (1) - m_j T_1^{(iv)} (0) \right\} \]

\[ + \frac{1}{h^2} \left\{ M_{j-1} T_2^{(iv)} (1) + M_j T_2^{(iv)} (0) \right\} \]

\[ = \frac{1}{h^4} \left\{ y_j T_0^{(iv)} (0) + y_{j+1} T_0^{(iv)} (1) \right\} \]

\[ - \frac{1}{h^3} \left\{ m_j T_1^{(iv)} (0) + m_{j+1} T_1^{(iv)} (1) \right\} \]

\[ + \frac{1}{h^2} \left\{ M_j T_2^{(iv)} (0) + M_{j+1} T_2^{(iv)} (1) \right\} \]

Simplifying this equation, we get, equation (5.4)

\[ \blacksquare \]

Note:

If \( N_j = S^{(iv)}_\Delta (x_j) \) and \( S_j = S_\Delta (x_j) \); \( j = 0, 1, 2, ..., N \), and if the partition \( \Delta \) is uniform length ‘\( h \)’ we obtain the relation. \( \{Cf. [1] (4.2.13)\} \)
\[
\frac{1}{120} \left[ N_{j-2} + 26N_{j-1} + 66N_j + 26N_{j+1} + N_{j+2} \right] = \frac{S_{j+2} - 4S_{j+1} + 6S_j - 4S_{j-1} + S_{j-2}}{h^4} \]
\[
\text{........................................(5.5)}
\]

Also from \(\{Cf [1] \ (4.1.28)\} \) we have

\[
N_{j-1} = \frac{360S_{j-1}}{h^4} - \frac{360S_j}{h^4} + \frac{192S'_{j-1}}{h^3} + \frac{168S'_j}{h^3} + \frac{36S''_{j-1}}{h^2} - \frac{24S''_j}{h^2} \text{...........(5.6)}
\]

\[
N_j = -\frac{360S_{j-1}}{h^4} + \frac{360S_j}{h^4} - \frac{168S'_{j-1}}{h^3} - \frac{192S'_j}{h^3} - \frac{24S''_{j-1}}{h^2} + \frac{36S''_j}{h^2} \text{...........(5.7)}
\]

Finally we state the convergence theorem for simple quintic splines.

5.2. Convergence Theorem for Simple Quintic Splines

Theorem 5.2

Let \( f \in C^4(a, b) \) and if \( \Delta : a = x_0 < \ldots \ldots , x_N = b \) is a uniform partition of \([a, b]\) and if \( S_\Delta(x) \) is the simple quintic spline interpolating to \( f \) at the mesh points and satisfying the end conditions

\[
S'_\Delta(a) = f'(a); \quad S''_\Delta(a) = f''(a); \\
S'_\Delta(b) = f'(b); \quad S''_\Delta(b) = f''(b);
\]

then

\[
f^{(k)}(x) - S^{(k)}_\Delta(x) = O \left( \| \Delta \|^5 \right) \text{ for } k = 0,1,2,3,4 \text{ as } \| \Delta \| \to 0,
\]

uniformly for \( x \in [a,b] \)