CHAPTER I

CONVERGENCE THEOREMS

1. Approximation of unbounded functions in \( C(0, \infty) \) by the Bernstein-type rational operators \( L_n \)

The Bernstein-type rational operators \( L_n \) (\( n \in \mathbb{N} \)) are defined on \( C(0, \infty) \), the class of all continuous functions on \( [0, \infty) \), by

\[
L_n(f; x) = (1 + x)^{-n} \sum_{k=0}^{n} \binom{n}{k} x^k \frac{f(k)}{n-k+1}
\]

(1.1)

These operators were introduced by Bleimann, Butzer and Hahn [11]. They studied the approximation properties of \( L_n \) when \( f \) belonged to \( C_B(0, \infty) \), the class of all continuous functions on \( [0, \infty) \) which are bounded there with norm

\[
\|f\|_{C_B} := \sup_{x \in [0, \infty)} |f(x)|
\]

We in this section consider the problem of determining the largest subclass of \( C(0, \infty) \) on which \( (L_n) \) defines a pointwise approximation process. Similar problems have been completely dealt with for the well-known operators of Szasz and Baskakov [26]

For \( F \) in \( C(0, \infty) \), \( F \geq 0 \) on \( [0, \infty) \), we denote by \( \{ C(0, \infty)/F(x) \} \), the class of all continuous functions \( f \) on \( [0, \infty) \) for which \( f/F \) is bounded in a neighbourhood of infinity. That is,

\[
\{ C(0, \infty)/F(x) \} = \{ f \in C(0, \infty)/f(x) = O(1) \ F(x), \ x \to \infty \}.
\]
We, in this section, prove that \((L_n)\) defines a pointwise approximation process on \(\mathcal{F} := \bigcap_{A > 0} \{ C[0, \infty)/ e^{Ax} \} \). That is, we prove that for \(f\) in \(\mathcal{F}\)

\[
\lim_{n \to \infty} L_n (f; x) = f(x),
\]

for each \(x, 0 < x < \infty\). We also show, by giving a counter example, that \((L_n)\) does not define a pointwise approximation process on \(\{ C[0, \infty)/ e^{Ax} \}\) however small \(A > 0\) is.

**THEOREM 1.1.** Let \(L_n \ (n=1,2, \ldots)\) be the sequence of operators given by expression (1.1).

Let \(\mathcal{F} := \bigcap_{A > 0} \{ C[0, \infty)/ e^{Ax} \} \). Then for any \(f\) in \(\mathcal{F}\)

\[
\lim_{n \to \infty} L_n (f; x) = f(x)
\]

for each \(x, 0 < x < \infty\).

To prove Theorem 1.1, we require an estimate for the sum

\[
\sum P_{n,k} (x),
\]

where

\[
P_{n,k}(x) = \binom{n}{k} x^k (1+x)^{-n}.
\]

To derive this estimate, we observe the connection between \(L_n\) and the well-known Bernstein operators \(B_n\), defined on \(C[0,1]\) by

\[
B_n (f; x) = \sum_{k=0}^{n} P_{n,k} (x) f\left( \frac{k}{n} \right) \quad (n \in \mathbb{N})
\]
where
\[ p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \]

Clearly
\[ p_{n,k}(x) = p_{n,k}\left(\frac{x}{1+x}\right), \quad 0 \leq x < \infty \]

For the sums \( \sum p_{n,k}(x) \) we have the following inequality due to S. Bernstein.

**Lemma 1.2.** ([32, p. 18]). Let
\[ p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \]

For any \( x, \ 0 \leq x \leq 1 \),
\[ \sum_{|k-nx| \geq 2z(nx(1-x))^{1/2}} p_{n,k}(x) \leq 2 \exp\left(-z^2\right) \]
if \( 0 \leq z \leq \frac{3}{2} (nx(1-x))^{1/2} \)

Due to the relation \( p_{n,k}(x) = p_{n,k}\left(\frac{x}{1+x}\right) \), Lemma 1.2. gives rise to the following lemma for the sums \( \sum p_{n,k}(x) \).

**Lemma 1.3.** Let
\[ p_{n,k}(x) = \binom{n}{k} x^k (1+x)^{-n} \]

Suppose \( 0 < \delta \leq x \). Then for all \( n \in \mathbb{N} \)
\[ \sum_{|\frac{k}{n-k} - x| \geq \delta} p_{n,k}(x) \leq 2 \exp\left(-\frac{\delta^2}{16} \frac{n}{x(1+x)^2}\right) \]
Proof.  Now

\[ \left| \frac{k}{n-k} - x \right| \geq \delta \]

implies

\[ \frac{k}{n} \leq \frac{x - \delta}{1+x-\delta} \quad \text{or} \quad \frac{k}{n} \geq \frac{x + \delta}{1+x+\delta}. \]

This implies that

\[ \left| \frac{k}{n} - \frac{x}{1+x} \right| \geq \frac{\delta}{(1+x)(1+x+\delta)} \geq \frac{\delta}{2(1+x)^2}, \]

since \(0 < \delta \leq x\). Also

\[ p_{n,k}(x) = p_{n,k}\left(\frac{x}{1+x}\right), \]

where \(p_{n,k}(x)\) is defined as in Lemma 1.2.

Hence

\[ \left| \frac{k}{n-k} - x \right| \geq \delta \]

Choose \(z\) such that

\[ 2z \left[ n \frac{x}{1+x} (1 - \frac{x}{1+x}) \right]^{\frac{1}{2}} = \frac{\delta}{2} \left( \frac{n}{(1+x)^2} \right)^{\frac{1}{2}} \]

That is,

\[ 2z \left( \frac{nx}{(1+x)^2} \right)^{\frac{1}{2}} = \frac{\delta}{2} \left( \frac{n}{(1+x)^2} \right)^{\frac{1}{2}} \]

Hence

\[ z = \frac{\delta}{4} \left( \frac{n}{x(1+x)^2} \right)^{\frac{1}{2}} \]
Since \( 0 < \delta \leq x \),

\[
0 < z \leq \frac{x}{4} \left( \frac{n}{x(1+x)^2} \right)^{\frac{1}{2}} = \frac{1}{4} \left[ n \frac{x}{1+x} \left( 1 - \frac{x}{1+x} \right) \right]^{\frac{1}{2}}
\]

Hence by Lemma 1.2.,

\[
\left| \frac{k}{n} - \frac{x}{1+x} \right| \geq \frac{\delta}{2(1+x)^2}
\]

It follows that

\[
\sum_{k=0}^{n-k-1} P_{n,k}(x) \leq 2 \exp \left( -\frac{\delta^2 n}{16x(1+x)^2} \right)
\]

Proof of Theorem 1.1. Suppose \( f \) belongs to \( \mathcal{F} \). Fix \( x > 0 \). Let \( \varepsilon > 0 \) be arbitrary. Choose \( \delta, \ 0 < \delta \leq x \) such that

\[
y \geq 0, \ |y-x| < 2\delta \text{ implies } |f(y)-f(x)| < \varepsilon.
\]

Now

\[
|L_n(f(t); x) - f(x)|
\]

\[
= |L_n(f(t) - f(x); x)|, \text{ since } L_n(1; x) = 1
\]

\[
\leq L_n(|f(t) - f(x)|; x)
\]

\[
= \sum_{k=0}^{n} P_{n,k}(x) |f \left( \frac{k}{n-k+1} \right) - f(x)|,
\]

where \( P_{n,k}(x) \) is defined as in Lemma 1.3.

Now

\[
\left| \frac{k}{n-k} - x \right| < \delta \text{ implies } \left| \frac{k}{n-k+1} - x \right| < 2\delta.
\]
for all large \( n \). This implies, by the choice of \( \delta \), that for such \( n \),

\[
|f\left(\frac{k}{n-k+1}\right) - f(x)| < \varepsilon
\]

Hence for all \( n \) sufficiently large,

\[
|L_n(f;x) - f(x)| < \varepsilon + \sum P_{n,k}(x) \left| f\left(\frac{k}{n-k+1}\right) - f(x) \right|
\]

Now, let

\[
F_f(x) = \max_{0 \leq t \leq x} |f(t)|
\]

Then for all \( n \) sufficiently large

\[
|L_n(f;x) - f(x)| < \varepsilon + 2 F_f(n) \sum P_{n,k}(x) \left| \frac{k}{n-k} - x \right| \geq \delta
\]

\[
\leq \varepsilon + 4 F_f(n) \exp\left(-\frac{\delta^2 n}{16x(1+x)^2}\right), \tag{1.2}
\]

by Lemma 1.3. Since \( f \in \mathcal{F} \), for each \( A > 0 \),

\[
f(x) = O(1) e^{Ax} \quad (x \to \infty).
\]

Also \( f(x) e^{-Ax} \) is bounded on compact subintervals of \([0, \infty)\). Hence

\[
f(x) = O(1) e^{Ax} \quad (0 \leq x < \infty).
\]

It follows that for each \( A > 0 \),

\[
F_f(x) = O(1) e^{Ax} \quad (0 \leq x < \infty).
\]
In particular there exists \( M > 0 \) such that
\[
F_f(t) \leq M \exp \left( \frac{\delta^2 t}{32 x(1+x)^2} \right) \quad (0 \leq t < \infty)
\] (1.3)

From estimates (1.2) and (1.3) it follows that for all \( n \) sufficiently large,
\[
|L_n (f;x) - f(x)| < \epsilon + 4M \exp \left( -\frac{\delta^2 n}{32 x(1+x)^2} \right) < 2\epsilon
\]

Since \( \epsilon > 0 \) is arbitrary, it follows that
\[
\lim_{n \to \infty} L_n (f; x) = f(x)
\]

Also for each \( n \), \( L_n (f; 0) = f(0) \). Hence for each \( x \), \( 0 \leq x < \infty \)
\[
\lim_{n \to \infty} L_n (f; x) = f(x)
\]

We close this section by giving a counter example to show that the class \( \mathcal{F} = \bigcap_{A > 0} \{ C[0,\infty); e^{Ax} \} \) in Theorem 1.1 cannot be replaced by the larger class \( \{ C[0,\infty); e^{Ax} \} \) however small \( A > 0 \) is.

**THEOREM 1.4.** For any \( A > 0 \),
\[
L_n (e^{At}; x) \to \infty \quad (n \to \infty)
\]

for all \( x \) sufficiently large.

**Proof.** Fix \( A > 0 \).
\[
L_n (e^{At}; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1+x)^{-n} e^{\frac{Ak}{n-k+1}}
\]
\[ x \left( \frac{x}{1+x} \right) e^A \to \infty \quad (n \to \infty) \]

for all \( x > (e^A - 1)^{-1} \)

2. **Approximation of unbounded functions in** \( C[0,\infty) \) **by the Bernstein type rational operators** \( R_n^{(\beta)} \).

Let \( a = (a_n) \) be a sequence of positive numbers. The Bernstein - type rational operators \( R_n(a) \) \((n \in \mathbb{N})\) are defined on \( C[0,\infty) \) by

\[
R_n(a; f; x) = (1+a_nx)^{-n} \sum_{k=0}^{n} \binom{n}{k} (a_nx)^k f\left(\frac{k}{n^{1-\beta}}\right) \quad (2.1)
\]

These operators were introduced by C. Balázs [2]. We denote by \( (R_n^{(\beta)}) \), the sequence of operators \( (R_n(a)) \) when \( a_n = n^{\beta-1} \). That is, \( R_n^{(\beta)} \) \((n \in \mathbb{N})\) are defined on \( C[0,\infty) \) by

\[
R_n^{(\beta)}(f; x) = (1+n^{\beta-1}x)^{-n} \sum_{k=0}^{n} \binom{n}{k} (n^{\beta-1}x)^k f\left(\frac{k}{n^{1-\beta}}\right) \quad (2.2)
\]

In [2], C. Balázs has shown that if \( a_n \to 0 \) and \( na_n \to \infty \) as \( n \to \infty \), then \( (R_n(a)) \) defines a pointwise approximation process on \( \{ C[0,\infty)/e^{Ax} \} \) for each \( A \geq 0 \). For the operators \( (R_n^{(\beta)}) \) \((0 < \beta < 1)\) more can be said. We, in this section, prove that for \( 0 < \beta < 1 \), \( (R_n^{(\beta)}) \) defines a pointwise approximation process on \( \{ C[0,\infty)/x^A x^{\frac{1}{1-\beta}} \} \) for each \( A, 0 \leq A < 1 \).
We also give a counter example to show that the above result cannot be extended to the case when \( A = 1 \).

**THEOREM 2.1.** Suppose \( 0 \leq A < 1 \), \( 0 < \beta < 1 \). Then for any \( f \) in
\[
\{ C[0,\infty)/ x^{Ax^{1-\beta}} \}
\]
\[
\lim_{n \to \infty} R_n^{(\beta)}(f; x) = f(x),
\]
for each \( x \), \( 0 \leq x < \infty \).

For proving Theorem 2.1, we need an estimate for the partial sum
\[
\left| \frac{k}{n^{\beta}} - \frac{x}{1+a_n x} \right| \geq \delta
\]
where
\[
q_{n,k}(x) = \binom{n}{k} \frac{(a_n x)^k}{(1+a_n x)^n} \frac{(-\frac{k}{n^{\beta}}) A \left( \frac{k}{n^{\beta}} \right)^{1-\beta}}{n^{\beta}}
\]
We first derive this estimate in two lemmas.

**LEMMA 2.2.** Let \( 0 < \beta < 1 \), \( 0 < \delta \leq x \). Then for all \( n \in \mathbb{N} \) such that \( n^{1-\beta} \geq 2x \),
\[
\left| \frac{k}{n^{\beta}} - \frac{x}{1+a_n x} \right| \geq \delta
\]
where
\[
r_{n,k}(x) = \binom{n}{k} (a_n x)^k (1+a_n x)^{-n} \quad a_n = n^{\beta-1}
\]
Proof

\[ r_{n,k}(x) = p_{n,k} \left( \frac{a_n x}{1+a_n x} \right), \]

where \( p_{n,k}(x) \) is defined as in lemma 1.2.

Hence

\[
\left| \frac{k}{n^\beta} - \frac{x}{1+a_n x} \right| \geq \delta \quad \text{and} \quad \left| k-\frac{a_n x}{1+a_n x} \right| \geq \delta n^\beta.
\]

Choose \( z \) such that

\[
2z \left[ \frac{a_n x}{1+a_n x} \left( 1- \frac{a_n x}{1+a_n x} \right) \right]^{1/2} = \delta n^\beta.
\]

That is,

\[
2z \frac{(n^\beta x)^{1/2}}{1+a_n x} = \delta n^\beta.
\]

Hence

\[
z = \frac{\delta}{2} \left( \frac{n^\beta x}{x} \right)^{1/2} \left( 1+a_n x \right)
\]

Since \( 0 < \delta < x \),

\[
z \leq \frac{x}{2} \left( \frac{n^\beta}{x} \right)^{1/2} \left( 1+a_n x \right)
\]

\[
= \frac{1}{2} \left[ \frac{n a_n x}{(1+a_n x)^2} \right]^{1/2} \left( 1+a_n x \right)^2
\]

\[
\leq \frac{9}{8} \left[ n \frac{a_n x}{1+a_n x} \left( 1- \frac{a_n x}{1+a_n x} \right) \right]^{1/2}.
\]
for \( n^{1-\beta} \geq 2x \). Hence applying Lemma 1.2.,

\[
\sum_{k} P_{n,k} \left( \frac{a_n x}{1+a_n x} \right) \leq 2 \exp(-z^2)
\]

for all \( n \in \mathbb{N} \) such that \( n^{1-\beta} \geq 2x \). Hence for \( 0 < \delta \leq x, \ n^{1-\beta} \geq 2x \),

\[
\sum_{k} r_{n,k}(x) \leq 2 \exp\left( -\frac{\delta^2 n^\beta}{4x} \right)
\]

for all \( n \in \mathbb{N} \) such that \( n^{1-\beta} \geq 2x \). Hence for \( a_n < b < 1, a_n < A < 1, \ \gamma > 0, \ \delta > 0 \). Then for each fixed \( x > 0 \),

\[
\sum_{k} q_{n,k}(x) = \sigma(1) n^{-\gamma} \quad (n \to \infty)
\]

where

\[
q_{n,k}(x) = \binom{n}{k} \left( \frac{a_n x}{1+a_n x} \right)^k \left( \frac{k}{n^\beta} \right) A \left( \frac{k}{n^\beta} \right)^{1-\beta} a_n = n^{\beta-1}
\]

**Proof** Let

\[ I = \{ k : k, \ \text{an integer,} \ 0 \leq k \leq n \} \]

\[ I_0 = \{ k \in I : \left| \frac{k}{n^\beta} - \frac{x}{1+a_n x} \right| \geq \delta \} \]

We have to estimate

\[
\sum_{k \in I_0} q_{n,k}(x)
\]
We split I into three disjoint sets as follows:

\[ I_1 := \{ k \in I : \frac{k+1}{n^\beta} \leq (1+\varepsilon)^2 \} \]

\[ I_2 := \{ k \in I - I_1 : k \leq \beta_1 n \} \), where \( \beta_1 = 2^{1-\frac{1}{\beta}} (1-\beta) \]

\[ I_3 := I - (I_1 \cup I_2) \]

Now let \( r_{n,k}(x) \) be defined as in Lemma 2.2. That is

\[ r_{n,k}(x) = \binom{n}{k} (a_n x)^k (1+a_n x)^{-n} \]

Then

\[ \frac{r_{n,k+1}(x)}{r_{n,k}(x)} = \frac{\frac{n-k}{k+1}}{a_n x} \leq \frac{n^\beta x}{k+1} \]

Hence

\[ q_{n,k+1}(x) / q_{n,k}(x) \leq \frac{n^\beta x}{k+1} \left( \frac{k+1}{n^\beta} \right)^A (\frac{k+1}{1-\beta})^{1-\beta} \left( \frac{k}{r^\beta} \right)^{-A} \left( \frac{k+1}{1-\beta} \right)^{1-\beta} \]

\[ = \frac{n^\beta x}{k+1} \left( \frac{k+1}{k} \right)^A (\frac{k}{n^\beta})^{1-\beta} \left( \frac{k+1}{n^\beta} \right)^A (\frac{k}{r^\beta})^{1-\beta} - (\frac{k}{r^\beta})^{1-\beta} \]

(2.3)

\[ (k+1)^{1-\beta} - k^{1-\beta} = (k+1)^{1-\beta} \left[ 1 - \left( \frac{k}{k+1} \right)^{1-\beta} \right] \]
\[
= (k+1)^{1/\beta} \left[ 1 - \left(1 - \frac{1}{k+1}\right)^{1/\beta} \right]
\]

\[
\leq (k+1)^{1/\beta} \frac{1}{1-\beta} \frac{1}{k+1}.
\]

using the following inequality: For \(a > 1\),

\[
(1-x)^a \geq 1-ax \quad (0 \leq x \leq 1).
\]

Hence

\[
(k+1)^{1/\beta} \frac{1}{1-\beta} \leq \frac{1}{1-\beta} (k+1)^{1/\beta}
\]

It follows that

\[
(k+1)^{1/\beta} - (\frac{k}{n^\beta})^{1/\beta} \leq \frac{1}{1-\beta} (\frac{k+1}{n})^{1/\beta}
\]

\[
\leq \frac{1}{1-\beta} (\frac{2k}{n})^{1/\beta}, \text{ for } k \neq 0.
\]

\[
\leq \frac{1}{2}, \text{ for } k \in I_2,
\]

since for such \(k\),

\[
2k \leq 2 \beta \leq \left(\frac{1-\beta}{2}\right) \frac{1-\beta}{\beta}
\]

Hence for \(k \in I_2\),

\[
(k+1)^{1/\beta} A(\frac{k+1}{n^\beta})^{1/\beta} - (\frac{k}{n^\beta})^{1/\beta} \leq (\frac{k+1}{n^\beta}) A/2
\]

\[
\leq (\frac{k+1}{n})^{1/2}, \quad (2.4)
\]
since $0 \leq A \leq 1$. Also for $k$ in $I_2$,

$$\left(\frac{k}{n}\right)^{1-\beta} \leq \frac{1}{k} \left(\frac{k}{n}\right)^{1-\beta} \leq k \left(\frac{\beta}{1-\beta}\right)^{1-\beta} \leq k,$$

since $\frac{\beta}{1-\beta} = 2^{-\frac{1}{\beta}} \left(1-\beta\right)^{1-\beta} < 1$. Hence for such $k$,

$$\left(\frac{k+1}{k}\right)^{A\left(\frac{k}{n}\right)^{1-\beta}} \leq (1 + \frac{1}{k})^{Ak} \leq e^A \leq e,$$  (2.5)

since $0 \leq A < 1$. Applying estimates (2.4) and (2.5) in inequality (2.3), it follows that for $k$ in $I_2$,

$$\frac{q_{n,k+1}(x)}{q_{n,k}(x)} \leq \frac{n^\beta}{k+1} \left(\frac{k+1}{n}\right)^{\frac{1}{2}} e$$

$$= \left(\frac{n^\beta}{k+1}\right)^{\frac{1}{2}} ex \leq 1,$$

since $k$ in $I_2$ implies $k$ not in $I_1$, so that

$$\frac{k+1}{n^\beta} \geq (1+ex)^2 \geq e^2x^2.$$  

Hence for all $k$ in $I_2$,

$$q_{n,k+1}(x) \leq q_{n,k}(x).$$

Let $k_0 = \min_{k \in I_2} k$. Then for $k$ in $I_2$,
q_{n,k}(x) \leq q_{n,k_0}(x)

= r_{n,k_0}(x) \frac{k_0}{n^\beta} A \left( \frac{k_0}{n^\beta} \right) \frac{1}{1-\beta}

Since \( k_0 = \min_{k \in I_2} k \), \( k_0 - 1 \) belongs to \( I_1 \).

Hence, using the definition of \( I_1 \),

\[ \frac{k_0}{n^\beta} \leq (1+ex)^2, \]

so that

\[ \frac{k_0}{n^\beta} A \left( \frac{k_0}{n^\beta} \right) \frac{1}{1-\beta} \leq (1+ex)^2 A(1+ex)^{1-\beta} = a(x), \text{ say.} \]

Hence for all \( k \) in \( I_2 \),

\[ q_{n,k}(x) \leq a(x) r_{n,k_0}(x) \]

It follows that

\[ \sum_{k \in I_2} q_{n,k}(x) \leq n a(x) r_{n,k_0}(x) \leq n a(x) \sum_{k \in I_2} r_{n,k}(x). \]

Also for \( k \) in \( I_2 \),

\[ \frac{k+1}{n^\beta} \geq (1+ex)^2 \geq 1 + ex. \]
Hence
\[
\frac{k}{n^\beta} - \frac{x}{1+a_n x} \geq \frac{k+1}{n^\beta} - x - 1 \geq (e-1)x \geq x.
\]

It follows that
\[
\sum_{k \in I_2} q_{n,k}(x) \leq n \alpha(x) \sum_{r \in \mathbb{I}} r_{n,k}(x)
\]
\[
|\frac{k}{n^\beta} - \frac{x}{1+a_n x}| \geq x
\]

Hence applying Lemma 2.2., for \( n^{1-\beta} \geq 2x \),
\[
\sum_{k \in I_2} q_{n,k}(x) \leq 2n \alpha(x) \exp \left( - \frac{n^\beta x}{4} \right)
\]
\[
= o(1) n^{-\gamma} \quad (n \to \infty)
\]  \tag{2.6}

Now if \( k \) belongs to \( I_1 \), then \( k < k_0 \). Hence for such \( k \),
\[
q_{n,k}(x) \leq r_{n,k}(x) \left( \frac{k_0}{n^\beta} \right) A \left( \frac{k_0}{n^\beta} \right) \frac{1}{1-\beta}
\]
\[
\leq \alpha(x) r_{n,k}(x)
\]

Hence
\[
\sum_{k \in I_1 \cap I_0} q_{n,k}(x) \leq \alpha(x) \sum_{r \in I_0} r_{n,k}(x)
\]

Now \( k \) in \( I_0 \),
\[
|\frac{k}{n^\beta} - \frac{x}{1+a_n x}| \geq \delta \geq \delta_0
\]
where $\delta_0 = \min(\delta, x)$. Then for $n^{1-\beta} \geq 2x$,

$$\sum_{k \in I_n \cap J_0} q_{n,k}(x) \leq \alpha(x) \sum_{r_{n,k}(x)} |\frac{k}{n^\beta} - \frac{x}{1+\alpha \cdot x}| \geq \delta_0$$

$$\leq 2 \alpha(x) \exp(-\frac{\delta_0^2 n^\beta}{4x})$$

$$= o(1) n^{-\gamma} \quad (n \to \infty) \quad (2.7)$$

Also for $k \leq n$,

$$\left(\frac{k}{n^\beta}\right) A \left(\frac{k}{\beta n^\beta}\right)^{1-\beta} \leq n(1-\beta)Ak \quad \quad (2.8)$$

since

$$\left(\frac{k}{n^\beta}\right)^{1-\beta} = k \left(\frac{k}{n}\right)^{1-\beta} \leq k.$$

Also

$$r_{n,k}(x) \leq \binom{n}{k} (a_n x)^k$$

$$\leq a_n^k (1+x)^n$$

$$= (1+x)^n n^{-(1-\beta)k} \quad \quad (2.9)$$

From estimates (2.8) and (2.9), for $k \leq n$

$$q_{n,k}(x) \leq \frac{(1+x)^n}{n(1-\beta)(1-A)k} \leq \left(\frac{1+x}{n(1-\beta)(1-A)\beta_1}\right)^n, \quad \text{for} \quad k \in I_3.$$
Hence

\[ \sum_{k \in I_3} q_{n,k}(x) \leq n \left( \frac{1+x}{n} \right)^{\frac{1+x}{(1-\beta)(1-A)\beta_1}} \]

\[ = o(1) \quad (n \to \infty) \quad (2.10) \]

Now

\[ I_0 = I \cap I_0 = (I_1 \cup I_2 \cup I_3) \cap I_0 \subset (I_1 \cap I_0) \cup I_2 \cup I_3. \]

Hence from estimates (2.6), (2.7) and (2.10), it follows that

\[ \sum_{k \in I_0} q_{n,k}(x) = o(1) \quad (n \to \infty) \]

That is,

\[ \sum q_{n,k}(x) = o(1) \quad (n \to \infty) \]

\[ |k - \frac{x}{1+a_n x}| \geq \delta \]

Proof of Theorem 2.1. Fix \( x > 0 \). Let \( \varepsilon > 0 \) be arbitrary. Choose \( \delta > 0 \), \( 0 < \delta \leq x \), such that

\[ y > 0, \quad |y-x| < 2\delta \quad \text{implies} \quad |f(y) - f(x)| < \varepsilon. \]

Now

\[ |R_n^{(\beta)}(f(t); x) - f(x)| \]

\[ = |R_n^{(\beta)}(f(t) - f(x); x)|, \text{ since } R_n^{(\beta)}(1; x) = 1 \]
\[ R_n^{(\beta)} ( \cdot f(t) - f(x) ; x) \]

\[ = \sum_{k = 0}^{n} r_{n,k} (x) \left| f \left( \frac{k}{n^{\beta}} \right) - f(x) \right|, \]

where \( r_{n,k} (x) \) is defined as in lemma 2.2. Now

\[ | \frac{k}{n^{\beta}} - \frac{x}{1+a_n x} | < \delta \text{ implies } | \frac{k}{n^{\beta}} - x | < \delta + \frac{a_n x^2}{1+a_n x} < 2 \delta, \]

for all \( n \) sufficiently large. This implies, by the choice of \( \delta \), that for all large \( n \),

\[ | f \left( \frac{k}{n^{\beta}} \right) - f(x) | < \varepsilon, \quad \text{whenever } | \frac{k}{n^{\beta}} - \frac{x}{1+a_n x} | < \delta \]

Hence for all \( n \) sufficiently large,

\[ | R_n^{(\beta)} (f; x) - f(x) | \]

\[ < \varepsilon + \sum_{k = 0}^{n} r_{n,k} (x) \left| f \left( \frac{k}{n^{\beta}} \right) - f(x) \right| \]

\[ \geq \frac{1}{1-\beta} \]

Since \( f \) belongs to \( \{ C[0, \infty) / x^{A_x^{1-\beta}} \} \)

\[ f(t) = O(1) \ t^{\frac{1}{1-\beta}} \quad (t \to \infty) \]

Assigning the function \( t^{\frac{1}{1-\beta}} \) the limiting value 1 at \( t = 0 \), it
becomes a continuous strictly positive function on \([0, \infty)\). It follows that
\[
f(t) = O(1) \, t^{1-\beta} \quad (0 \leq t < \infty).
\]
Hence
\[
\left| \frac{k}{n^\beta} - \frac{x}{1+a_n x} \right| \geq \delta
\]
\[
= O(1) \sum \left| \frac{k}{n^\beta} - \frac{x}{1+a_n x} \right| \geq \delta
\]
\[
= o(1) \, n^{-1} \quad (n \to \infty) \quad (2.12)
\]
Using Lemmas 2.2 and 2.3. From estimates (2.11) and (2.12) it follows that
\[
|R_n^{(\beta)}(f; x) - f(x)| < 2 \varepsilon,
\]
for all \(n\) sufficiently large. Since \(\varepsilon > 0\) is arbitrary, it follows that
\[
\lim_{n \to \infty} R_n^{(\beta)}(f; x) = f(x) \quad (2.13)
\]
Here we had fixed \(x > 0\). For \(x = 0\), since \(R_n^{(\beta)}(f; 0) = f(0)\), the limit in (2.13) holds trivially. Hence for each \(x, 0 \leq x < \infty\),
\[
\lim_{n \to \infty} R_n^{(\beta)} (f; x) = f(x) \]  

We conclude this section by giving a counter example to show that Theorem 2.1 cannot be extended to the case when \( A=1 \).

THEOREM 2.4. Suppose \( 0 < \beta < 1 \). Then for all \( x > 1 \),

\[
\lim_{n \to \infty} R_n^{(\beta)} \left( \frac{1}{t^{1-\beta}} ; x \right) = \infty .
\]

Proof. Let

\[
f(x) = x^{1-\beta}
\]

Then

\[
R_n^{(\beta)} (f; x) \geq f(n^{1-\beta}) \left( \frac{a_n x}{1+a_n x} \right)^n
\]

\[
= n^{(1-\beta)n} \left( \frac{a_n x}{1+a_n x} \right)^n
\]

\[
= \left( \frac{x}{1+a_n x} \right)^n
\]

Now, if \( x > 1 \), then \( \frac{x}{1+a_n x} > 1 \) for all large \( n \). Hence if \( x > 1 \),

\[
\left( \frac{x}{1+a_n x} \right)^n \to \infty \quad \text{as} \quad n \to \infty
\]

Hence for all \( x > 1 \),

\[
\lim_{n \to \infty} R_n^{(\beta)} \left( \frac{1}{t^{1-\beta}} ; x \right) = \infty .
\]
3. Approximation of analytic functions by the operators \( L_n \) and \( R_n^{(\beta)} \)

In this section we are concerned with the behaviour of the rational functions

\[
L_n (f; z) = (1+z)^{-n} \sum_{k=0}^{n} \binom{n}{k} z^k \frac{f\left(\frac{k}{n-k+1}\right)}{n-k+1}
\]  

(3.1)

and

\[
R_n^{(\beta)} (f; z) = (1+a_n z)^{-n} \sum_{k=0}^{n} \binom{n}{k} (a_n z)^k f\left(\frac{k}{n^{\beta}}\right),
\]  

(3.2)

where \( a_n = n^{\beta-1} \), \((0 < \beta < 1)\)

for complex values of \( z \) outside the interval \( 0 \leq z < \infty \). We now assume that \( f(z) \) is defined and analytic in a certain region containing \([0, \infty)\). For the Bernstein operators similar problem was discussed by Wright [65], Kantorovitch [29] and later by S.Bernstein [8], [9], [10]. B.Wood and S. Eisenberg has discussed similar problems for the various Bernstein - type operators [24], [63], [64].

We observe that for the operators \( L_n \) defined by expression 3.1.,

\[
L_n (t; z) = (1+z)^{-n} \sum_{k=0}^{n} \binom{n}{k} z^k \frac{k}{n-k+1}
\]

\[
= (1+z)^{-n} \sum_{k=1}^{n} \binom{n}{k-1} z^k
\]

\[
= z - z \left(\frac{z}{1+z}\right)^n
\]
Hence

\[ L_n(t; z) \to z \quad (n \to \infty), \]

if and only if \( z \) lies in the half plane \( H \) where

\[ H := \{ z : \frac{Z}{1+Z} < 1 \}. \]

We, in this section, prove that if \( f(z) \) is any function analytic in the half plane \( H \) and satisfying certain suitable growth restriction, then \( L_n(f; z) \) converges to \( f(z) \) uniformly on compact subsets of \( H \). Regarding the convergence of the operators \( R_n^{(\beta)}(f; z) \), we prove that if \( f(z) \) is an entire function, that is, a function analytic in the entire complex plane and if \( f(z) \) satisfies some suitable growth restriction, then \( R_n^{(\beta)}(f; z) \) converges to \( f(z) \) uniformly on compact subsets of the complex plane.

For arriving at these results we mainly depend on the following theorem of Vitali.

**THEOREM (Vitali, §6 , p. 168)**

Let \( \{ f_n(z) \} \) be a sequence of functions each analytic in a region \( \Omega \) and let

\[ |f_n(z)| \leq M \quad (n \in \mathbb{N}, \ z \in \Omega). \]

Suppose \( f_n(z) \) tend to a limit as \( n \) tends to infinity on a set of points having a limit point inside \( \Omega \). Then \( f_n(z) \) tends uniformly to a limit in any region bounded by a contour interior to \( \Omega \), the limit being therefore an analytic function of \( z \).
Throughout this section we refer to the above theorem by the name Vitali's theorem. We, in this section, also make use of the following inequality.

**Lemma 3.1.** [13, p. 13]. Suppose 
\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \]
is analytic in \(|z| \leq R\). For \(0 < r \leq R\), let \(M_1(r; f)\) and \(M_2(r; f)\) be defined as follows:

\[ M_1(r; f) = \max_{|z|=r} |f(z)| \]
\[ M_2(r; f) = \left[ \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \right]^{\frac{1}{2}} \]

Then for \(0 < r < R\),

\[ M_2(r; f) \leq M_1(r; f) \leq \left( \frac{R+r}{R-r} \right)^{\frac{1}{2}} M_2(R; f) \]

We also require certain estimates for \(L_n((\frac{t}{1+t})^m; z)\) and \(R_n^{(\beta)}(t^m; z)\). We derive them next.

**Lemma 3.2.** Suppose \(a > 0\), \(m \in \mathbb{N}\).

(i) For all \(z\) such that \(|\frac{z}{1+z}| \leq a\) and for all \(n \in \mathbb{N}\),

\[ |L_n((\frac{t}{1+t})^m; z)| \leq L_n((\frac{t}{1+t})^m; \frac{a}{1-a}) \]

(ii) For all \(z\) such that \(|z| \leq a\) and for all \(n \in \mathbb{N}\), \(n \geq (4a)^{\frac{1}{1-\beta}}\)

\[ |R_n^{(\beta)}(t^m; z)| \leq R_n^{(\beta)}(t^m; 2a) \]
Proof. Let \( \{ B_n \} \) be the well-known Bernstein operators defined for functions \( f \) whose domain contains \([0, 1]\) by

\[
B_n(f; z) = \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} f\left(\frac{k}{n}\right).
\]

It is known that (See [24, proof of Theorem 2.1]) \( B_n(t^m; z) \) has non-negative Taylor coefficients. Hence

\[
|B_n(t^m; z)| \leq B_n(t^m; a) \quad (|z| \leq a; n \in \mathbb{N})
\]  

(3.3)

Now

\[
L_n\left((\frac{t}{1+t})^m; z\right) = (1+z)^{-n} \sum_{k=0}^{n} \binom{n}{k} z^k \left(\frac{k}{n+1}\right)^m
\]

\[
= \left(\frac{n}{n+1}\right)^m B_n\left(t^m; \frac{z}{1+z}\right)
\]

Hence using estimate (3.3), for \( \left|\frac{z}{1+z}\right| < a \) and for all \( n \in \mathbb{N} \),

\[
|L_n((\frac{t}{1+t})^m; z)| \leq \left(\frac{n}{n+1}\right)^m B_n(t^m; a)
\]

\[
= L_n((\frac{t}{1+t})^m; \frac{a}{1-a})
\]

This proves (i) of lemma 3.2.

Suppose \( |z| \leq a \) and \( n \geq (4a)^{1-\beta} \). Then

\[
\left|\frac{a_n z}{1+a_n z}\right| \leq \frac{a_n a}{1-a_n a} \leq \frac{a_n a}{1+2a_n a} \cdot \frac{1+2a_n a}{1-a_n a} \leq \frac{2a_n a}{1+2a_n a}
\]

(3.4)

since \( a_n a \leq \frac{1}{4} \).
Also

\[
R_n^{(\beta)}(t^m; z) = (1+a_n z)^{-n} \sum_{k=0}^{n} \binom{n}{k} (a_n z)^k \frac{k^m}{n^\beta}
\]

\[= n^{(1-\beta)m} B_n(t^m; \frac{2a_n z}{1+2a_n z})
\]

From expression (3.5), using (i) of lemma 3.2 and estimate (3.4), it follows that

\[|R_n^{(\beta)}(t^m; z)| \leq n^{(1-\beta)m} B_n(t^m; \frac{2a_n z}{1+2a_n z})
\]

\[= R_n^{(\beta)}(t^m; 2a)
\]

This proves (ii) of Lemma 3.2.

We are now in a position to prove the two main theorems of this section. We state them first.

**THEOREM 3.3.** Let \( f(z) \) be analytic in the half plane

\[H = \{ z : \left| \frac{z}{1+z} \right| < 1 \} = \{ z : \text{Real } z > - \frac{1}{2} \}
\]

For each \( r, \ 0 < r < 1 \), let

\[M_0(r; f) = \max_{\left| \frac{z}{1+z} \right| = r} |f(z)|
\]

Suppose

\[\limsup_{r \to 1^-} [M_0(r; f)]^{1-r} \leq 1.
\]
Then $L_n(f; z)$ converges to $f(z)$ uniformly on compact subsets of $H$.

**THEOREM 3.4.** Let $f(z)$ be an entire function. For each $r > 0$, let

$$M_1(r; f) = \max_{|z| = r} |f(z)| .$$

Suppose for some $\beta$, $0 < \beta < 1$,

$$\limsup_{r \to \infty} \frac{\log M_1(r; f)}{\frac{1}{r^{1-\beta}} \log r} < 1 .$$

Then $R_n^{(\beta)}(f; z)$ converges to $f(z)$ uniformly on compact subsets of the complex plane.

**Proof of Theorem 3.3.** Since $f(z)$ is analytic in the half plane

$$H = \{ z : \frac{z}{1+z} < 1 \} ,$$

$f\left(\frac{z}{1-z}\right)$ is analytic in the disc $|z| < 1$. $f\left(\frac{z}{1-z}\right)$ has therefore a Taylor series expansion, say

$$f\left(\frac{z}{1-z}\right) = \sum_{m=0}^{\infty} c_m z^m , \quad |z| < 1 .$$

Hence for all $z$ in $H$, $f(z)$ has an expansion

$$f(z) = \sum_{m=0}^{\infty} c_m \left(\frac{z}{1+z}\right)^m .$$

Since the series $\sum_{m=0}^{\infty} c_m z^m$ represents an analytic function in $|z| < 1$, the series $\sum_{m=0}^{\infty} |c_m| z^m$ also represents a function analytic in $|z| < 1$. 
Hence
\[ f_1(z) = \sum_{m=0}^{\infty} |c_m| \left( \frac{z}{1+z} \right)^m \]
is analytic in the half-plane H.

For \( 0 < r < 1 \), let
\[ M_0(r; f_1) = \max \{ |f_1(z)| : \left| \frac{z}{1+z} \right| = r \} \]
\[ = \max \{ |f_1\left( \frac{z}{1-z} \right)| : |z| = r \} . \]

Then applying Lemma 3.1.,
\[ M_0(r; f_1) \leq \left( \frac{1+3r}{1-r} \right)^{\frac{1}{2}} \left\{ \sum_{m=0}^{\infty} |c_m|^2 \left( \frac{1+r}{2} \right)^{2m} \right\}^{\frac{1}{2}} \]
\[ \leq \left( \frac{4}{1-r} \right)^{\frac{1}{2}} \max \{ |f\left( \frac{z}{1-z} \right)| : |z| = \frac{1+r}{2} \} \]
\[ = \left( \frac{4}{1-r} \right)^{\frac{1}{2}} \max \{ |f(z)| : \left| \frac{z}{1+z} \right| = \frac{1+r}{2} \} \]
\[ = \left( \frac{4}{1-r} \right)^{\frac{1}{2}} M_0\left( \frac{1+r}{2} ; f \right) . \]

Also
\[ \left( \frac{4}{1-r} \right)^{\frac{1}{2}} (1-r) \rightarrow 1 \quad \text{as} \quad r \rightarrow 1^- . \]

Hence
\[ \limsup_{r \rightarrow 1^-} [M_0(r; f_1)]^{1-r} \leq \limsup_{r \rightarrow 1^-} [M_0\left( \frac{1+r}{2} ; f \right)]^{1-r} \]
\[ = \limsup_{r \rightarrow 1^-} [M_0(r; f)]^{2(1-r)} \]
\[ \leq 1. \]
by assumption of Theorem 3.3. Hence given any $A > 0$,

$$f_1(x) = O(1) e^{Ax} \ (0 < x < \infty, \ x \to \infty)$$

Since $|f(x)| \leq f_1(x)$ for $x$ in $[0, \infty)$, it follows that both the functions $f/[0, \infty)$ and $f_1/[0, \infty)$ belong to $\{ C[0, \infty) / e^{Ax} \}$ for each $A > 0$. (Here $f/[0, \infty)$ stands for the function $f$ with domain restricted to $[0, \infty)$).

Hence $f/[0, \infty)$ and $f_1/[0, \infty)$ belong to $\bigcap_{A>0} \{ C[0, \infty) / e^{Ax} \}$. It follows, using Theorem 1.1, that for each $x$, $0 \leq x < \infty$,

$$\lim_{n \to \infty} L_n(f; x) = f(x) \quad (3.6)$$

and

$$\lim_{n \to \infty} L_n(f_1; x) = f_1(x) \quad (3.7)$$

Clearly $z = -1$ is the only singularity of the rational function $f(z)$. Also

$$f(\frac{k}{n-k+1}) = \sum_{m=0}^{\infty} c_m \left( \frac{k}{n+1} \right)^m \ (k = 0, \ldots, n; \ n \in \mathbb{N})$$

Hence for all $z \neq -1$,

$$L_n(f; z) = \sum_{k=0}^{n} \binom{n}{k} z^k (1+z)^{-n} f(\frac{k}{n-k+1})$$

$$= \sum_{m=0}^{\infty} c_m \sum_{k=0}^{n} \binom{n}{k} z^k (1+z)^{-n} (\frac{k}{n+1})^m$$

$$= \sum_{m=0}^{\infty} c_m L_n \left( (\frac{t}{1+t})^m ; z \right).$$
Hence if $|\frac{z}{1+z}| \leq a < 1$, by (i) of Lemma 3.2,

$$|L_n(f; z)| \leq \sum_{m=0}^{\infty} |c_m| L_n\left(\left(\frac{t}{1+t}\right)^m ; \frac{a}{1-a}\right)$$

$$= L_n(f_1 ; \frac{a}{1-a})$$

$$\rightarrow f_1(\frac{a}{1-a}) \text{ as } n \rightarrow \infty,$$

using expression (3.7). Hence $L_n(f; z)$ is uniformly bounded in

$|\frac{z}{1+z}| \leq a$ for each $a$, $0 < a < 1$. That is, for each $a$, $0 < a < 1$,

$$\sup \{ |L_n(f; z)| : |\frac{z}{1+z}| \leq a; \ n \in \mathbb{N} \} < \infty \quad (3.8)$$

From (3.6) and (3.8), it now follows, by Vitali's theorem, that

$L_n(f; z)$ converges to $f(z)$ on compact subsets of the half plane $H$. ##

**Proof of Theorem 3.4.** Since $f(z)$ is an entire function, it has a Taylor series expansion

$$f(z) = \sum_{m=0}^{\infty} c_m z^m,$$

the series converging absolutely for each $z$ in the complex plane. Since the series $\sum_{m=0}^{\infty} c_m z^m$ and the series $\sum_{m=0}^{\infty} |c_m| z^m$ have the same radii of convergence, it follows that

$$f_1(z) = \sum_{m=0}^{\infty} |c_m| z^m$$
is also an entire function. For \( r > 0 \), let

\[
M_1(r; f_1) = \max_{|z| = r} |f_1(z)|.
\]

Then by Lemma 3.1.,

\[
M_1(r; f_1) \leq (2r + 1)^{\frac{1}{2}} \left\{ \sum_{m=0}^{\infty} |c_m|^2 (r+1)^{2m} \right\}^{\frac{1}{2}} \\
\leq (2r + 1)^{\frac{1}{2}} M_1(r+1; f)
\]

Hence

\[
\log M_1(r; f_1) \leq \frac{1}{2} \log (2r + 1) + \log M_1(r+1; f)
\]

Also

\[
\frac{\log (2r + 1)}{\frac{1}{r^{1-\gamma}} \log r} \to 0 \text{ as } r \to \infty
\]

Hence

\[
\limsup_{r \to \infty} \frac{\log M_1(r; f_1)}{\frac{1}{r^{1-\gamma}} \log r} \leq \limsup_{r \to \infty} \frac{\log M_1(r+1; f)}{\frac{1}{r^{1-\gamma}} \log r} = \limsup_{r \to \infty} \frac{\log M_1(r+1; f)}{\frac{1}{(r+1)^{1-\beta}} \log (r+1)} < 1,
\]

by assumption of Theorem 3.4.
That is
\[
\lim_{r \to \infty} \sup_{r} \frac{\log M_{1}(r; f_{1})}{1 - \frac{1}{1 - \beta} \log r} \leq A < 1,
\]
for some \( A, \ 0 < A < 1 \). Let \( A < A_{1} < 1 \). Then for all \( r \) sufficiently large,
\[
\log M_{1}(r; f_{1}) \leq A_{1} r^{1 - \beta} \log r
\]
Hence
\[
M_{1}(r; f_{1}) = O(1) r^{A_{1} r^{1 - \beta}} \quad (r \to \infty)
\]
It follows that
\[
f_{1}(r) = O(1) r^{A_{1} r^{1 - \beta}} \quad (0 \leq r < \infty, \ r \to \infty)
\]
Since \( |f(r)| \leq f_{1}(r) \) on \([0, \infty)\), it follows that both the functions
\( f/(0, \infty) \) and \( f_{1}/(0, \infty) \) belong to \( \{ C[0, \infty) / x^{1/(1 - \beta)} \} \). Since \( A_{1} < 1 \),
using Theorem 2.1., it follows that for each \( x, 0 \leq x < \infty \),
\[
\lim_{n \to \infty} R_{n}^{(\beta)} (f; x) = f(x) \quad (3.9)
\]
and
\[
\lim_{n \to \infty} R_{n}^{(\beta)} (f_{1}; x) = f_{1}(x) \quad (3.10)
\]
Clearly \( z = -a_{n}^{-1} = -n^{1 - \beta} \) is the only singularity of the rational
function \( R_{n}^{(\beta)} f; z \). Also
\[
f_{n}^{(\beta)} = \sum_{m=0}^{\infty} c_{m} (\frac{k}{\beta})^{m}.
\]
Hence

\[ R_n(\beta)(f; z) = \sum_{k=0}^{\infty} \binom{n}{k} (a_n z)^k (1 + a_n z)^{-n} f\left(\frac{k}{n^{\beta}}\right) \]

\[ = \sum_{m=0}^{\infty} c_m \sum_{k=0}^{n} \binom{n}{k} (a_n z)^k (1 + a_n z)^{-n} \left(\frac{k}{n^{\beta}}\right)^m \]

\[ = \sum_{m=0}^{\infty} c_m R_n(\beta)(t^m; z). \]

Hence if \(|z| \leq a\), by (ii) of Lemma 3.2.,

\[ |R_n(\beta)(f; z)| \leq \sum_{m=0}^{\infty} |c_m| |R_n(\beta)(t^m; z)| \]

\[ \leq \sum_{m=0}^{\infty} |c_m| R_n(\beta)(t^m; 2a), \]

for all \(n \in \mathbb{N}, \ n \geq (4a)^{1-\beta}\). It follows that for all \(n \geq (4a)^{1-\beta}\) and for all \(z\) such that \(|z| \leq a\),

\[ |R_n(\beta)(f; z)| \leq R_n(\beta)(f_1; 2a) + f_1(2a) \text{ as } n \to \infty \]

using expression (3.10). Hence for any \(a > 0\),

\[ \sup \{ |R_n(\beta)(f; z)| : |z| \leq a; \ n \geq (4a)^{1-\beta} \} < \infty \quad (3.11) \]

From (3.9) and (3.11), it now follows by Vitali's theorem, that

\(R_n(\beta)(f; z)\) converges uniformly to \(f(z)\) on compact subsets of the complex plane.