CHAPTER II

DIRECT THEOREMS

Introduction. Let \((L_n, (R_n^\beta)) (0 < \beta < 1)\) be the sequences of operators introduced respectively in sections 1 and 2 of Chapter I. We are now interested in determining the following function classes.

(i) For \(A > 0, 0 < \alpha \leq 1,\)

\[
L_A^\alpha = \{ f \in \mathcal{C}[0,\infty)/x^A : \frac{L_n(f;x) - f(x)}{(1+x)^A} = O(1) \cdot \frac{x(1+x^2)^\alpha}{n} \mbox{ (} x \geq 0 \mbox{ )} \}
\]

(ii) For \(A > 0, 0 < \alpha \leq 1,\)

\[
L_A^{0,\alpha} = \{ f \in \mathcal{C}[0,\infty)/x^A : ||L_n f - f||_A = O(1) \cdot n^{-\alpha} , n \in \mathbb{N} \}
\]

where

\[
||f||_A := \sup_{x \geq 0} \frac{|f(x)|}{(1+x)^A}
\]

(iii) For \(0 < \beta < 1, 0 < \alpha \leq 1,\)

\[
R_\beta^\alpha = \{ f \in \mathcal{C}_B [0,\infty) : ||R_n^\beta f - f||_{\mathcal{C}_B} = O(1) \cdot n^{-\beta_0 \alpha} , n \in \mathbb{N} \}
\]

where \(\beta_0 = \min (\beta, 1-\beta)\) and

\[
||f||_{\mathcal{C}_B} = \sup_{x \geq 0} |f(x)|
\]
(iv) For $0 < \alpha < b < \infty$, $0 < \alpha \leq 1$,

$$L^\alpha_{[a,b]} = \{ f \in C[0,\infty) : \|L_n f - f\|_{[a,b]} = O(1) \ n^{-\alpha}, \ n \in \mathbb{N} \}$$

where

$$\|f\|_{[a,b]} = \max_{a \leq x \leq b} |f(x)| .$$

The theorems that characterize such function classes are known as 'approximation theorems'. In the first three cases, the approximation over the whole interval $[0,\infty)$ is taken into account. The approximation theorems we obtain in these cases are therefore 'global' approximation theorems. In the last case however only a proper subinterval $[a,b]$ of $(0,\infty)$ is considered. Hence the approximation theorem we obtain in this case is a 'local' approximation theorem. We shall prove in Chapter V that in each of these cases when we take $\alpha = 1$ and replace $O(1)$ by $o(1)$ the corresponding function class we obtain is only a trivial class of functions, in most cases consisting only of linear functions and in some cases just constants. The case $\alpha = 1$ is therefore called an optimal case or saturation case (See §9).

An approximation theorem is usually divided into two parts - (1) a theorem giving sufficient conditions for a function $f$ to belong to such a function class called the 'direct theorem' and (2) a theorem giving necessary conditions for a function $f$ to belong to the function class called the 'inverse theorem'. In this chapter we obtain the various direct theorems. The corresponding inverse theorems, mostly
for the non-optimal case $0 < \alpha < 1$, is derived in Chapter III. The inverse theorems corresponding to the optimal case $\alpha = 1$ is derived in Chapter V.

1. **A Global Direct Theorem for the operators $L_n$ when $f$ belongs to $\{C[0,\infty)/x^{A}\}(A \geq 0)$**

In this section, we are concerned with the Bernstein-type rational operators $L_n (n \in \mathbb{N})$ introduced in Chapter I, Section 1. For these operators, Bleimann, Butzer and Hahn [11] has obtained the following result.

**THEOREM.** For $f \in C_B[0,\infty)$, $x \in [0,\infty)$ and $n \geq N(x) = 24(1+x),$

$$|L_n(f;x) - f(x)|$$

$$\leq c \left\{ \omega \left( \frac{\phi(x)}{n^2} ; f \right) + \frac{\phi^2(x)}{n} \right\} ||f||_{C_B}$$

for some constant $c > 0$, where

\[ \omega(t;f) = \sup_{x \geq 0} \frac{|f(x) - 2f(x+h) + f(x+2h)|}{h^2}, \quad 0 < h \leq t \]

and

\[ \phi(x) = \sqrt{x(1+x)} \]

We in this section extend the above result to $f$ in $\{C[0,\infty)/x^{A}\}(A \geq 0)$ and from this deduce a direct theorem for the operators $L_n$ (See also [28]).
We first derive the following properties of the operator $L_n$.

**Lemma 1.1.** Let $A \geq 0$

(i) $L_n(1; x) = 1$

(ii) $L_n(t-x); x) = -x\left(\frac{x}{1+x}\right)^n$

(iii) $L_n((1+t)^A; x) = O(1) (1+x)^A \quad (x \geq 0; \ n \in \mathbb{N})$

(iv) $L_n((\frac{t}{1+t}) - \frac{x}{1+x})^2 (1+t)^A; x) = O(1) \frac{x(1+x)^A}{n} \quad (0 \leq x \leq h \quad \text{if} \quad n \in \mathbb{N})$

(v) $L_n((t-x)^2 (1+t)^A; x) = O(1) \frac{x(1+x)^A}{n} \quad (0 \leq x \leq n \quad \text{if} \quad n \in \mathbb{N})$

*Proof.* Statements (i) and (ii) follow trivially. However for the sake of completeness we prove these also.

Now

$$L_n(f; x) = \sum_{k=0}^{n} P_{n,k}(x) f\left(\frac{k}{n-k+1}\right)$$

where

$$P_{n,k}(x) = \binom{n}{k} x^k (1+x)^{-n}$$

$$L_n(1; x) = \sum_{k=0}^{n} P_{n,k}(x) = (1+x)^n (1+x)^{-n} = 1.$$ 

This proves (i) of Lemma 1.1.

$$\binom{n}{k} \frac{k}{n-k+1} = \binom{n}{k-1} (k = 1, \ldots, n)$$

Hence

$$P_{n,k}(x) \frac{k}{n-k+1} = \binom{n}{k-1} x^k (1+x)^{-n} = x P_{n,k-1}(x),$$
So that

\[ L_n(t; x) = \sum_{k=0}^{n} P_n(k) \frac{k}{n-k+1} \]

\[ = x \sum_{k=1}^{n} P_n(k-1)(x) \]

\[ = x(1 - P_{n,n}(x)) \]

\[ = x(1 - (\frac{x}{1+x})^n) \]

Hence

\[ L_n(t-x; x) = L_n(t; x) - x, \text{ since } L_n(1; x) = 1 \]

\[ = -x(\frac{x}{1+x})^n \]

This proves (ii) of Lemma 1.1. Let \( N = \lfloor A \rfloor + 1 \) where \( \lfloor A \rfloor \) denotes the integral part of \( A \).

\[ \binom{n}{k} \left( \frac{n+1}{n-k+1} \right)^N \leq N! \binom{n}{k} \frac{(n+1)^N}{(n-k+1) \ldots (n-k+N)} \]

\[ \leq N! \binom{n+N}{k} \]

Hence

\[ L_n((1+t)^N; x) \leq N! \sum_{k=0}^{n} \binom{n+N}{k} x^k (1+x)^{-n} \]

\[ \leq N! (1+x)^N \]

Consequently

\[ L_n((1+t)^A; x) \leq \left\{ L_n((1+t)^N; x) \right\}^{A/N} \left\{ L_n(1; x)^{1-A/N} \right\} \]

by Holder's inequality
This proves (iii) of Lemma 1.1. Again by applying inequality (1.1)

\[
L_n(\left(\frac{t}{1+t}\right) - \frac{x}{1+x} )^2 (1+t)_N; x)
\]

\[
\leq \frac{N!}{(1+x)^n} \sum_{k=0}^{n} \binom{n+N}{k} x^k \left( \frac{k}{n+1} - \frac{x}{1+x} \right)^2.
\]

Also

\[
\binom{n+N}{k} \left( \frac{k}{n+1} - \frac{x}{1+x} \right)^2
\]

\[
= \frac{(n+N)(n+N-1)}{(n+1)^2} \binom{n+N-1}{k-2} \frac{n+N}{(n+1)^2} \binom{n+N-1}{k-1} + \frac{2x}{1+x} \frac{n+N}{n+1} \binom{n+N-1}{k-1} + \binom{n+N}{k} \frac{x^2}{(1+x)^2}
\]

Hence

\[
L_n(\left(\frac{t}{1+t}\right) - \frac{x}{1+x} )^2 (1+t)_N; x)
\]

\[
\leq N! (1+x)^N \left\{ \left( \frac{N-N}{n+1} \right)^2 - 2 \frac{n+N}{n+1} + 1 \right\} \frac{x^2}{(1+x)^2} + \frac{n+N}{(n+1)^2} \frac{x}{1+x}
\]

\[
= N! (1+x)^N \left\{ \left( \frac{N-1}{n+1} \right)^2 \frac{x^2}{(1+x)^2} + \frac{n+N}{(n+1)^2} \frac{N+n}{(n+1)^2} \frac{x}{1+x} \right\}
\]

\[
\leq N! (1+x)^N \left\{ \frac{N-N}{n+1} \frac{x}{(1+x)^2} + \frac{n+N}{n+1} \frac{x}{(1+x)^2} \right\}, \text{ for } n \geq x
\]

\[
\leq (N+2)! \frac{x}{n} (1+x)^{N-2} \quad (1.3)
\]
Hence for any $A \geq 0$, taking $N = \lfloor A \rfloor + 1$,

$$L_n((\frac{t}{1+t} - \frac{x}{1+x})^2 (1+t)^A; x)$$

$$\leq \left\{ L_n((\frac{t}{1+t} - \frac{x}{1+x})^2 (1+t)^N; x) \right\}^{A/N} \left\{ L_n((\frac{t}{1+t} - \frac{x}{1+x})^2; x) \right\}^{1-A/N},$$

by Holder's inequality

$$\leq \left\{ (N+2)! \frac{x}{n} (1+x)^{N-2} \right\}^{A/N} \left\{ 2 \frac{x}{n} (1+x)^{-2} \right\}^{1-A/N}$$

for $n \geq x$ ,

using estimate (1.3)

$$\leq (N+2)! \frac{x}{n} (1+x)^{A-2}$$

$$\leq ([A]+x)! \frac{x}{n} (1+x)^{A-2}$$

This proves (iv) of Lemma 1.1. Lastly (v) follows from (iv) since

$$\frac{t-x}{2} = (\frac{1}{1+t} - \frac{x}{1+x})^2 (1+t)^2 (1+x)^2$$

We now derive an estimate for the degree of approximation of

$$L_n(f; x) - f(x).$$

**THEOREM 1.2.** Let $A > 0$. There exists some constant $M > 0$ such that

for all positive integers $n \geq x(1+x)^2$ and for all $f$ in $\{ C[0,\infty)/x^A \}$.

$$\frac{L_n(f;x)-f(x)}{(1+x)^A} \leq M \{ ||f||_A \frac{x}{n} + \omega^*(\frac{\Phi(x)}{n^{1/2}}; f; A) \}$$

where

$$||f||_A = \sup_{x \geq 0} \frac{|f(x)|}{(1+x)^A}.$$
\[
\phi(x) = \sqrt{x} (1 + x)
\]

\[
\omega^* (\delta; f; A) = \sup_{0 < h \leq \delta, 0 \leq x \leq \infty} \frac{|\Delta_h^2 (f; x)|}{(1 + x)^A}
\]

\[
\Delta_h^2 (f; x) = f(x+2h) - 2f(x+h) + f(x)
\]

**Proof.** For a given \( f \) in \( \{ C[0, \infty)/x \} \) and for a fixed \( \delta, 0 < \delta \leq 1 \), defined \( f_\delta (x) \) on \([0, \infty)\) as in [20], by

\[
f_\delta(x) = 4\delta^{-2} \left\{ \int_0^{\delta/2} \int_0^{\delta/2} (2f(x+u+v) - f(x+2u+2v)) \, du \, dv \right\}
\]

(1.4)

Now if

\[
f_1(x) = \int_0^x f(t) \, dt \quad \text{and} \quad f_2(x) = \int_0^x f_1(t) \, dt,
\]

then it can be easily seen that \( f_\delta (x) \) has the following two expressions.

\[
f_\delta(x) = \delta^{-2} \left\{ 8 \int_0^{\delta/2} (f_1(x+\delta/2 + u) - f_1(x+u)) \, du \
- \int_0^{\delta} (f_1(x+\delta+u) - f_1(x+u)) \, du \right\}
\]

(1.5)

and

\[
f_\delta(x) = \delta^{-2} \left( 8 \Delta_\delta^2 (f_2; x) - \Delta_\delta^2 (f_2; x) \right)
\]

(1.6)

From expressions (1.4), (1.5) and (1.6) we see that \( f_\delta \) is twice continuously differentiable on \([0, \infty)\) and that
\[ \|f(x) - f_\delta(x)\| \leq \sup_{0 < h \leq \delta} |\Delta^2_h (f; x)| \tag{1.7} \]

\[ \|f'_\delta(x)\| \leq \frac{5}{\delta} \sup_{0 < h \leq \delta} |f(t+h) - f(t)| \tag{1.8} \]

and

\[ \|f''_\delta(x)\| \leq \frac{5}{\delta^2} \sup_{0 < h \leq \delta} |\Delta^2_h (f; x)| \tag{1.9} \]

Let \( \omega^* (\delta; f; A) \) be defined as in the theorem. Then from estimates (1.7) to (1.9), it follows that

\[ |f(x) - f_\delta(x)| \leq \omega^* (\delta; f; A) (1+x)^A \tag{1.10} \]

\[ |f_\delta'(x)| = \frac{O(1)}{\delta} \|f\|_A (1+x + 2\delta)^A \]

\[ = \frac{O(1)}{\delta} \|f\|_A (1+x)^A, \text{ for } 0 < \delta \leq 1 \tag{1.11} \]

and

\[ |f_\delta''(x)| = \frac{O(1)}{\delta^2} \omega^* (\delta; f; A) (1+x)^A \tag{1.12} \]

By the well-known Taylor's remainder formula of calculus,

\[ f_\delta(t) - f_\delta(x) = f_\delta'(x) (t-x) + \frac{1}{2} f_\delta''(\xi_t) (t-x)^2 \]

for some \( \xi_t \) between \( t \) and \( x \). Hence

\[ L_n(f_\delta; x) = L_n(f_\delta(t) - f_\delta(x); x) = f_\delta'(x) L_n(t-x; x) + \frac{1}{2} L_n(f_\delta''(\xi_t) (t-x)^2; x) \tag{1.13} \]

Fix \( n \in \mathbb{N}, n \geq x(1+x)^2 \). Take \( \delta^2 = \frac{x}{n} (1+x)^2 \)

Now by (ii) of lemma 1.1.,

\[ L_n(t-x; x) = -x \left( \frac{x}{1+x} \right)^n \]
Hence

\[ \left| f'(x) L_n(t-x; x) \right| = \left| f'(x) \right| x \left( \frac{x}{1+x} \right)^n \]

\[ = O(1) \left| f \right| A \frac{x}{\delta} \left( \frac{x}{1+x} \right)^n (1+x)^A, \]

by estimate (1.11). Also

\[ \frac{X}{\delta} \left( \frac{x}{1+x} \right)^n \leq \frac{X}{\delta^2} \left( \frac{x}{1+x} \right)^n, \text{ Since } 0 < \delta \leq 1 \]

\[ = \frac{n x^n}{(1+x)^{n+2}} \]

\[ = \frac{2x}{n+1} \left( \frac{n+1}{2} \right) \frac{x^{n-1}}{(1+x)^{n+2}} \]

\[ \leq \frac{2x}{n+1} \]

Hence

\[ f'(x) L_n(t-x; x) = O(1) \left| f \right| A \frac{x}{n+1} (1+x)^A \]

(1.14)

Now

\[ \left| f''(\xi_t) \right| \leq \left| f''_0 \right| A (1+\xi_t)^A \]

\[ = \frac{O(1)}{\delta^2} w^* (\delta; f; A) (1+\xi_t)^A, \]

using estimate (1.12)

\[ = \frac{O(1)}{\delta^2} w^* (\delta; f; A) ((1+t)^A + (1+x)^A), \]

since \( \xi_t \) is between \( t \) and \( x \). Hence
\[
\left| L_n(f(\xi_\delta) (t-x)^2; x) \right|
\]
\[
= \frac{O(1)}{\delta^2} \omega^* (\delta; f; A) L_n ((1+t)^2 ((1+t)^A + (1+x)^A); x)
\]
\[
= \frac{O(1)}{\delta^2} \omega^* (\delta; f; A) \frac{x}{n} (1+x)^A + 2
\]
using estimate (v) of Lemma 1.1.
\[
= O(1) \omega^* (\delta; f; A) \ (1+x)^A
\]
(1.15)
since \( \delta^2 = \frac{x}{n} (1+x)^2 \). Using estimates (1.14) and (1.15) in expression (1.13) it follows that
\[
L_n(f_\delta; x) - f_\delta(x) = O(1) \left\{ \left\| f \right\|_A \frac{x}{n+1} + \omega^* (\delta; f; A) \right\} (1+x)^A
\]
(1.16)
Also
\[
\left| L_n (f-f_\delta; x) \right| \leq \left\| f-f_\delta \right\|_A L_n ((1+t)^A; x)
\]
\[
\leq \omega^* (\delta; f; A) L_n ((1+t)^A; x) ,
\]
using estimate (1.10)
\[
= O(1) \omega^* (\delta; f; A) \ (1+x)^A
\]
(1.17)
by (iii) of Lemma 1.1. Also by estimate (1.10),
\[
|f(x) - f_\delta(x)| \leq \omega^* (\delta; f; A) \ (1+x)^A
\]
(1.18)
From estimates (1.16) to (1.18) it follows that for \( n \in \mathbb{N} \), \( n \geq x(1+x)^2 \) and for all \( f \) in \( \{ C[0,\infty)/x^A \} \)
\[
L_n(f; x) - f(x) = O(1) \left\{ \left\| f \right\|_A \frac{x}{n+1} + \omega^* \left( \frac{\phi(x)}{n^A}; f; A \right) \right\} (1+x)^A ,
\]
since \( \left(\frac{x}{n}\right)^{\frac{1}{2}}(1+x) = \frac{\phi(x)}{n^{\frac{1}{2}}} = \delta \)  

We close this section by deducing from Theorem 1.2. the following direct theorem for the operators \( L_n \).

**COROLLARY 1.3.** Suppose \( f \) belongs to \( \{ C[0, \infty) / x^A \} \) for some \( A > 0 \). Suppose for some \( \gamma, \ 0 < \gamma \leq 1 \),

\[
\frac{\Delta_h^2 f(x)}{(1+x)^A} = O(1) h^2 \gamma \quad (0 < h \leq 1) \quad 0 < x < \infty
\]

Then

\[
\frac{L_n f(x) - f(x)}{(1+x)^A} = O(1) \left( \frac{x}{n} (1+x)^2 \right)^{\gamma} \quad (n \in \mathbb{N}, x \geq 0)
\]

**Proof.** By assumption,

\[
\frac{\Delta_h^2 f(x)}{(1+x)^A} = O(1) h^2 \gamma \quad (0 < h \leq 1) \quad 0 < x < \infty
\]

Let \( \omega^* (\delta; f; A) \) be defined as in Theorem 1.2.

Then

\[
\omega^* (\delta; f; A) = O(1) \delta^2 \gamma \quad (0 < \delta \leq 1)
\]

Now, by Theorem 1.2., for \( n \geq x(1+x)^2 \),

\[
\frac{|L_n f(x) - f(x)|}{(1+x)^A} \leq M \{ ||f||_A \frac{x}{n} + \omega^* \left( \frac{\phi(x)}{n^{\frac{1}{2}}} ; f; A \right) \}
\]

where \( \phi(x) = \sqrt{x} (1+x) \)
\[ \frac{L_n(f; x) - f(x)}{(1+x)^A} = O(1) \left( \frac{x}{n} \right) (1+x)^2 \gamma \] in this case since for \( n \geq x (1+x)^2 \),

\[ \frac{x}{n} \leq \frac{x}{n} (1+x)^2 \leq \left( \frac{x}{n} \right) (1+x)^2 \gamma , \]

It follows that, for \( n \geq x (1+x)^2 \),

\[ \frac{L_n(f; x) - f(x)}{(1+x)^A} = O(1) \left( \frac{x}{n} \right) (1+x)^2 \gamma \tag{1.19} \]

Also since \( f \in \{ C[0,\infty)/x^A \} \), for any \( n \in \mathbb{N} \)

\[ \frac{|L_n(f; x) - f(x)|}{(1+x)^A} \leq \frac{||f||_A}{(1+x)^A} \{ L_n((1+t)^A; x) + (1+x)^A \} \]

\[ = O(1) \text{, using (iii) of lemma 1.1.} \]

\[ = O(1) \left( \frac{x}{n} \right) (1+x)^2 \gamma \text{, for } n \leq x (1+x)^2 \tag{1.20} \]

From estimates (1.19) and (1.20), it follows that for all \( n \in \mathbb{N} \) and all \( x \geq 0 \),

\[ \frac{L_n(f; x) - f(x)}{(1+x)^A} = O(1) \left( \frac{x}{n} \right) (1+x)^2 \gamma \]
2. A Direct Theorem for the Operators $L_n$ associated with the approximation in the norm over the space \( C[0,\infty)/x^A \) \((A \geq 0)\)

V. Totik in [58] has obtained the following results for the Bernstein-type rational operators $L_n$.

**THEOREM A.** Suppose $f$ belongs to $C_B[0,\infty)$, the class of all continuous bounded functions on $[0,\infty)$. Then

$$\|L_n f - f\|_{C_B} = o(1) \, n^{-1} \quad (n \to \infty)$$

if and only if $f$ is a constant.

(Here $\|f\|_{C_B} = \sup_{t \geq 0} |f(t)|$)

Also

$$\|L_n f - f\|_{C_B} = O(1) \, n^{-1} \quad (n \in \mathbb{N})$$

if and only if $f$ has an absolutely continuous derivative with

$$x(1+x)^2 \quad |f''(x)| \leq K_f \quad (x > 0)$$

almost everywhere.

**THEOREM B.** Suppose $f$ belongs to $C[0,\infty]$, the class of all continuous functions on $[0,\infty)$ having a finite limit at $x = \infty$. Let

$$\|f\|_C = \sup_{0 \leq t \leq \infty} |f(t)|$$

Suppose

$$0 < \gamma < 1.$$ Then

$$\|L_n f - f\|_C = O(1) \, n^{-\gamma} \quad (n \in \mathbb{N})$$
if and only if

\[ |\Delta_h^2(f; x)| \leq K_f \left( \frac{h}{\phi(x)} \right)^2 \gamma \quad (x > 0) \quad 0 < h \leq x \]

where \( \phi(x) = \sqrt{x(1+x)} \),

\[ \Delta_h^2(f; x) = f(x+2h) - 2f(x+h) + f(x) \]

(ii) \[ |f(x) - f(x)_{\infty}| \leq K_f x^{-\gamma} \quad (x > 0) \]

We extend these results to functions in \( \{ C[0, \infty)/x^A \} \) \((A \geq 0)\). In this section, we extend the direct part of these results (The corresponding inverse part for the non-optimal case \( 0 < \gamma < 1 \) is dealt with in Chapter III, Section 2 and the inverse part for the optimal case \( \gamma = 1 \) is dealt with in Chapter V, Section 3). In this section we also derive an estimate for \( \|L^h f - f\|_A \) in terms of various modulii of smoothness of the function \( f \).

We first derive several auxiliary results needed in the sequel.

The following expression for \( \Delta_{nh}^2(f; x) \) in terms of \( \Delta_h^2(f; t) \), \( x \leq t \leq x+2h \) \((n-1)\) is due to G.G. Lorentz [32].

**Lemma 2.1.** (c.f. [32, p. 47]). Let \( n \) be any positive integer and \( h > 0 \). Then

\[ \Delta_{nh}^2(f; x) = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} \Delta_h^2(f; x+k_1h + k_2h) \]

From the above expression for \( \Delta_{nh}^2(f; x) \) we now deduce the following
LEMMA 2.2 Suppose for some $A > 0$ f belongs to $\{ C[0, \infty)/x^A \}$. Let

$\phi(x)$ be any positive increasing function on $(0, \infty)$. For $\delta > 0$, let

$$\Omega_2 (\delta; \phi; f) := \sup_{0 < h \leq \delta} \frac{|\Delta_{h}^2 (f; x)|}{x^A (1+x+2h\phi(x))^A}$$

Then for any $h$, $\delta > 0$,

$$\Omega_2 (\delta; \phi; f) \leq 3(1+ \frac{\delta^2}{h^2}) \Omega_2 (h; \phi; f).$$

Proof. From the expression for $\Delta_{nh}^2(f; x)$ given in Lemma 2.1., we see that

$$|\Delta_{nh}^2(f; x)| \leq n^2 \sup_{t \geq x} |\Delta_{h}^2 (f; t)| : x \leq t \leq x+2h \ (n-1)$$

$$\leq n^2 (1+x+2nh)^A \sup_{t \geq x} \frac{|\Delta_{h}^2 (f; t)|}{(1+t+2h)^A}$$

Hence

$$\frac{|\Delta_{nh}^2(f; x)|}{(1+x+2nh)^A} \leq n^2 \sup_{t \geq x} \frac{|\Delta_{h}^2 (f; t)|}{(1+t+2h)^A}$$

Now since $\phi$ is increasing, for $t \geq x$,

$$h = \frac{h}{\phi(x)} \phi(x) \leq \frac{h}{\phi(t)} \phi(t)$$

Hence for $t \geq x$,

$$\frac{|\Delta_h^2(f; t)|}{(1+t+2h)^A} \leq \Omega_2 \left( \frac{h}{\phi(x)} \right) ; \phi; f)$$

Hence

$$\frac{|\Delta_{nh}^2(f; x)|}{(1+x+2nh)^A} \leq n^2 \Omega_2 \left( \frac{h}{\phi(x)} \right) ; \phi; f)$$
Replacing $h$ by $h \phi(x)$,

$$\frac{\left| \Delta_{nh\phi(x)}^{2}(f; x) \right|}{(1+x+2nh\phi(x))} \leq n^{2} \Omega_{2}(h; \phi; f)$$

Hence

$$\Omega_{2}(nh; \phi; f) \leq n^{2} \Omega_{2}(h; \phi; f)$$

Let $h$, $\delta \geq 0$. Let $n = \lceil \frac{\delta}{h} \rceil + 1$, where $\lceil \frac{\delta}{h} \rceil$ denotes the integral part of $\frac{\delta}{h}$. Then

$$\Omega_{2}(\delta; \phi; f) \leq \Omega_{2}(nh; \phi; f)$$

$$\leq n^{2} \Omega_{2}(h; \phi; f)$$

$$\leq (1+\frac{\delta}{n})^{2} \Omega_{2}(h; \phi; f)$$

Hence

$$\Omega_{2}(\delta; \phi; f) \leq 3(1+\frac{\delta^{2}}{n^{2}}) \Omega_{2}(h; \phi; f).$$

**LEMMA 2.3.** Let $n$ be any positive integer and $h > 0$. Then

$$\Delta_{h}^{2}(f; x) = \Delta_{2^{-n}h}^{2}(f; x) + 2 \sum_{k=1}^{n} \Delta_{2^{-k}h}^{2}(f; x+2^{-k}h)$$

$$+ \sum_{k=1}^{n} \Delta_{2^{-k}h}^{2}(f; x+2^{-k+1}h). \quad (2.1)$$
Proof. In Lemma 2.1, taking $n = 2$,

$$\Delta_{2h}^2 (f; x) = \Delta_h^2 (f; x) + 2 \Delta_{h}^2 (f; x+h) + \Delta_h^2 (f; x+2h)$$

Replacing $h$ by $h/2$ in the above expression

$$\Delta_{h/2}^2 (f; x) = \Delta_{h/2}^2 (f; x) + 2 \Delta_{h/2}^2 (f; x+2^{-1}h) + \Delta_{h/2}^2 (f; x+h) \quad (2.2)$$

Thus expression (2.1) holds for $n = 1$. Suppose now that expression (2.1) holds for some $n = m$. Then

$$\Delta_{h}^2 (f; x) = \Delta_{2^{-m}h}^2 (f; x) + 2 \sum_{k=1}^{m} \Delta_{2^{-k}h}^2 (f; x+2^{-k}h)$$

$$+ \sum_{k=1}^{m} \Delta_{2^{-k}h}^2 (f; x+2^{-k+1}h) \quad (2.3)$$

Also, by expression (2.2),

$$\Delta_{2^{-m}h}^2 (f; x) = \Delta_{2^{-m+1}h}^2 (f; x) + 2 \Delta_{2^{-m+1}h}^2 (f; x+2^{-1}h)$$

$$+ \Delta_{2^{-m+1}h}^2 (f; x+2^{-m}h) \quad (2.4)$$

Adding expressions (2.3) and (2.4) and cancelling out the common term $\Delta_{2^{-m}h}^2 (f; x)$ we get

$$\Delta_{2^{-m}h}^2 (f; x) = \Delta_{2^{-m+1}h}^2 (f; x) + 2 \sum_{k=1}^{m+1} \Delta_{2^{-k}h}^2 (f; x+2^{-k}h)$$

$$+ \sum_{k=1}^{m+1} \Delta_{2^{-k}h}^2 (f; x+2^{-k+1}h)$$
Hence if expression (2.1) holds for \( n = m \), it also holds for \( n = m+1 \). We have already seen that it holds for \( n=1 \). Hence by the principle of mathematical induction it follows that expression (2.1) holds for all \( n \in \mathbb{N} \).

LEMMA 2.4. Suppose \( 0 < \gamma \leq 1 \). Assume that \( f \) in \( \{ C[0,\infty)/x^A \} \) satisfies the following condition.

\[
\Delta_h^2(f; x) = O(1) \left( \frac{h^2}{(x(1+x))^2} \right)^\gamma (1+x+2h)^A \quad (0 < h \leq x, \quad 0 < x < \infty).
\]

Then

(i) \[ \Delta_h^2(f; x) = O(1) \left( \frac{h}{(1+x)^2} \right)^\gamma (1+x+2h)^A \quad (x > 0, h > 0) \]

and

(ii) \[ \Delta_h^2(f; x) = O(1) x^{-\gamma} (1+x+2h)^A \quad (x > 0, h > 0) \]

Proof. Let \( x > 0, \ h > 0 \). Then for any \( n \in \mathbb{N} \), by Lemma 2.3.,

\[
|\Delta_h^2(f;x)| \leq |\Delta_{2^{-n}h}^2(f;x)| + 2 \sum_{k=1}^{n} |\Delta_{2^{-k}h}^2(f; x+2^{-k+1}h)| + \sum_{k=1}^{n} |\Delta_{2^{-k}h}^2(f; x+2^{-k+1}h)|
\]

\[
\leq |\Delta_{2^{-n}h}^2(f;x)| + O(1) \sum_{k=1}^{n} \left\{ \frac{2^{-h}}{(x+2^{-k}h)(1+x+2^{-k}h)^2} \right\} \gamma (1+x+2h)^A, \quad (2.5)
\]

using the assumption of Lemma 2.4.
\[ \Delta^2_{2^{-n}h} (f; x) = | \Delta^2_{2^{-n}h} (f; x) | + O(1) \sum_{k=1}^{n} \left( \frac{2^k h}{(1+x)^2} \right)^\gamma (1+x+2h)^A \]

Taking the limit as \( n \to \infty \),

\[ | \Delta^2_h (f; x) | = O(1) \sum_{k=1}^{\infty} 2^{-k \gamma} \left( \frac{h}{(1+x)^2} \right)^\gamma (1+x+2h)^A \]

\[ = O(1) \left( \frac{h}{(1+x)^2} \right)^\gamma (1+x+2h)^A, \]

since \( \sum_{k=1}^{\infty} 2^{-k \gamma} < \infty \). This proves (i) of Lemma 2.4. Again by assumption of Lemma 2.4,.

\[ \Delta^2_h (f; x) = O(1) x^{-\gamma} (1+x+2h)^A \quad (0 < h \leq x) \] \hspace{1cm} (2.6)

Now let \( h > x \). Choose \( n \in \mathbb{N} \) such that \( 2^{-n} h \leq x < 2^{-(n-1)}h \).

Let \( h = 2^{-n}h \). Then \( h_0 \leq x < 2h_0 \).

From estimate (2.5),

\[ | \Delta^2_h (f; x) | \leq | \Delta^2_{2^{-n}h} (f; x) | + O(1) \sum_{k=1}^{n} \left( 2^{-k h} \right)^\gamma (1+x+2h)^A \] \hspace{1cm} (2.7)

Now

\[ \sum_{k=1}^{n} \left( 2^{-k h} \right)^\gamma = \sum_{k=1}^{n} \left( 2^{-(n-k) h_0} \right)^\gamma = \sum_{k=0}^{n} 2^{-k \gamma} h_0^{-\gamma} \]

\[ = O(1) h_0^{-\gamma}, \text{ since } \sum_{k=0}^{\infty} 2^{-k \gamma} < \infty \]

\[ = O(1) x^{-\gamma}, \text{ since } h_0 > x/2 \] \hspace{1cm} (2.8)

Also since \( h_0 \leq x \), by estimate (2.6),

\[ \Delta^2_{2^{-n}h} (f; x) = \Delta^2_{h_0} (f; x) \]

\[ = O(1) x^{-\gamma} (1+x+2h)^A \] \hspace{1cm} (2.9)
From estimates (2.7) to (2.9), it follows that for \( h > x > 0 \),

\[
\Delta^2_h (f; x) = O(1) \left( x^{-\gamma} (1+x+2h)^A \right) \quad (2.10)
\]

From estimates (2.6) and (2.10), (ii) of Lemma 2.4 follows.  

##

### LEMMA 2.5.

For any positive integer \( m \),

\[
L_n((1+t)^{-m}; x) = O(1) \left( (1+x)^{-m} + n^{-m} \right) \quad (x \geq 0)
\]

\( n \in \mathbb{N} \)

**Proof.**

\[
L_n((1+t)^{-m}; x) = \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{(1+x)^n} \left( \frac{n-k+1}{n+1} \right)^m
\]

For \( 0 \leq k \leq n-m \),

\[
\binom{n}{k} \left( \frac{n-k+1}{n+1} \right)^m = O(1) \binom{n}{k} \frac{(n-k) \ldots (n-k-m+1)}{(n+1)^m}
\]

\[
= O(1) \binom{n-m}{k}
\]

For \( k = n-m+1, \ldots, n \)

\[
\binom{n}{k} \left( \frac{n-k+1}{n+1} \right)^m = O(1) \binom{n}{k} n^{-m}.
\]

Hence

\[
L_n((1+t)^{-m}; x) = O(1) \left\{ \frac{1}{(1+x)^m} \sum_{k=0}^{n-m} \binom{n-m}{k} \frac{x^k}{(1+x)^n-m} + n^{-m} \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{(1+x)^n} \right\}
\]

\[
= O(1) \left\{ (1+x)^{-m} + n^{-m} \right\}.
\]
LEMMA 2.6. Let $L_n^*$ $(n \in \mathbb{N})$ be the sequence of operators defined on $C(0, \infty)$ by

$$L_n^*(f; x) = \sum_{k=1}^{n} P_{n,k}(x) f\left(\frac{k}{n-k+1}\right),$$

where $P_{n,k}(x) = \binom{n}{k} x^k (1+x)^{-n}$.

Let $\phi(x) = \sqrt{x} (1+x)$. Then

(i) $L_n^* \left( (\phi^2(t))^{-1}; x \right) = O(1) \left( \phi(x)^{-1} \right)$ $(n \in \mathbb{N}, 0 < x \leq n)$

(ii) $L_n^* \left( \frac{(t-x)^2}{\phi^2(t)} ; x \right) = O(1) n^{-1}$ $(n \in \mathbb{N}, 0 < x \leq n)$

Proof. Now for $n \in \mathbb{N}$, $n = 1, \ldots, n-3$

$$\binom{n}{k} \frac{(n-k+1)^3}{k(n+1)^2} = O(1) \binom{n-2}{k+1} \tag{2.11}$$

For $n \in \mathbb{N}$, $k = n-2, n-1, n$: $0 < x \leq n$,

$$\binom{n}{k} \frac{(n-k+1)^3}{k(n+1)^2} = O(1) \frac{x}{x(1+x)^2} \binom{n}{k} \tag{2.12}$$

From estimates (2.11) and (2.12), for $n \in \mathbb{N}$, $0 < x \leq n$, $k = 1, \ldots, n$, taking $P_{n,k}(x) := 0$ $(k > n)$, we have

$$P_{n,k}(x) \frac{(n-k+1)^3}{k(n+1)^2} = O(1) \frac{1}{x(1+x)^2} P_{n-2,k+1}(x) + P_{n,k}(x) \tag{2.13}$$

From estimate (2.13), summing for $k = 1, \ldots, n$ it follows that for $n \in \mathbb{N}$, $0 < x \leq n$,

$$L_n^* \left( \frac{1}{\phi^2(t)} ; x \right) = O(1) \left( \frac{1}{\phi^2(x)} \right) \left( L_{n-2}^* (1; x) + L_n^* (1; x) \right) = O(1) \frac{1}{\phi^2(x)},$$

since $L_n^* (1; x) \leq L_n (1; x) = 1$. This proves (i) of Lemma 2.6.

Now

$$L_n^* \left( \frac{(t-x)^2}{\phi^2(t)} ; x \right) = L_n^* \left( \frac{(t-x)^2}{t(1+t)^2} ; x \right)$$
\[(n) \frac{n-k+1}{k} \leq 4 \binom{n}{k^*}, \quad k = 1, 2, \ldots, n-1.\]

Hence for \(k = 1, \ldots, n-1,\)

\[
P_{n,k}(x) \frac{n-k+1}{k} \left( \frac{k}{n+1} - \frac{x}{1+x} \right)^2 \leq \frac{4}{x} P_{n,k+1}(x) \left( \frac{k}{n+1} - \frac{x}{1+x} \right)^2
\]

\[
\leq \frac{12}{x} P_{n,k+1}(x) \left( \frac{k+1}{n+1} - \frac{x}{1+x} \right)^2 + \frac{1}{(n+1)^2}
\]

(2.15)

using the inequality: \((a+b)^2 \leq 3(a^2 + b^2),\) \(a \geq 0, b \geq 0.\) Also,

\[
P_{n,k}(x) \frac{n-k+1}{k} \left( \frac{k}{n+1} - \frac{x}{1+x} \right)^2 \leq \frac{1}{x} P_{n,k}(x) \left( \frac{k}{n+1} - \frac{x}{1+x} \right)^2,
\]

for \(k = n, n > x\)

(2.16)

From estimates (2.15) and (2.16), for \(n \in \mathbb{N}, 0 < x \leq n,\)

\[
L_n \left( \frac{1}{t} \left( \frac{t}{1+t} - \frac{x}{1+x} \right)^2 ; x \right) \leq \frac{13}{x} \left\{ L_n \left( \left( \frac{t}{1+t} - \frac{x}{1+x} \right)^2 ; x \right) + \frac{1}{(n+1)^2} \right\}
\]

\[
= O(1) \left( \frac{1}{n(1+x)^2} + \frac{1}{x(n+1)^2} \right),
\]

using (iv) of Lemma 1.1. Also for \(\frac{1}{n} \leq x \leq n,\)

\[
(n+1)^{-1} x^{-1} \leq \min \left\{ x^{-1} (1+x)^{-1}, (1+x)^{-1} \right\} \leq 2(1+x)^{-2}
\]

Hence for \(n \in \mathbb{N}, \frac{1}{n} \leq x \leq n,\)

\[
L_n \left( \frac{1}{t} \left( \frac{t}{1+t} - \frac{x}{1+x} \right)^2 ; x \right) = O(1) \frac{1}{n(1+x)^2}
\]

(2.17)

Also \(0 < x < 1/n, \quad \frac{1}{t} < \frac{1}{x}\) for \(t = \frac{k}{n-k+1} \) (k=1, ..., n). Hence

\[
L_n \left( \frac{1}{t} \left( \frac{t}{1+t} - \frac{x}{1+x} \right)^2 ; x \right) < \frac{1}{x} L_n \left( \left( \frac{t}{1+t} - \frac{x}{1+x} \right)^2 ; x \right) = O(1) \frac{1}{n(1+x)^2}
\]

(2.18)

using (iv) of Lemma 1.1. From estimates (2.14), (2.17) and (2.18), (ii) of Lemma 2.6 follows
We now derive an estimate for

\[ \|L_n f - f\|_A = \sup_{x \geq 0} \frac{|L_n(f;x) - f(x)|}{(1+x)^A} \]

in terms of various modulii of smoothness of the function \( f \).

**THEOREM 2.7.** For all \( f \) in \( \{ C[0, \infty)/x^A \} (A \geq 0) \) and all \( n \in \mathbb{N} \),

\[ \|L_n f - f\|_A = O(1) \left[ \Omega_0(n^{-1}; f) + \Omega_2(n^{-1/2}; f) + \Omega_2(n^{-1}; \psi; f) \right] \]

where \( \phi(x) = \sqrt{x} (1+x) \), \( \psi(x) = (1+x)^2 \),

\( \Omega_2(\delta; \phi; f), \Omega_2(\delta; \psi; f) \) are defined as in Lemma 2.2.

and

\[ \Omega_0(\delta; f) := \sup_{y \geq x \geq 1/\delta} \frac{|f(y) - f(x)|}{(1+y)^A} \]

**Proof.**

**Case (i) \( x > n \).** For \( 0 \leq t \leq n \),

\[ |f(t) - f(x)| \leq |f(t) - f(n)| + |f(n) - f(x)| \]

\[ \leq |\Delta^2_{n-t} (f;t)| + |f(n) - f(2n-t)| + |f(n) - f(x)| \]

\[ \leq \left\{ \Omega_2 \left( \frac{n^{4/4}}{(1+t)^4} \psi; f \right) + 2 \Omega_0 (n^{-1}; f) \right\} (1+2x)^A, \]

using the definitions of \( \Omega_2 \) and \( \Omega_0 \).

\[ = O(1) \left\{ (1+ \frac{n^4}{(1+t)^4} \psi; f) + \Omega_0 (n^{-1}; f) \right\} (1+x)^A \]

(2.19)

using Lemma 2.2.
Also by Lemma 2.5.,
\[ n^4 L_n((1+t)^{-4}; x) = O(1) n^4 ((1+x)^{-4} + n^{-4}) \]
\[ = O(1) \text{, since } n < x \text{ in this case} \]  
\[ (2.20) \]

From estimates (2.20) and (2.19), it follows that for \( x > n \),
\[ |L_n(f; x) - f(x)| \leq L_n (|f(t) - f(x)|; x) \]
\[ = O(1) \{ \Omega_2 (n^{-1}; f) + \Omega_0 (n^{-1}; f) \} (1+x)^A \]  

Case (ii) \( 0 < x < n \). Define \( f_\delta \) as in the previous section by
\[ f_\delta(x) = \frac{4}{\delta^2} \int_0^{\delta/2} \int_0^{\delta/2} (2f(x+u+v) - f(x+2u+2v)) \, du \, dv. \]

Let
\[ g_\delta = f - f_\delta. \]

Now as in the previous section
\[ f(t) - f(x) - f_\delta'(x) (t-x) \]
\[ = (f_\delta(t) - f_\delta(x) - f_\delta'(x) (t-x)) + g_\delta(t) - g_\delta(x) \]
\[ = f_\delta''(\xi) (t-x)^2 + g_\delta(t) - g_\delta(x), \]
for some \( \xi \) between \( t \) and \( x \). Using estimates (1.7) and (1.9) of the previous section, for any \( t \)
\[ |g_\delta(t)| \leq \sup_{0 < h \leq \delta} |\Delta_{h}^2 (f; t)| \]
and
\[ |f_\delta''(\xi)| \leq \frac{9}{\delta^2} \sup_{0 < h \leq \delta} |\Delta_{h}^2 (f; \xi)| \]
It follows that
\[ f(t) - f(x) - f'(x)(t-x) \]
\[ = O(1) \left( \frac{(t-x)^2}{\delta^2} + 1 \right) \sup_{0 < h < \delta} \{ |\Delta_h^2(f; u)| : u \text{ between } t \text{ and } x \} \] \tag{2.21}

Hence using the definition of \( \Omega_2 \), for \( t \neq 0 \),
\[ f(t) - f(x) - f'(x)(t-x) \]
\[ = O(1) \left( \frac{(t-x)^2}{\delta^2} + 1 \right) \left\{ \Omega_2\left(\frac{\delta}{\phi(t)} ; \phi ; f \right) (1+x+2\delta)^A + \Omega_2\left(\frac{\delta}{\phi(x)} ; \phi ; f \right) (1+t+2\delta)^A \right\} \]
\[ = O(1) \left( \frac{(t-x)^2}{\delta^2} + 1 \right) \left\{ (1 + \frac{\phi^2(x)}{\phi^2(t)}) (1+x+2\delta)^A + (1+t+2\delta)^A \right\} \]
\[ \cdot \Omega_2\left(\frac{\delta}{\phi(x)} ; \phi ; f \right) \] \tag{2.22}

using Lemma 2.2.

Take
\[ \delta = \frac{\phi(x)}{n^{\frac{1}{2}}} = \frac{x^{\frac{1}{2}}}{n^{\frac{1}{2}}} (1+x) \]

Since \( n \geq x \), \( \delta < 1+x \). Hence
\[ 1+x+2\delta < 3(1+x); (1+t+2\delta) < 3(1+t) \text{ if } t \geq x. \]

Hence from estimate (2.22) it follows that, for \( t \neq 0 \),
\[ f(t) - f(x) - f'(x)(t-x) \]
\[ = O(1) \left( \frac{(t-x)^2}{\delta^2} + 1 \right) \left\{ (1 + \frac{\phi^2(x)}{\phi^2(t)}) (1+x)^A + (1+t)^A \right\} \Omega_2(n^{-\frac{1}{2}}; \phi; f) \] \tag{2.23}

Let \( L_n^* \) be defined as in Lemma 2.6.
Then from estimate (2.23),

\[ L_n^*(f(t) - f(x) - f'_\delta(x) (t-x); x) \]

\[ = O(1) \left( \frac{1}{\delta^2} L_n((t-x)^2; x) + \phi \frac{2}{x^2} L_n^*(\frac{(t-x)^2}{\phi^2(t)}; x) \right) \]

\[ + \phi^2(x) L_n^*(\frac{1}{\phi^2(t)}; x) + L_n(1;x) (1+x)^A \]

\[ + (\frac{1}{\delta^2} L_n((t-x)^2(1+t)^A); x) + L_n((1+t)^A;x) \Omega_2(n^{-2};\phi;f) \]

\[ = O(1) \Omega_2(n^{-2};\phi;f) (1+x)^A \quad (2.24) \]

using (i), (iii), (v) of Lemma 1.1. and (i), (ii) of Lemma 2.6. and the fact that \( \delta^2 = \frac{x}{n} (1+x)^2 \), \( n \geq x \).

For \( t = 0 \), using estimate (2.21),

\[ f(0) - f(x) - f'_\delta(x) (0-x) \]

\[ = O(1) \left( 1 + \frac{x^2}{\delta^2} \right) \sup_{0 < h \leq \delta} |\Delta_\delta^2 (f; u)| \]

\[ \sup_{0 < u \leq x} \Omega_2(\frac{\delta}{\Psi(u)}; \psi; f) (1+u+2\delta)^A, \]

where \( \Psi(x) \equiv (1+x)^2 \)

\[ = O(1) \left( 1 + \frac{x^2}{\delta^2} \right) \Omega_2(\delta; \psi; f) (1+x)^A \]

\[ = O(1) \left( 1 + \frac{x^2}{\delta^2} \right) (1+n^2\delta^2) \Omega_2(n^{-1};\psi; f) (1+x)^A \]
Also

\[
(1 + \frac{x^2}{\delta^2}) (1 + n^2 \delta^2) = 1 + \frac{x^2}{\delta^2} + n^2 \delta^2 + n^2 x^2
\]

\[
= 1 + \frac{nx}{(1+x)\delta^2} + nx (1+x)^2 + n^2 x^2
\]

\[
\leq 2(1 + nx + n^2 x^2) (1+x)^2
\]

\[
= O(1) (1 + (\frac{n-2}{1}) x + (\frac{n-2}{2}) x^2) (1+x)^2
\]

\[
= O(1) (1+x)^n
\]

Hence

\[
(f(0) - f(x)) P_{n,0}(x) = O(1) \Omega_2(n^{-1}; \psi; f) (1+x)^A
\]  

(2.25)

From estimates (2.24) and (2.25),

\[
L_n (f(t); x) - f(x) - f'(x) L_n(t-x; x)
\]

\[
= O(1) \{ \Omega_2(n^{-1}; \phi; f) + \Omega_2(n^{-1}; \psi; f) \} (1+x)^A
\]  

(2.26)

Now

\[
|f'_{\delta}(x)| \leq |f'_{\delta}(x) - f'_{\delta}(n)| + |f'_{\delta}(n)|
\]

\[
= |f'',(\xi)| (n-x) + |f'_{\delta}(n)|, \text{ for some } \xi, \ x < \xi < n
\]

\[
\leq \frac{9n^2}{\delta^2} \sup_{0 < h \leq \delta} \Delta^2_h (f; \xi) + \frac{5}{\delta} \sup_{n < u < n+\delta} \sup_{0 < h \leq \delta} |f(u+h) - f(u)|,
\]

using estimates (1.8) and (1.9). Hence, by the definitions of \( \Omega_2 \) and \( \Omega_0 \)
\[ |f'_\delta(x)| \leq 9 \left\{ \frac{n}{\delta^2} \Omega_2 \left( \frac{\delta}{\phi(\xi)} ; \phi ; f \right) (1+\xi + 2\delta)^A + \frac{1}{\delta} \Omega_0 \left( n^{-1} ; f \right) (1 + n + 2\delta)^A \right\} \]

\[ = O(1) \frac{n+1}{\delta^2} \left\{ \Omega_2 \left( n^{-\frac{1}{2}} ; \phi ; f \right) + \Omega_0 \left( n^{-1} ; f \right) \right\} (1+n)^A \quad (2.27) \]

since \( x < \xi < n; \quad \delta = \frac{\phi(x)}{n^{\frac{1}{2}}}; \quad \delta \leq 1+x \leq 1+n \)

Using (ii) of Lemma 1.1.,

\[ \frac{n+1}{\delta^2} (1+n)^A \mid L_n(t-x; x) \mid = \frac{n+1}{\delta^2} (1+n)^A x \left( \frac{x}{1+x} \right)^n \]

\[ = \frac{(1+n)^{A+2}}{(1+x)^2} \left( \frac{x}{1+x} \right)^n \]

\[ \leq \frac{1}{(1+x)^2} L_n \left( (1+t)^{A+2} ; x \right) = O(1) (1+x)^A, \]

using (iii) of Lemma 1.1. Using the above estimate in estimate (2.27), we have

\[ |f'_\delta(x)| L_n(t-x; x) = O(1) \left\{ \Omega_2 \left( n^{-\frac{1}{2}} ; \phi ; f \right) + \Omega_0 \left( n^{-1} ; f \right) \right\} (1+x)^A \quad (2.28) \]

From estimates (2.26) and (2.28), it follows that for \( n \geq x, \)

\[ L_n(f(t); x) - f(x) \]

\[ = O(1) \left\{ \Omega_2 \left( n^{-\frac{1}{2}} ; \phi ; f \right) + \Omega_2 \left( n^{-1} ; \psi ; f \right) + \Omega_0 \left( n^{-1} ; f \right) \right\} (1+x)^A \]
It now follows by cases (i) and (ii) that for all \( n \in \mathbb{N} \) and \( 0 < x < \infty \) and all \( f \) in \( \{ C[0,\infty)/x^A \} \)

\[
L_n(f; x) - f(x) = O(1) \{ \Omega_2(n^{\frac{1}{2}}; f) + \Omega_2(n^{-1}; f) + \Omega_0(n^{-1}; f) \} (1+x)^A
\]

Hence for all \( f \) in \( \{ C[0,\infty)/x^A \} \) and all \( n \in \mathbb{N} \),

\[
||L^f_n - f||_A = O(1) \{ \Omega_2(n^{\frac{1}{2}}; f) + \Omega_2(n^{-1}; f) + \Omega_0(n^{-1}; f) \}
\]

From Theorem 2.7, we now obtain the following direct theorem for the operators \( L_n \).

**THEOREM 2.8.** Suppose \( 0 < \gamma \leq 1 \). Assume that \( f \) in \( \{ C[0,\infty)/x^A \} \) (\( A > 0 \)) satisfies the following conditions

\[ (i) \quad \frac{\Delta^2 h(f;x)}{(1+x+2h)^A} = O(1) \left( \frac{h^2}{x(1+x)^2} \right)^\gamma \quad (x > 0) \quad (0 < h \leq x) \]

and

\[ (ii) \quad \frac{f(y)-f(x)}{(1+y)^A} = O(1) \quad x^{-\gamma} \quad (y \geq x > 0) \]

Then

\[
||L^f_n - f||_A = O(1) n^{-\gamma} \quad (n \in \mathbb{N})
\]

**Proof.** Replacing the \( h \) in assumption (i) of the theorem by

\[ h\phi(x) = h \sqrt{x(1+x)}, \text{ we get} \]


\[ \Delta^2_{h \phi(x)} (f; x) \left( \frac{1}{1+x+2h \phi(x)} \right)^A = O(1) \ h^2 \gamma \quad (0 < h \phi(x) \leq x) \quad (2.29) \]

Again from assumption (i) of the theorem, applying (i) of Lemma 2.4.,

\[ \Delta^2_{h \phi(x)} (f; x) \left( \frac{1}{1+x+2h \phi(x)} \right)^A = O(1) \left( \frac{h}{(1+x)^2} \right) \gamma \quad (x > 0, h > 0) \]

Replacing \( h \) by \( h \phi(x) = h \sqrt{x} (1+x) \) in the above estimate,

\[ \Delta^2_{h \phi(x)} (f; x) \left( \frac{1}{1+x+2h \phi(x)} \right)^A = O(1) \ h \gamma \quad (h > 0) \]

since

\[ h \phi(x) \geq x \quad \text{implies} \quad h \frac{\sqrt{x}}{1+x} \leq h^2 \]

From estimates (2.29) and (2.30),

\[ \Delta^2_{h \phi(x)} (f; x) \left( \frac{1}{1+x+2h \phi(x)} \right)^A = O(1) \ h^2 \gamma \quad (h > 0) \]

Consequently

\[ \Omega_2 (\delta; \phi ; f) = O(1) \delta^2 \gamma \quad (\delta > 0) \quad (2.31) \]

where \( \Omega_2(\delta; \phi ; f) \) is defined as in Theorem 2.6. by

\[ \Omega_2(\delta; \phi ; f) = \sup_{\substack{x > 0 \quad 0 < h \leq \delta}} \left| \frac{\Delta^2_{h \phi(x)} (f; x)}{(1+x+2h \phi(x))^A} \right| \]
Also from assumption (ii) of the theorem, it follows that

$$\Omega_0(\delta; f) = O(1) \delta^\gamma \quad (\delta > 0)$$  \hspace{1cm} (2.32)

where \( \Omega_0(\delta; f) \) is also defined as in Theorem 2.7. by

$$\Omega_0(\delta; f) = \sup_{y \geq x \geq 1/\delta} \frac{|f(y) - f(x)|}{(1+y)^A}$$

From estimates (2.31) and (2.32) it follows that

$$\Omega_2(n^{-\frac{1}{2}}; \phi; f) = O(1) n^{-\gamma} \quad (n \in \mathbb{N})$$ \hspace{1cm} (2.33)

$$\Omega_0(n^{-1}; f) = O(1) n^{-\gamma} \quad (n \in \mathbb{N})$$ \hspace{1cm} (2.34)

From estimates (2.33) and (2.34), applying Theorem 2.7., it follows that

$$||L_n f - f||_A = O(1) n^{-\gamma} \quad (n \in \mathbb{N})$$  

We conclude this section by deducing from Theorem 2.8. the following direct theorem for the operators \( L_n \) when \( f \) belongs to \( C_B[0,\infty) \).

**THEOREM 2.9.** Suppose \( 0 < \gamma \leq 1 \). Assume that \( f \) in \( C_B[0,\infty) \) satisfies the following condition

$$\Delta_h^2 (f; x) = O(1)(\frac{h^2}{x(1+x)^2})^\gamma \quad 0 < x < \infty \quad 0 < h \leq x$$

Then

$$||L_n f - f||_{C_B} = O(1) n^{-\gamma} \quad (n \in \mathbb{N})$$
Proof. By assumption,
\[ \Delta_h^2 f(x; x) = O(1)(\frac{h^2}{x(1+x)^2}) \gamma \quad (0 < x < \infty) \]
Hence; applying (ii) of lemma 2.4.,
\[ \Delta_h^2 f(x; x) = O(1) x^{-\gamma} \quad (x > 0) \]
Hence, if \( y > x > 0 \) and \( h > 0 \), then
\[
|f(x) - f(x+h) - f(y) + f(y+h)| \\
\leq |f(x) - 2f(\frac{x+y+h}{2}) + f(y+h)| + |f(x+h) - 2f(\frac{x+y+h}{2}) + f(y)| \\
= O(1) x^{-\gamma} \quad (2.35)
\]
Hence if \( m, n \) are positive integers and \( m > n \), then for a fixed \( h > 0 \),
\[
|f(n) - f(n+h) - f(m) + f(m+h)| = O(1) n^{-\gamma}
\]
Hence, given any \( \varepsilon > 0 \),
\[
|(f(n) - f(n+h)) - (f(m) - f(m+h))| < \varepsilon
\]
for \( m, n \) sufficiently large. It follows that \( \{ f(n) - f(n+h) \} \) \( n=1,2,\ldots \)
is a Cauchy sequence. Since every Cauchy sequence of real numbers converge
\[ C_h = \lim_{n \to \infty} (f(n) - f(n+h)) \text{ exists.} \]
Now, by estimate (2.35), for \( n \in \mathbb{N}, n \geq x \),
\[
|f(x) - f(x+h) - (f(n) - f(n+h))| = O(1) x^{-\gamma}
\]
Taking the limit as $n \to \infty$,

$$|f(x) - f(x+h) - C_h| = O(1) x^{-\gamma} \quad (x > 0, \ h > 0)$$  \hspace{1cm} (2.36)

In particular, this implies that

$$\lim_{x \to \infty} (f(x) - f(x+h)) = C_h.$$

Now

$$f(x) - f(x+h) \to C_h \quad (x \to \infty)$$

$$f(x+h) - f(x+2h) \to C_h \quad (x \to \infty)$$

$$\vdots$$

$$f(x+(n-1)h) - f(x+nh) \to C_h \quad (x \to \infty)$$

Adding, we get, for each positive integer $n$,

$$f(x) - f(x+nh) \to n C_h \quad (x \to \infty)$$

Hence for all $n \in \mathbb{N}$,

$$n|C_h| = \lim_{x \to \infty} |f(x)-f(x+nh)| \leq 2 \|f\|_{C_B}$$

It follows that

$$|C_h| \leq \frac{2}{n} \|f\|_{C_B} \quad (n \in \mathbb{N})$$

Taking the limits as $n \to \infty$, we get

$$C_h = 0 \quad \text{for all} \quad h > 0.$$

This together with estimate (2.36) gives

$$f(x) - f(x+h) = O(1) x^{-\gamma} \quad (x > 0, \ h > 0)$$
Hence

\[ f(y) - f(x) = O(1) \frac{x^\gamma}{y} \quad (y > x > 0) \quad (2.37) \]

Also, by assumption of the theorem, we have

\[ \Delta^2_h(f; x) = O(1) \left( \frac{h^2}{x(1+x)^2} \right)^\gamma \quad (0 < x < \infty) \quad (0 < h \leq x) \quad (2.38) \]

From estimates (2.37) and (2.38) using Theorem 2.8 with \( A = 0 \), it follows that

\[ \left\| L_n f - f \right\|_{C_B^n} = O(1) n^{-\gamma} \quad (n \in \mathbb{N}) \]

REMARK. V. Totik has obtained approximation theorems for the operators \( L_n \) for functions in \( C_B[0, \infty) \) — Theorems A and B stated at the beginning of this section. Theorem 2.9 is in improvement over the direct part of Theorem B since it dispenses with the assumption (ii) of Theorem B. Also we remark that the proof of Theorem 2.9 remains valid if we replace the space \( C_B[0, \infty) \) by the space \( \{C[0, \infty)/x^A \}_{0 \leq A < 1} \). Hence the assumption (ii) of Theorem 2.8 may be dropped for \( 0 \leq A < 1 \). However the assumption (ii) of Theorem 2.8 cannot in general be dropped. This may be shown by the following example.

Take

\[ f(x) = x \log (1+x) \]

Then

\[ f'(x) = \frac{x}{1+x} + \log (1+x) \]

so that

\[ f''(x) = \frac{1}{(1+x)^2} + \frac{1}{1+x} \]
\[ \Delta_h^2 (f; x) = f''(\xi) h^2 , \text{ for some } \xi, \quad x < \xi < x+2h. \]
\[ = \left( \frac{1}{(1+\xi)^2} + \frac{1}{1+\xi} \right) h^2 \]

Since \( \xi > x \),
\[ |\Delta_h^2 (f; x)| \leq \frac{2h^2}{1+x} \leq \frac{2h^2}{x(1+x)^2} (1+x)^2 \]

Hence \( f(x) = x \log (1+x) \) satisfies condition (i) of Theorem 2.8, with \( A = 2, \gamma = 1 \). Also
\[ f(2x) - f(x) = 2x \log (1+2x) - x \log (1+x) \]
\[ = x \log \frac{(1+2x)^2}{1+x} \]
\[ \geq x \log (1+x) \]

Now
\[ \frac{x \log (1+x)}{(1+2x)^2} \neq O(1) x^{-1} \quad (x > 0) \]

Hence
\[ \frac{f(2x) - f(x)}{(1+2x)^2} \neq O(1) x^{-1} \quad (x > 0) \]

Consequently
\[ \frac{f(y) - f(x)}{(1+y)^2} \neq O(1) x^{-1} \quad (y > x > 0) \]

Thus \( f(x) = x \log (1+x) \) does not satisfy condition (ii) of Theorem 2.8, with \( A = 2, \gamma = 1 \). Now by Theorem V.3.6, (proved later in Chapter V)
\[ \left\| L_n f - f \right\|_A = O(1) n^{-1} \quad (n \in \mathbb{N}) \]
only if \( f \) satisfies conditions (i) and (ii) of Theorem 2.8, with \( \gamma = 1 \).

It follows that for the function \( f(x) = x \log(1+x) \), the statement

\[
||L_n f - f||_2 = O(1) \frac{n^{-1}}{n} \quad (n \in \mathbb{N})
\]

does not hold.

### 3. Direct Theorem associated with Uniform Approximation for the operators \( R^{(n)}_n \)

Here we are concerned with the approximation of the Bernstein type rational operators \( R^{(n)}_n \) introduced in Chapter I, Section 2. In [3] Catherine Bala'zs and J. Szabados, obtained weighted estimates for

\[
|R^{(n)}_n (f;x) - f(x)|, \quad 0 < \beta \leq 2/3
\]

and they treated certain questions on the uniform convergence of \( R^{(n)}_n f \) on \([0, \infty)\).

V. Totik in [58] solved the saturation problem for the operators \( R^{(n)}_n \) on intervals \((a, \infty)\), \( a > 0 \). He observed that it would be interesting to study the saturation around \( x = 0 \). He also conjectured the conditions under which the non-optimal approximation order can be achieved. Here these results are proved completely. Incidentally we show that many of the assumptions of Totik are superfluous and can be dispensed with. (The details of the work done by V. Totik on \( R^{(n)}_n \) is given in the introduction of the thesis).
In this section we prove the direct part of these results. We first estimate the rate of convergence of \( R_n^\beta(f;x) - f(x) \) in terms of a particular modulus of smoothness of the function \( f \). We then deduce a direct theorem for \( R_n^\beta \).

For arriving at these results we make use of an interpolation theorem proved by V. Totik in [60]. We state below a special of this theorem.

**Lemma 3.1.** (See [60, Theorem 7]). Let \( f \in C_B[0,\infty) \). Define

\[
\omega_2(\delta;f) = \sup_{0 < h \leq \delta} \left| \Lambda_{h\phi(x)}^2 (f;x-h\phi(x)) \right|
\]

where \( \phi(x) = \sqrt{x} \). Let

\[
D = \{ g \in C_B[0,\infty) / g' \text{ absolutely continuous on compact subintervals of } (0,\infty) \}
\]

Let

\[
K(t^2;f) = \inf_{g \in D} \left\{ \| f - g \| + t^2 \| g'' \| \right\}
\]

Then there exists constants \( M > 0 \) and \( t_0 > 0 \) such that for all \( t, 0 < t \leq t_0 \) and all \( f \) in \( C_B[0,\infty) \),

\[
\frac{1}{M} K(t^2;f) \leq \omega_2(t;f) \leq M K(t^2;f)
\]

From lemma 3.1. we deduce the following.
**Lemma 3.2.** There exists a constant \( M_1 > 0 \) such that for all \( f \) in \( \mathcal{C}_B[0, \infty) \) and all \( t, 0 < t \leq 1 \)

\[
K(t^2; f) \leq M_1 \omega_2(t; f)
\]

where \( K(t^2; f) \) and \( \omega_2(t; f) \) are defined as in Lemma 3.1.

**Proof.** By Lemma 3.1, there exists constants \( M > 0, 0 < t_0 \leq 1 \) such that for \( 0 < t < t_0 \) and all \( f \) in \( \mathcal{C}_B[0, \infty) \)

\[
\frac{1}{M} K(t^2; f) \leq \omega_2(t; f) \leq M K(t^2; f)
\]

Hence for all \( f \) in \( \mathcal{C}_B[0, \infty) \)

\[
K(t^2; f) \leq M \omega_2(t; f) \quad (0 < t \leq t_0)
\]

Suppose \( 0 < t \leq 1 \), \( t > t_0 \). Then for all \( g \in \mathcal{D} \),

\[
\|f - g\| + t^2 \|\phi''g''\| \leq \frac{t^2}{t_0^2} \{ \|f - g\| + t_0^2 \|\phi''g''\| \}
\]

Taking the infimum over \( g \) in \( \mathcal{D} \)

\[
K(t^2; f) \leq \frac{1}{t_0^2} K(t_0^2; f) \quad \text{, since } 0 < t \leq 1
\]

\[
\leq M t_0^{-2} \omega_2(t_0; f)
\]

\[
\leq M t_0^{-2} \omega_2(t; f)
\]

Hence

\[
K(t^2; f) \leq M t_0^{-2} \omega_2(t; f) \quad (0 < t \leq 1).
\]
LEMMA 3.3. Let $D$ be defined as in Lemma 3.1. Then for all $g \in D$

$$|\Delta_h^2 (g; x)| \leq 3 \||\phi^2 g''||_C^h (x > 0; \ h > 0)$$

Proof. Let $g \in D$. Then for $x > 0$, $h > 0$,

$$|\Delta_h^2 (g; x)| = \left| \int_0^h \int_0^h g''(x+u+v) \, du \, dv \right|$$

$$\leq \||\phi^2 g''|| \frac{h^2}{x},$$

(3.1)

since $\phi^2(x) = x$; and $\xi \geq x$. Now, let $x > 0$, $h > 0$. By Lemma 2.3.

for any $n \in N$,

$$\Delta_h^2 (g; x) = \Delta_{2^{-n}h}^2 (g; x) + 2 \sum_{k=1}^{n} \Delta_{2^{-k}h}^2 (f; x+2^{-k}h)$$

$$+ \sum_{k=1}^{n} \Delta_{2^{-k}h}^2 (f; x+2^{-k+1}h)$$

Hence, from estimate (3.1), for all $n \in N$,

$$|\Delta_h^2 (g; x)| \leq |\Delta_{2^{-n}h}^2 (g; x)| + 3 \||\phi^2 g''||_C^h \sum_{k=1}^{n} \frac{(2^{-k}h)^2}{x+2^{-k}h}$$

$$\leq |\Delta_{2^{-n}h}^2 (g; x)| + 3 \||\phi^2 g''||_C^h \sum_{k=1}^{n} 2^{-k}h$$

Taking the limits as $n \to \infty$,

$$|\Delta_h^2 (g; x)| \leq 3 \||\phi^2 g''||_h (x > 0, \ h > 0)$$

since

$$\sum_{k=1}^{n} 2^{-k} \leq \sum_{k=1}^{\infty} 2^{-k} = 1 \text{ and } \Delta_{2^{-n}h}^2 (g; x) \to 0 (n \to \infty).$$

##
We also require the following properties of $R_n^{(\beta)}$.

**Lemma 3.4.** Let $x_n = \frac{x}{1 + a_n x}$, $a_n = n^{\beta - 1}$.

For all $n \in \mathbb{N}$ and $x \geq 0$,

(i) $R_n^{(\beta)}(1; x) = 1$

(ii) $R_n^{(\beta)}(t-x_n; x) = 0$

(iii) $R_n^{(\beta)}((t-x_n)^2; x) \leq \frac{x_n}{n^{\beta}}$

**Proof.** Properties (i) and (ii) of Lemma 3.4 have been proved by Katalin Balazs (see [2], Lemma 2.1). But for the sake of completeness we prove these also.

Now

$$R_n^{(\beta)}(f; x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(a_n x)^k}{(1+a_n x)^n} f\left(\frac{k}{n^{\beta}}\right), \quad a_n = n^{\beta - 1}$$

Hence

$$R_n^{(\beta)}(1; x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(a_n x)^k}{(1+a_n x)^n} \equiv 1$$

This proves (i) of Lemma 3.4.

$$R_n^{(\beta)}(t; x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(a_n x)^k}{(1+a_n x)^n} \cdot \frac{k}{n^{\beta}}$$

$$= \frac{n}{n^{\beta}} \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{(a_n x)^k}{(1+a_n x)^n}$$

$$= \frac{x}{1 + a_n x} = x_n.$$
Hence

\[ R_n^\beta(t; x) - x_n = 0. \]

Also

\[ R_n^\beta(1; x) = 1, \] by (i) of Lemma 3.4.

Hence

\[ R_n^\beta(t-x_n; x) = 0. \]

This proves (ii) of Lemma 3.4.

\[ R_n^\beta((t-x_n)^2; x) = R_n^\beta(t^2; x) - 2x R_n^\beta(t; x) + x_n^2 R_n^\beta(1; x) \]

\[ = R_n^\beta(t^2; x) - x_n, \]

using (i) and (ii) of Lemma 3.4. Also

\[ R_n^\beta(t^2; x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(a_n x)^k}{(1+a_n x)^n} \frac{k^2}{n^{2\beta}} \]

\[ = \sum_{k=0}^{n} \binom{n}{k} \frac{k(k-1)+k}{n^{2\beta}} \]

\[ = \frac{n(n-1)}{n^{2\beta}} \binom{n-2}{k-2} + \frac{n}{n^{2\beta}} \binom{n-1}{k-1} \]

Hence

\[ R_n^\beta(t^2; x) = \frac{n(n-1)}{n^{2\beta}} \frac{(a_n x)^2}{(1+a_n x)^n} + \frac{n}{n^{2\beta}} \frac{a_n x}{1+a_n x} \]

\[ \leq \frac{x^2}{(1+a_n x)^2} + \frac{1}{n^\beta} \frac{x}{1+a_n x} \]

\[ = x_n^2 + \frac{x_n}{n^\beta} \]
It follows that

\[ R_n^{(\beta)} \left( (t-x_n)^2; x \right) = R_n^{(\beta)} \left( t^2; x \right) - x_n^2 \leq \frac{x_n}{n^{\beta}} \]

This proves (iii) of Lemma 3.4.

We now estimate the rate of convergence of

\[ R_n^{(\beta)} (f; x) - f \left( \frac{x}{1+a_n x} \right) \]

**Theorem 3.5.** Suppose \( 0 < \beta < 1 \). Then for all \( f \) in \( C_B[0, \infty) \) and all \( n \in \mathbb{N} \)

\[ ||R_n^{(\beta)} f - f_n||_{C_B} = O(1) \omega_2 (n^{-\beta/2}; f), \]

where

\[ f_n(x) = f \left( \frac{x}{1+a_n x} \right), \quad a_n = n^{\beta-1} \]

and

\[ \omega_2(\delta; f) = \sup_{x > h/\sqrt{x}} \Delta_2^{h/\sqrt{x}} (f; x-h/\sqrt{x}) \]

\[ 0 < h \leq \delta \]

**Proof.** Let \( D \) is defined as in Lemma 3.1. That is,

\[ D = \{ g \in C_B[0, \infty)/g' \text{ absolutely continuous on compact subintervals of } (0, \infty) \} \]

Let \( g \in D \). Fix \( x > 0 \). Suppose

\[ 0 \leq t < \frac{1}{2} x_n, \quad \text{where } x_n = \frac{x}{1+a_n x}, \quad a_n = n^{\beta-1} \]
Now
\[ g(t) - g(x_n) - g'(x_n) (t-x_n) \]
\[ = \Delta^2 \frac{(g; t) - (g(2x_n-t) - g(x_n)) + g'(x_n) (x_n-t)}{2}, \]
\[ = \Delta^2 \frac{(g; t) - (g'(\xi) - g'(x_n)) (x_n-t)}{2}, \]
for some $\xi$ between $x_n$ and $2x_n-t$. Hence for $0 < t < \frac{1}{2} x_n$,
\[ |g(t) - g(x_n) - g'(x_n) (t-x_n)| \]
\[ \leq |\Delta^2 \frac{(g; t)}{2}| + |g''(\xi)_{\omega}| (x_n-t)^2, \]
for some $\xi_{\omega}$ between $x_n$ and $2x_n-t$
\[ \leq |\Delta^2 \frac{(g; t)}{2}| + ||\phi^2 g''|| \frac{(x_n-t)^2}{x_n}, \]

since $(\phi^2 g'')(\xi_{\omega}) = \xi_{\omega} g''(\xi_{\omega})$ and $\xi_{\omega} > x_n$.
\[ \leq 3 ||\phi^2 g''|| (x_n-t) + ||\phi^2 g''|| \frac{(x_n-t)^2}{x_n} \]

using Lemma 3.3.
\[ \leq 7 ||\phi^2 g''|| \frac{(x_n-t)^2}{x_n}, \quad (3.2) \]

Since $\frac{x_n-t}{x_n} \geq \frac{1}{2}$ for $0 \leq t < \frac{1}{2} x_n$.

For $t \geq \frac{1}{2} x_n$,
\[ |g(t) - g(x_n) - g'(x_n) (t-x_n)| \]
\[ = |(g'(\xi_1) - g'(x_n)) (t-x_n)|, \]
for some $\xi_1$, between $t$ and $x_n$. 

\[ \text{II} \]
for some $\xi_2$ between $t$ and $x_n$

\[
\leq \frac{1}{\xi_2} g''(t-x_n^2) \leq 2 \frac{1}{x_n^2} g''(t-x_n^2) \cdot \frac{x_n^2}{x_n^2},
\]

since $\xi_2 \geq \min\{t, x_n\} \geq \frac{x_n}{2}$.

From estimates (3.2) and (3.3), it follows that for all $t \geq 0$,

\[
|g(t) - g(x_n) - g'(x_n)(t-x_n)| \leq 7 \frac{1}{x_n} g''(t-x_n^2).
\]

Hence

\[
|R_n^{(\beta)} (g(t); x) - g(x_n) R_n^{(\beta)} (1; x) - g'(x_n) R_n^{(\beta)} (t-x_n; x)|
\]

\[
\leq 7 x_n^{-1} \frac{1}{\phi^2 g''} R_n^{(\beta)} ((t-x_n)^2; x)
\]

(3.4)

Now applying (i), (ii) and (iii) of lemma 3.4, it follows that

\[
|R_n^{(\beta)} (g;x) - g(x_n)| \leq 7 n^{-\beta} \frac{1}{\phi^2 g''} \cdot \frac{n \in \mathbb{N}}{x \geq 0}
\]

(3.5)

Now let $f \in C_B[0,\infty)$. Then for any $g$ in $D$,

\[
|R_n^{(\beta)} (f; x) - R_n^{(\beta)} (g;x)| \leq R_n^{(\beta)} (|f-g|; x)
\]

\[
\leq ||f-g||.
\]

Since $R_n^{(\beta)} (1; x) = 1$. Also

\[
|f(x_n) - g(x_n)| \leq ||f-g||
\]

(3.6)

From estimates (3.4) to (3.6),

\[
|R_n^{(\beta)} (f; x) - R_n^{(\beta)} (f(x_n))| \leq 2 ||f-g|| + 7 n^{-\beta} ||\phi^2 g''||.
\]
Taking the infimum over all \( g \) in \( D \),
\[
|R_n^{(\beta)}(f; x) - f(x_n)| \leq 7 K(n^{-\beta}; f)
\]  
where \( K(t^2; f) \) is defined as in Lemma 3.1.

By Lemma 3.2, there exists a constant \( M_1 > 0 \) such that for all \( f \) in \( C_B[0, \infty) \) and all \( t, 0 < t \leq 1, \)
\[
K(t^2; f) \leq M_1 \omega_2(t; f)
\]  
Using estimates (3.7) and (3.8), for all \( f \) in \( C_B[0, \infty) \) and all \( n \in \mathbb{N}, \)
\[
|R_n^{(\beta)}(f; x) - f(x_n)| \leq 7 M_1 \omega_2(n^{-\beta/2}; f) \quad (x > 0)
\]  
The above estimate holds trivially for \( x = 0 \). Hence
\[
\|R_n^{(\beta)} f - f_n\|_{C_B} = O(1) \omega_2(n^{-\beta/2}; f)
\]  
for all \( f \) in \( C_B[0, \infty), n \in \mathbb{N} \), where
\[
f_n(x) = f(x_n) = f\left(\frac{x}{1+a_n x}\right)
\]  

From Theorem 3.5, we now deduce the following direct theorem for the operators \( R_n^{(\beta)} \).

**THEOREM 3.6.** Suppose \( 0 < \beta < 1; 0 < \gamma \leq 1 \).
Let \( \beta_0 = \min \{ \beta, 1-\beta \} \). Assume that \( f \) in \( C_B[0, \infty) \) satisfies the following two conditions.

(i) \[ \Delta^2_h(f; x) = O(1) \left( \frac{h^2}{x+h} \right)^{\beta_0} \gamma (x > 0, h > 0) \]

(ii) \[ \Delta^1_h(f; x) = O(1) \left( \frac{h}{x(x+h)} \right)^{\beta_0} \gamma (x > 0, h > 0) \]

(Here \( \Delta^2_h(f; x) = f(x) - 2f(x+h) + f(x+2h); \Delta^1_h(f; x) = f(x+h) - f(x) \))
Then
\[ |R_n^{(\beta)} f - f|_{C_B} = O(1) n^{-\beta_0} \gamma \quad (n \in N) \]

Proof. By assumption (i),
\[ \Delta_h^2(f; x) = O(1) \left( \frac{h^2}{x+h} \right)^{\beta_0} \gamma \quad (x > 0; h > 0) \]

Hence
\[ \Delta_h^2(f; x-h) = O(1) \left( \frac{h^2}{x} \right)^{\beta_0} \gamma \quad (x > h > 0) \]

Consequently
\[ \Delta_{h/\sqrt{x}}^2(f; x-h/\sqrt{x}) = O(1) \frac{2^{\beta_0} \gamma}{h^{\beta}} \quad (x > h > 0) \]

Hence
\[ \omega_2(h; f) = O(1) \frac{2^{\beta_0} \gamma}{h^{\beta}} \quad (h > 0) \]

where \( \omega_2(h; f) \) is defined as in Theorem 3.5. It follows that
\[ \omega_2(n^{-\beta/2}; f) = O(1) n^{-\beta_0} \gamma \quad (n \in N) \]

Hence by applying Theorem 3.5, taking \( x_n = \frac{x}{1+a_n x} \),
\[ R_n^{(\beta)}(f; x) - f(x_n) = O(1) \omega_2(n^{-\beta/2}; f) \]
\[ = O(1) n^{-\beta_0} \gamma \quad (n \in N, x > 0) \] (3.9)

Also by assumption (ii) of the theorem
\[ \Delta_n^1 (f; x) = O(1) \left( \frac{h}{x(x+h)} \right)^{1-\beta_0} \gamma \quad (h > 0) \]

Hence

\[ f(x) - f(x_n) = O(1) \left( \frac{x-x_n}{xx_n} \right)^{1-\beta_0} \gamma \quad (n \in \mathbb{N}, x > 0) \]

Now

\[ x-x_n = x - \frac{x}{1+a_n x} = \frac{a_n x^2}{1+a_n x} = a_n xx_n \]

Hence for \( x > 0 \) and \( n \in \mathbb{N} \),

\[ f(x) - f(x_n) = O(1) a_n^{1-\beta_0} \gamma = O(1) n^{-\beta_0} \gamma \quad (3.10) \]

From estimates (3.9) and (3.10),

\[ R_n^{(\beta)} f; x) - f(x) = O(1) n^{-\beta_0} \gamma \quad (n \in \mathbb{N}, x > 0) \]

Hence

\[ \| R_n^{(\beta)} f - f \|_{C_B} = O(1) n^{-\beta_0} \gamma \quad (n \in \mathbb{N}) \]

4. A Local Direct Theorem for the Operators \( L_n \).

As expressed by V. Totik in [58], if \( f \) is smooth on \([a, b]\), we can hope to get a good approximation by \( L_n f \) only on subsets of \((a, b)" far" from a or b, since the values of \( f \) outside \([a, b]\) can spoil the order of approximation near a or b. Conversely if we know the order of approximation of \( L_n f - f \) on \([a, b]\) we clearly can infer the smoothness properties of \( f \) only on parts of \((a, b)" far" from the end points.\]
We, in this section, prove a local direct theorem for the operators \( L_n \) when \( f \) belongs to \( \bigcap_{A>0} \{ C[0, \infty)/e^{Ax} \} \). (Here \( L_n(n \in \mathbb{N}) \) are the operators introduced in Chapter I, Section 1).

**DEFINITION 4.1.** Suppose \( 0 < \gamma \leq 2 \), let \( I \) be a subinterval of \( \mathbb{R} \) and let \( f \in C(I) \). We say that \( f \) belongs to \( \text{Lip}^\gamma \) on \( I \) if

\[
\omega_2(\delta; f; I) = O(1)\delta^\gamma \quad (0 < \delta \leq 1)
\]

where

\[
\omega_2(\delta; f; I) = \sup_{x, x+2h \in I, 0 < h \leq \delta} |\Delta_h^2(f; x)|
\]

**THEOREM 4.2.** Suppose \( 0 < \gamma \leq 2 \). Let

\[
\mathcal{F} = \bigcap_{A>0} \{ C[0, \infty)/e^{Ax} \}.
\]

Suppose \( f \) in \( \mathcal{F} \) belongs to \( \text{Lip}^\gamma \) on \([a, b]\) where \([a, b]\) is a compact subinterval of \((0, \infty)\). Then

\[
n^{\gamma/2} (L_n(f; x) - f(x)) = O(1) \quad (n \in \mathbb{N})
\]

uniformly on compact subintervals of \((a, b)\).

For arriving at the above local direct theorem, we make use of the following result due to A.F. Timan.

**LEMMA 4.3.** (See [55, p. 121]). Let \( f \in C[a, b] \).

Suppose \( f(a) = f(b) = 0 \). Then \( f \) has an extension \( f^* \) onto \((-\infty, \infty)\) which is periodic of period \( 2(b-a) \) such that

\[
\omega_2(\delta; f; (-\infty, \infty)) \leq 5 \omega_2(\delta; f; [a, b]) \quad (0 < \delta \leq \frac{b-a}{2})
\]
We also require the following property of the operator $L_n$.

**Lemma 4.4.** Let $f$ belong to $\mathcal{F}$ where $\mathcal{F}$ is defined as in Theorem 4.2. Suppose $f$ vanishes on an open interval $(a, b)$ contained in $(0, \infty)$. Then

$$n^2 L_n(f; x) = O(1) \quad (n \in \mathbb{N})$$

uniformly on compact subintervals of $(a, b)$.

**Proof.** Let $a < a_1 < b_1 < b$. Then there exists $\epsilon > 0$ such that $[a_1 - \epsilon, b_1 + \epsilon]$ is also contained in $(a, b)$.

For all $x$ in $[a_1, b_1]$ and all $x$ sufficiently large

$$|\frac{k}{n-k} - x| < \frac{\epsilon}{2} \quad \Rightarrow \quad |\frac{k}{n-k+1} - x| < \epsilon$$

$$\Rightarrow \quad \frac{k}{n-k+1} \text{ belongs to } (x - \epsilon, x + \epsilon) \quad (a, b)$$

$$\Rightarrow \quad f(\frac{k}{n-k+1}) = 0, \text{ since } f \text{ vanishes on } (a, b)$$

Hence for all $x$ in $[a_1, b_1]$ and all $x$ sufficiently large

$$L_n(f; x) = \sum P_{n,k}(x) f(\frac{k}{n-k+1}), \quad (4.1)$$

where $P_{n,k}(x) = \binom{n}{k} x^k (1+x)^{-n}$.

Since $f$ belongs to $\mathcal{F} = \bigcap_{A>0} \{ C(0, \infty)/e^{Ax} \}$, for each $A > 0$,

$$f(x) = O(1) \quad e^{Ax} \quad (x \to \infty)$$
Also for any \( A > 0 \), \( f(x) e^{-Ax} \) is bounded on compact sub-intervals of \([0, \infty)\). Hence for each \( A > 0 \),

\[
f(x) = O(1) e^{Ax} \quad (0 \leq x < \infty).
\]  

(4.2)

Using estimate (4.2) in expression (4.1) we get, for each \( A > 0 \),

\[
L_n(f; x) = O(1) e^{An} \sum_{|\frac{k}{n-k} - x| \geq \frac{\varepsilon}{2}} P_{n,k}(x),
\]  

(4.3)

for all \( x \) in \([a, b]\) and all \( n \) sufficiently large.

Now by Theorem 1.1.1., for all \( x \) in \([a_1, b_1]\),

\[
\left| \frac{k}{n-k} - x \right| \geq \frac{\varepsilon}{2} \quad P_{n,k}(x) \leq 2\exp\left(-\frac{1}{16}\frac{\varepsilon}{2n}\right) \frac{\varepsilon^2 n}{4b_1(1+b_1)^2}
\]

\[
\leq 2 \exp\left(-A_0n\right), \text{ say} \quad (4.4)
\]

where \( A_0 = \frac{2^{-6} \varepsilon^2}{b_1(1+b_1)^2} \). Taking \( A = A_0/2 \) in estimate (4.3) and applying estimate (4.4), it follows that for all \( x \) in \([a_1, b_1]\) and all \( n \) sufficiently large, say \( n > N_0 \),

\[
L_n(f;x) = O(1) e^{-A_0/2 n}
\]

\[
= O(1) n^{-2} \quad (4.5)
\]

Also

\[
\sup_{a_1 \leq x \leq b_1} \quad n^2 |L_n(f;x)| < \infty
\]

(4.6)

From estimates (4.5) and (4.6) it follows that

\[
n^2 L_n(f;x) = O(1) \quad (a_1 \leq x \leq b_1)
\]

##
Proof of Theorem 4.2. Let \( l(x) \) be the unique linear function coinciding with \( f \) at \( a \) and \( b \). For \( a \leq x \leq b \), let

\[
f_1(x) := f(x) - l(x)
\]

Then \( f_1 \) belongs to \( C[a,b) \) and \( f_1(a) = f_1(b) = 0 \). Hence, by Lemma 4.3., \( f_1 \) has a periodic extension \( f_1^* \) onto \( (-\infty, \infty) \) of period \( 2(b-a) \) such that

\[
\omega_2(\delta; f_1^*; (-\infty, \infty)) \leq 5 \omega_2(\delta; f_1; [a,b]) \tag{4.7}
\]

\[
(0 < \delta \leq \frac{b-a}{2})
\]

By assumption \( f \) belongs to \( \text{Lip}^* \gamma \) on \( [a, b] \).

Since \( l \) is linear, \( \Delta_h^2(l; x) = 0 \)

Hence \( f_1 = f - l \) also belongs to \( \text{Lip}^* \gamma \) on \( [a,b] \).

Hence

\[
\omega_2(\delta; f_1; [a,b]) = O(1) \delta^\gamma \quad (0 < \delta \leq 1)
\]

Hence applying estimate (4.7)

\[
\omega_2(\delta; f_1^*; (-\infty, \infty)) = O(1) \delta^\gamma \quad (0 < \delta \leq 1)
\]

In particular

\[
\Delta_h^2(f_1^*; x) = O(1) h^\gamma \quad (0 < \frac{x}{h} \leq 1) \tag{4.8}
\]

Also since \( f_1 = f - l \) is continuous on \( [a,b] \) and \( f_1^* \) is an extension of \( f_1 \) which is periodic, \( f_1^* \) is a continuous bounded function on \( (-\infty, \infty) \).
Hence $f_1^*$ restricted to $[0, \infty)$ belongs to $C_B(0, \infty)$. Hence from estimate (4.8), applying Corollary 1.3. with $A = 0$, it follows that

$$L_n(f_1^*; x) - f_1^*(x) = O(1) \frac{x}{n} (1+x)^{\gamma/2} \quad (x > 0) \quad (n \in \mathbb{N}) \quad (4.9)$$

Since $f_1^*(x) = f_1(x)$ on $[a,b]$,

$f_1^*-f_1$ vanishes on the open interval $(a,b)$. Let $a < a_1 < b_1 < b$. Then, by Lemma 4.4.,

$$L_n(f_1^*-f_1; x) = O(1) n^{-2} \quad \left( \frac{n}{a_1 \leq x \leq b_1} \right) \quad (n \in \mathbb{N}) \quad (4.10)$$

From estimates (4.9) and (4.10), since $f_1^* = f_1$ on $[a,b]$,

$$L_n(f_1; x) - f_1(x) = O(1) n^{-\gamma/2} \quad \left( a_1 \leq x \leq b_1 \right) \quad (n \in \mathbb{N}) \quad (4.11)$$

Let $l(x) = ax + b$. Then

$$\left| L_n(1; x) - l(x) \right| = \left| a \left( L_n(t; x) - x \right) \right|$$

$$= \left| -a' x \left( \frac{x}{1-x} \right)^n \right|$$

$$\leq \left| a \right| b \left( \frac{1}{1+b} \right)^n$$

for $x$ in $[a,b]$. Hence

$$L_n(1; x) - l(x) = O(1) n^{-\gamma/2} \quad \left( \frac{a \leq x \leq b}{n \in \mathbb{N}} \right) \quad (4.12)$$

Since $f(x) = f_1(x) + l(x)$, from estimates (4.11) and (4.12) it follows that

$$L_n(f; x) - f(x) = O(1) n^{-\gamma/2} \quad \left( \frac{a_1 \leq x \leq b_1}{n \in \mathbb{N}} \right)$$
That is, for all $n \in \mathbb{N}$,

$$n^{\gamma/2} \left( L_n(f;x) - f(x) \right) = O(1)$$

uniformly on $[a_1, b_1]$, where $[a_1, b_1]$ is an arbitrary compact subinterval of $(a, b)$. 

##