CHAPTER V

SATURATION

For the well known operators $B_n$ of Bernstein, E. Voronovskaja [61] had proved that if $f$ is a function bounded in $[0,1]$ and if $f''(x)$ exists at a certain point $x$ in $[0,1]$, then

$$\lim_{n \to \infty} n(B_n(f;x) - f(x)) = \frac{x(1-x)}{2} f''(x)$$

We in sections 1 and 2 of this chapter obtain Voronovskaja type theorems for the operators $(L_n)$ and $(R_n^{(\beta)})$ respectively. For the operators $R_n^{(\beta)}$ separate Voronovskaja type theorems for the cases $0 < \beta \leq \frac{1}{2}$ and $2/3 \leq \beta < 1$ are known [58, p. 246]. We, in section 2, derive a Voronovskaja type theorem for the operators $R_n^{(\beta)}$ which takes into account all the cases $0 < \beta < 1$ simultaneously.

Using the Voronovskaja type theorems for the operators $L_n$ and $R_n^{(\beta)}$ we then in sections 3 and 4 determine the function classes $L^{1,1}_{A}$, $L^{0,1}_{A}$, $L^{1}_{[a,b]}$, $R^{\beta,1}$ where $L^\alpha_A$, $L^\alpha_{[a,b]}$ etc. are defined as in the introduction of chapter II. That is,

$$L^{0,1}_{A} = \{ f \in C[0,\infty) : ||L_n f - f||_A = O(1) n^{-1} \}$$

where $||f||_A = \sup_{0 \leq t < \infty} |f(t)|$

$$L^{1}_{[a,b]} = \{ f \in C[0,\infty) : ||L_n f - f||_{[a,b]} = O(1) n^{-1} \}$$

where $||f||_{[a,b]} = \sup_{a \leq t \leq b} |f(t)|$
and
\[ R^{\beta,1} = \{ f \in C_B[0,\infty) : ||R^\beta_n f - f|| = O(1) n^{-\beta_0} \} , \]
where
\[ ||f|| = \sup_{0 \leq t < \infty} |f(t)| ; \quad \beta_0 = \min\{\beta, 1 - \beta\} \]
We also show that in each case when \( O(1) \) is replaced by \( o(1) \) the function class we obtain is only a trivial one, so that the function classes \( L^1_A, L^{0,1}_A, R^{\beta,1} \) are indeed the various saturation classes.

We here like to point out that V. Totik had characterized the saturation classes \( L^1_0 \) and \( R_{[a,\infty)}^{1} \) \((a > 0) (\{57\}, \{58\}) \) where
\[ L^1_0 = \{ f \in C_B[0,\infty) : ||L_n f - f|| = O(1) n^{-1} \} \]
where
\[ ||f|| = \sup_{0 \leq t < \infty} |f(t)| \]
and
\[ R^{\beta,1}_{[a,\infty)} = \{ f \in C_B[0,\infty) : ||R^\beta_n f - f||_{[a,\infty)} = O(1) n^{-\beta_0} \} \]
He had observed that it would be interesting to solve the saturation problem around the origin for the \( R^\beta_n \). This is indeed what we do by characterizing \( R^{\beta,1} \).
1. Voronovskaja type Theorems for the Operators \( L_n \)

In [57], V. Totik had mentioned the following Voronovskaja type theorem for the operators \( L_n \) [57, p.91].

**THEOREM** Let \( f \in C_\beta(0,\infty) \). Suppose \( f \) has a second derivative at \( x_0 \). Then

\[
\lim_{n \to \infty} n (L_n(f;x_0) - f(x_0)) = \frac{1}{2} f''(x_0) x_0 (1+x_0)^2
\]

We in this section extend the above theorem to functions in

\[
\mathcal{F} = \bigcap_{A > 0} \{ C(0,\infty) / e^{A x} \}
\]

**THEOREM 1.1.** Let \( f \) belongs to \( \mathcal{F} \) where

\[
\mathcal{F} = \bigcap_{A > 0} \{ C(0,\infty) / e^{A x} \}.
\]

Suppose \( f \) has a second derivative at \( x_0 \). Then

\[
\lim_{n \to \infty} n (L_n(f;x_0) - f(x_0)) = \frac{1}{2} f''(x_0) x_0 (1+x_0)^2
\]

**Proof.** Since \( f \) has a second derivative at \( x_0 \), by the Taylor formula for remainder

\[
f(x) = f(x_0) + f'(x) (x-x_0) + \frac{1}{2} f''(x_0) (x-x_0)^2 + \varepsilon(x) (x-x_0)^2,
\]

where \( \varepsilon(x) \) is a continuous function on \([0,\infty)\) and

\[
\lim_{x \to x_0} \varepsilon(x) = 0.
\]
Hence given \( \varepsilon > 0 \), there exists an open interval \((x_0 - \delta, x_0 + \delta)\) containing \( x_0 \) such that \( |\varepsilon(t)| < \varepsilon_0 \) whenever \( t \in (x_0 - \delta, x_0 + \delta) \).

Let

\[
g(t) = \max \left\{ 0, (|\varepsilon(t)| - \varepsilon_0)(t-x_0)^2 \right\}
\]

since \( f \) belongs to \( \mathcal{F} \), by equation (1.1), \( \varepsilon(t)(t-x_0)^2 \) also belongs to \( \mathcal{F} \). Hence \( g \in \mathcal{F} \). Also \( g \) vanishes on \((x_0 - \delta, x_0 + \delta)\). Hence by Lemma II.4.4.

\[
L_n(g; x_0) = o(1) \quad n^{-1} \quad (n \to \infty)
\]

Now

\[
|\varepsilon(t)| (t-x_0)^2 = \varepsilon_0(t-x_0)^2 + (|\varepsilon(t)| - \varepsilon_0)(t-x_0)^2
\]

\[
\leq \varepsilon_0 (t-x_0)^2 + g(t)
\]

Hence

\[
|L_n(\varepsilon(t)(t-x_0)^2; x_0)|
\]

\[
\leq \varepsilon_0 L_n((t-x_0)^2; x_0) + L_n(g; x_0)
\]

\[
= \varepsilon_0 L_n(t-x_0)^2; x_0) + o(1) \quad n^{-1} \quad (n \to \infty)
\]

\[
\leq O(1) \quad \varepsilon_0 \frac{x_0(1+x_0)^2}{n} + o(1) \quad n^{-1} \quad (n \to \infty)
\]

using (v) of lemma II.1.1. Since \( \varepsilon_0 > 0 \) is arbitrary it follows that

\[
\lim_{n \to \infty} n L_n(\varepsilon(t)(t-x_0)^2; x_0) = 0 \quad (1.2)
\]
From expressions (1.1) and (1.2),

\[ L_n(f(t); x_o) = f(x_o) L_n(1; x_0) + f'(x_o) L_n(t-x_0; x_0) \]

\[ + \frac{1}{2} f''(x_o) L_n((t-x_0)^2; x_0) + o(1)n^{-1} \quad (n \to \infty) \]

Also from (i) and (ii) of Lemma II.1.1,

\[ L_n(1; x) = 1; \quad L_n(t-x; x) = -x \left( \frac{x}{1+x} \right)^n = o(1)n^{-1} \quad (n \to \infty) \]

Further it is known that (see [11])

\[ L_n((t-x_0)^2; x_0) = \frac{x_0(1+x_0)^2}{n} + o(1)n^{-1} \quad (n \to \infty) \]

It follows that

\[ L_n(f; x_0) = f(x_0) + \frac{x_0(1+x_0)^2}{n} f''(x_0) + o(1)n^{-1} \quad (n \to \infty) \]

Consequently

\[ \lim_{n \to \infty} n \left( L_n(f; x_0) - f(x_0) \right) = \frac{1}{2} \frac{x_0(1+x_0)^2}{n} f''(x_0) \]

2. Voronovskaja type Theorem for the Operators \( R_n^{(\beta)} \)

Let \( a = \{ a_n \} \) be a sequence of positive numbers decreasing to zero as \( n \) increases to infinity. For a function \( f \) defined on \([0, \infty)\)

Let

\[ R_n(a; f; x) := (1+a_n x)^{-n} \sum_{k=0}^{n} \binom{n}{k} (a_n x)^k f \left( \frac{k}{n a_n} \right) \]
For the operators $R_n(a)$, Katalin Balasz [2] has obtained the following asymptotic formula.

**THEOREM A.** Let $f$ be a function defined on $[0, \infty)$ such that
\[ f(t) = O(1) e^{at} (t \to \infty) \] where $a$ is a fixed real number. Suppose $n^{\frac{1}{2}} a_n \to \infty$ as $n \to \infty$. Then at each point $t = x$ at which $f''(t)$ exists and is finite,

\[ R_n(a; f; x) = f(t) + a_n f'(x) g_1(x) + a_n f''(x) g_2(x) + a_n \rho_n, \]

where $\rho_n \to 0$ as $n \to \infty$ and

\[ g_1(x) = -\frac{x^2}{1 + a_n x}; \quad g_2(x) = \frac{n a_n^2 x^4 + a_n^{-1} x}{2n a_n (1 + a_n x)^2} \]

Now consider the particular case when $a_n = \frac{\beta}{n^2} - 1$ ($0 < \beta < 1$). Then the operator $R_n(a)$ coincides with the operator $R_n^{(\beta)}$ and the condition $n^{\frac{1}{2}} a_n \to \infty$ is then equivalent to the condition $\beta > \frac{1}{2}$. Consequently from Theorem A, we arrive at the following corollary.

**COROLLARY B.** Let $f$ be a function defined as $[0, \infty)$ for which
\[ f(t) = O(1) e^{\alpha t} (t \to \infty) \] where $\alpha$ is a fixed real number. Suppose $\frac{1}{2} < \beta < 1$. Then at each point $t = x$ at which $f''(t)$ exists and is finite,

\[ R_n^{(\beta)} (f; x) = f(x) - \frac{x^2 f'(x)}{n^{1-\beta}} + O(1) \frac{1}{n^{1-\beta}} \quad (n \to \infty) \]

##
Also for the operators $R_n^{(\beta)}$, the following Voronovskaja type theorems are known (see [58, Lemma 2]).

**THEOREM C.** Let $f$ be a bounded function on $[0, \infty)$.

(i) If $\frac{2}{3} \leq \beta < 1$ and $f$ is differentiable at the point $x_0$, then

$$
\lim_{n \to \infty} n^{1-\beta} (R_n^{(\beta)} (f; x_0) - f(x_0)) = -x_0^2 f'(x_0)
$$

(ii) If the second derivative of $f$ exists at the point $x_0$ then we have

$$
\lim_{n \to \infty} n^{\beta} (R_n^{(\beta)} (f; x_0) - f(x_0)) = \begin{cases} 
-x_0^2 f'(x_0) + \frac{x_0}{2} f''(x_0) & \text{for } \beta = \frac{1}{2} \\
\frac{x_0}{2} f''(x_0) & \text{for } 0 < \beta < \frac{1}{2}
\end{cases} ##
$$

We in this section prove the following Vorovskaja type theorem for the operators $R_n^{(\beta)}$.

**THEOREM 2.1.** Let $f$ be a function defined on $[0, \infty)$ such that

$$
f(t) = O(1) \ t^A t^{1/(1-\beta)} \quad (t \geq 0) \text{ for some } A, \beta \text{ where } 0 < A, \beta < 1.
$$

Let $x$ be a point in $(0, \infty)$ at which $f'(x)$ exists. Assume further that the following limit exists

$$
\lim_{t \to x} \frac{f(t) - f(x) - (t-x)f'(x)}{|t-x|^{1/\beta_1}} = r_\beta(x) < \infty
$$

where

$$
\beta_1 = \max \{ 1, \frac{2\beta}{\beta} \} , \quad \beta_0 = \min \{ \beta, 1-\beta \}.
$$
Then

\[ R_n^{(\beta)} (f;x) - f(x) = - \frac{x^2}{n^{1-\beta}} f'(x) + A_\beta \frac{r_\beta(x)}{\beta \gamma_0} x^{\frac{\beta \gamma_0}{\beta}} + o(1) n^{-\beta \gamma_0} \quad (n \to \infty) \]

where

\[ A_\beta = \pi^{-\frac{\beta}{2}} 2^{\beta/\gamma} \frac{\beta!}{\beta^{\beta/\gamma}} \Gamma \left( \frac{\beta \gamma_0}{\beta} + \frac{\beta}{2} \right) \]

REMARKS. Theorem 2.1. has the following advantages over the other Voronovskaja type theorems proved for the operators \( R_n^{(\beta)} \)

1. The growth restriction \( f(t) = O(1) t^{1/(1-\beta)} \) (t > 0)
where \( 0 < A, \beta < 1 \) cannot be replaced by a weaker growth restriction (see Theorem 1.2.4)

2. The condition

\[ \lim_{t \to x} \frac{f(t)-f(x)-(t-x)f'(x)}{|t-x|^{\gamma_1}} = r_\beta(x) < \infty \]

cannot be replaced by a weaker condition.

Also from Theorem 2.1. we make the following observations.

(i) For \( \beta \geq \frac{2}{3} \), from the assumption that \( f'(x) \) exists it follows that \( r_\beta(x) \) exists and is equal to zero.

(ii) For \( 1/2 < \beta < 2/3 \), the condition that \( r_\beta(x) \) exists is equivalent to the condition that
\[
\lim_{t \to x} \frac{f(t) - f(x) - f'(x)(t-x)}{|t-x|^{2(1-\beta)/\beta}} \quad \text{exists}
\]

and in this case \( r_{\beta}(x) \) is equal to the above limit.

(ii) For \( 0 < \beta \leq \frac{1}{3} \), the condition that \( r_{\beta}(x) \) exists is equivalent to the condition that \( f''(x) \) exists and in this case \( r_{\beta}(x) = \frac{1}{2} f''(x) \)

Also for \( 0 < \beta \leq \frac{1}{3} \),

\[
A_{\beta} = \pi^{-\frac{1}{2}} 2 \Gamma \left( \frac{3}{2} \right) = \pi^{-\frac{1}{2}} \Gamma \left( \frac{1}{2} \right) = 1
\]

using the well-known properties of the \( \Gamma \) operator.

**Proof of Theorem 2.1.** Since

\[
r_{\beta}(x) = \lim_{t \to x} \frac{f(t) - f(x) - (t-x) f'(x)}{|t-x|^{1+\beta}}
\]

\( f \) can be expressed as follows. For \( t \geq 0 \),

\[
f(t) = f(x) + (t-x) f'(x) + r_{\beta}(x) |t-x|^{1+\beta} + \varepsilon(t;x) |t-x|^{\beta} \quad (2.1)
\]

where

\[
\varepsilon(t;x) \to 0 \quad (t \to x)
\]

Consequently

\[
R_{n}^{(\beta)}(f;x) = f(x) + f'(x) \cdot R_{n}^{(\beta)}(t-x;x) + r_{\beta}(x) R_{n}^{(\beta)}(|t-x|^{1+\beta}; x) + R_{n}^{(\beta)}(\varepsilon(t;x) |t-x|^{1}; x) \quad (2.2)
\]
Now

\[ R_n(\beta ; t-x) = -\frac{a_n x}{1+a_n x} \]
\[ = -\frac{x}{n^{1-\beta}} + o(1) n^{-(1-\beta)} \]
\[ = \frac{x}{n^{1-\beta}} + o(1) n^{-\beta_0} \]  

(2.3)

For \( \beta > \frac{2}{3} \), \( \beta_1 = \max \{ 1, \frac{2\beta_0}{\beta} \} = 1 \)

Also since \( f'(x) \) exists,

\[ \frac{f(t)-f(x)-(t-x)f'(x)}{t-x} \rightarrow 0 \text{ as } t \rightarrow x \]

Hence for \( \beta \geq 2/3 \),

\[ r_\beta (x) = \lim_{t \rightarrow x} \frac{t(t)-f(x)-(t-x)f'(x)}{|t-x|} = 0 \]  

(2.4)

For \( 0 < \beta < \frac{2}{3} \), \( \beta_1 = \frac{2\beta_0}{\beta} \)

Now

\[ (t-x)^2 - (t-x_n)^2 = (x-x_n)^2 - 2(t-x_n)(x-x_n) \]

Hence

\[ |(t-x)^2 - (t-x_n)^2| \leq (x-x_n)^2 + 2|t-x_n||x-x_n|, \]
so that
\[ || t-x | - | t-x_n | | \leq | (t-x)^2 - (t-x_n)^2 | \]

\[ \leq (x-x_n)^{\frac{\beta}{\beta_0}} + 2(x-x_n)^{\frac{\beta}{\beta_0}} | t-x_n |^{\frac{\beta}{\beta_0}} \]

Hence
\[ | R_n^{(\beta)} (|t-x|^\beta ; x) - R_n^{(\beta)} (|t-x_n|^\beta ; x) | \leq \frac{2\beta_0}{\beta} R_n^{(\beta)} (1; x) + 2(x-x_n)^{\frac{\beta}{\beta_0}} R_n^{(\beta)} (|t-x_n|^\beta ; x) \]

Also by (i) of Lemma II.3.4,
\[ R_n^{(\beta)} (1; x) = 1 \]

Further, by Holder's inequality,
\[ R_n^{(\beta)} (|t-x_n|^\beta ; x) \leq \frac{\beta}{\beta_0} R_n^{(\beta)} ((t-x_n)^2 ; x) \]

\[ \leq \frac{\beta_0}{2\beta} \], by (iii) of Lemma II.3.4.
It follows that

\[ |R_n^{(\beta)} (|t-x|^{-\beta}; x) - R_n^{(\beta)} (|t-x|^{-\beta}; x)| \]

\[
\leq \left( x-x_n \right)^\beta + 2 \left( x-x_n \right)^\beta \left( \frac{x_n}{n^\beta} \right)^{2\beta} \\
= \left( \frac{a_n x^2}{1 + a_n x} \right)^{2\beta} + 2 \left( \frac{a_n x^2}{1 + a_n x} \right)^{\beta} \left( \frac{x}{n^\beta} \right)^{2\beta} \\
since x_n = \frac{x}{1 + a_n x} \\
\leq \left( a_n x^2 \right)^{2\beta} + 2 \left( a_n x^2 \right)^{\beta} \left( \frac{x}{n^\beta} \right)^{2\beta} \\
= o(1) n^{-\beta} for 0 < \beta < \frac{2}{3},
\]

since for 0 < \beta < \frac{2}{3}, \quad \frac{1-\beta}{\beta} > \frac{1}{2} so that

\[
-a_n^{-\beta} = (n^{\beta-1})^{-\beta} = o(1) n^{-\beta} \\
\text{and}
\]

\[
a_n^{-\frac{\beta}{2}} = n^{(\beta-1)} \frac{\beta}{2} = o(1) n^{-\beta} \\
\]

From estimate (2.5) it now follows that for 0 < \beta < 2/3

\[
R_n^{(\beta)} (|t-x|^{-\beta}; x) = R_n^{(\beta)} (|t-x_n|^{-\beta}; x) + o(1) n^{-\beta} (n^{+\infty})
\]
Now, for $0 < \beta < \frac{2}{3}$, $\beta_1 = \max \{1, \frac{2\beta}{\beta}\}$ is equal to $\frac{2\beta}{\beta}$. Hence for $0 < \beta < \frac{2}{3}$,

$$R(n) \left(|t-x|^{\beta_1}; x\right) = R(n) \left(|t-x|^{\beta}; x\right)$$

$$= R(n) \left(|t-x|^{\beta}; x\right) + o(1) n^{-\beta} (n \to \infty),$$

using estimate (2.6)

$$= \pi^{-\frac{1}{2}} (2\pi)^{-\beta} \frac{\beta_0}{2\beta} \Gamma \left(\frac{\beta_0}{\beta} + \frac{1}{2}\right) n^{-\beta} + o(1) n^{-\beta} (n \to \infty)$$

using Theorem IV.1.1.

Hence for $0 < \beta < 2/3$

$$R(n) \left(|t-x|^{\beta_1}; x\right) = A_\beta x^{\frac{\beta_0}{\beta}} n^{-\beta_0} + o(1) n^{-\beta_0} (n \to \infty) \quad (2.7)$$

where

$$A_\beta = \pi^{-\frac{1}{2}} \frac{\beta_0}{2\beta^2} \Gamma \left(\frac{\beta_0}{\beta} + \frac{1}{2}\right) \quad (2.8)$$

From estimate (2.2), (2.3), (2.4) and (2.7), it follows that for $0 < \beta < 1$,

$$R(n) (f(x) = f(x) - \frac{2}{n^{1-\beta}} f'(x) + r_\beta (x) A \frac{\beta_0}{\beta} n^{-\beta}$$

$$+ R(n) \left(e(t;x) |t-x|^{\beta}; x\right) + o(1) n^{-\beta} (n \to \infty), \quad (2.9)$$
where \( \varepsilon(t;x) \to 0 \) as \( t \to x \)

Now, given \( \varepsilon_0 > 0 \); there corresponds a \( \delta > 0 \) such that

\[
\varepsilon(t,x) < \varepsilon_0 \text{ whenever } |t-x| < \delta
\]

Also since \( f(t) = O(1) t^{At^{1/(1-\beta)}} \) \( (t \geq 0) \), from equation (2.1) it follows that

\[
\varepsilon(t;x) |t-x|^\beta_1 = O(1) t^{At^{1/(1-\beta)}} \quad (t \geq 0)
\]

Consequently

\[
R_n^{(\beta)} (\varepsilon(t;x) |t-x|^\beta_1; x)
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} (a_n x)^k (1+a_n x)^{n-k} \varepsilon_0 \left| \frac{k}{n^\beta} - x \right|^\beta_1 \]

\[
< \varepsilon_0 \sum_{k=0}^{n} r_{n,k}(x) \left| \frac{k}{n^\beta} - x \right|^\beta_1 + O(1) \sum_{k=0}^{n} q_{n,k}(x) \left| \frac{k}{n^\beta} - x \right| \geq \delta
\]

where

\[
r_{n,k}(x) = \binom{n}{k} \frac{(a_n x)^k}{(1+a_n x)^n} \quad q_{n,k}(x) = r_{n,k}(x) \left( \frac{k}{n^\beta} \right)
\]

\[
R_n^{(\beta)} (\varepsilon(t;x) |t-x|^\beta_1; x)
\]
Also

\[
\begin{align*}
\sum_{n} r_n, k(x) \left| \frac{k}{n^\beta} - x \right|^{\beta_1} < \delta \\
\leq R_n^{(\beta)} \left( 1 - |t - x|^{\beta_1}; x \right) \\
\leq \left\{ R_n^{(\beta)} \left( (t-x)^2; x \right) \right\}^{\frac{1}{2}\beta_1} \\
= \left\{ R_n^{(\beta)} \left( (t-x_n)^2; x \right) + (x_n - x)^2 R_n^{(\beta)}(1; x) + 2(x_n - x) R_n^{(\beta)}(t-x_n; x) \right\}^{\frac{1}{2}\beta_1} \\
\leq \left\{ \frac{x}{n^\beta} + \frac{a_n^2 x^4}{(1+a_n x)^2} \right\}^{\frac{1}{2}\beta_1}, \text{ using Lemma II.3.4.} \\
\leq x^{\frac{1}{2}\beta_1} n^{-\frac{1}{2}\beta_1} + n^{-\beta_1(1-\beta)} x^{2\beta_1}
\end{align*}
\]

Also since \( \beta_1 = \max \{ 1, \frac{2\beta_0}{\beta} \} \),

\[\frac{1}{2}\beta_1 \geq \beta_0 : \beta_1(1-\beta) = \begin{cases} 1-\beta = \beta_0, & \frac{2}{3} \leq \beta < 1 \\ 2\beta_0 \frac{(1-\beta)}{\beta} \geq \beta_0, & 0 < \beta < \frac{2}{3} \end{cases} \]
Consequently

\[ \Sigma \frac{r_{n,k}(x)}{|x|^{\beta}} \leq \frac{1}{n^{\beta}} \]

Also using Lemma I.2.

\[ \Sigma q_{n,k}(x) = o(1) \quad n \rightarrow \infty \] (2.12)

From estimates (2.10), (2.11) and (2.12), using the fact that \( \varepsilon_0 > 0 \) is arbitrary, it follows that

\[ R_n^{(\beta)}(\varepsilon(t;x) |t-x|^{\beta_1}; x) = o(1) \quad n \rightarrow \infty \] (2.13)

From estimates (2.9) and (2.13), it follows that for \( 0 < \beta < 1 \),

\[ R_n^{(\beta)} f(x) = f(x) - \frac{x^2 f'(x)}{n^{1-\beta}} + A_\beta R_\beta'(x) \quad n \rightarrow \infty \]

where \( A_\beta \) is given by expression (2.8).
In this section, we first derive a local saturation theorem for the operators \( L_n \). For arriving at this theorem we make use of the parabola technique of De Vore stated below.

**Lemma 3.1** (see [19, p. 124]). Let \( g \in C[a,b] \) with \( g(a) = g(b) = 0 \) and \( g(x_0) > 0 \) for some \( x_0 \in (a,b) \). Then there is a parabola \( Q(x) = ax^2 + bx + c \) with \( a < 0 \), such that

\[
Q(x) \geq g(x) \quad \text{on} \quad [a,b]
\]

and

\[
Q(y) = g(y) \quad \text{for some} \quad y \in (a,b)
\]

The following lemma is an easy consequence of Lemma 3.1.

**Lemma 3.2.** Let \( g \in C[a,b] \). Suppose for some \( \xi \in (a,b) \), \( (\xi, g(\xi)) \), lies above the line connecting \( (a, g(a)) \) and \( (b, g(b)) \). Then there exists a point \( x_0 \in (a,b) \) and a parabola

\[
Q(x) = a(x-x_0)^2 + b(x-x_0) + g(x_0)
\]

such that

\[
Q(x) \geq g(x) \quad \text{on} \quad [a,b]
\]

**Proof.** Let \( h(x) \) denote the unique linear function assuming the values \( h(a) \) and \( h(b) \) at \( a \) and \( b \) respectively. Let \( g_1(x) = g(x) - h(x) \).

Then \( g_1 \) belongs to \( C[a,b] \) and \( g_1(a) = g_1(b) = 0 \). Also

\[
g_1(\xi) = g(\xi) - h(\xi) > 0
\]
Hence by Lemma 3.1., there exists parabola $Q_1(x) = a_1x^2 + b_1x + c_1$ with $a_1 < 0$ such that

$$Q_1(x) \geq g_1(x) \text{ on } [a,b]$$

and

$$Q_1(x_0) = g_1(x_0), \text{ for some } x_0 \in (a,b).$$

Let $Q(x) = Q_1(x) + h(x)$. Then $Q(x)$ has all the stated properties. ##

**LEMMA 3.3.** Let $g$ belong to $\mathcal{F}$ where

$$\mathcal{F} = \bigcap_{A > 0} \{ C[0, 1)/e^{Ax} \}$$

Suppose for all $x$ in some interval $[a, b]$ contained in $(0,\infty)$

$$\limsup_{n \to \infty} n (L_n (g;x) - g(x)) \geq 0$$

Then $g$ is convex on $[a, b]$

**Proof.** If $g$ is not convex on $[a, b]$, then for some $a \leq x_0 < \xi < y_0 \leq b$, the point $(\xi, g(\xi))$ lies above the line connecting $(a, g(a))$ and $(b, g(b))$. Then by Lemma 3.2., there exists a point $\eta$, $x_0 < \eta < y_0$ and a parabola

$$Q(x) = a(x - \eta)^2 + b(x - \eta) + c$$

such that

$$Q(x) \geq g(x) \text{ on } [x_0, y_0]$$
Now

\[ L_n(g; \eta) - g(\eta) = L_n(g(t) - g(\eta); \eta), \quad \text{since} \quad L_n(1; \eta) = 1 \]

\[ = L_n Q(t) - g(\eta); \eta) + L_n(g(t) - Q(t); \eta) \]

\[ = a L_n((t-n)^2; \eta) + b L_n(t-n; \eta) + L_n(g(t) - Q(t); \eta) \]

Now it is known that (see [11], [28])

\[ L_n(t-n; \eta) = o(1) \frac{n^{-1}}{n} \quad (n \to \infty) \]

\[ L_n(t^2-n^2; \eta) = \frac{n(1+n)}{n} + o(1) \frac{n^{-1}}{n} \quad (n \to \infty) \]

Hence

\[ L_n((t-n)^2; \eta) = L_n(t^2-n^2; \eta) + 2n L_n(t-n; \eta) \]

\[ = \frac{n(1+n)}{n} + o(1) \frac{n^{-1}}{n} \quad (n \to \infty) \]

It follows that

\[ L_n(g(\eta) - g(\eta) = \frac{a}{n} \eta(1+\eta) + L_n(g(t) - Q(t); \eta) + o(1) \frac{n^{-1}}{n} (n \to \infty) \]

Let

\[ P(x) = \max \{ g(x) - Q(x); 0 \} \]

Then since \( g(x) \leq Q(x) \) on \([x_0, y_0]\),

\[ P(x) = 0 \text{ on } [x_0, y_0]. \]
That is, $P$ vanishes on an open interval containing $n$. Also $P \in F$, since $g \in F$ and $Q$ being a polynomial of degree 2 also belongs to $F$. Hence by Lemma II.4,

$$L_n(P;\eta) = o(1) \, n^{-1} \quad (n \to \infty)$$

Hence

$$L_n(g(t) - Q(t);\eta) \leq L_n(P;\eta) = o(1) \, n^{-1} \quad (n \to \infty) \quad (3.2)$$

From estimates (3.1) and (3.2)

$$L_n(g;\eta) - g(\eta) \leq \frac{a}{n} \, \eta (1+\eta)^2 + o(1) \, n^{-1} \quad (n \to \infty)$$

Hence

$$\limsup_{n \to \infty} n (L_n(g;\eta) - g(\eta)) \leq a \, \eta (1+\eta)^2 < 0,$$

a contradiction to the assumption of the Lemma. Hence $g$ must be convex on $[a,b]$. 

THEOREM 3.4. Let $f \in F$ where

$$F = \bigcap_{A > 0} \{ C[0,\infty)/e^{Ax} \}$$

Suppose

$$L_n(f;x) - f(x) = O(1) \, n^{-1} \quad (n \to \infty)$$

uniformly on $[a,b]$ where $[a,b]$ is some subinterval of $(0,\infty)$. Then $f$ belongs to Lip$^*$ 2 on $[a,b]$. 

##
(ii) Suppose for each \( x \) in \((a,b)\)

\[
L_n(f;x) - f(x) = o(1) n^{-1} \quad (n \to \infty)
\]

Then \( f \) is linear on \([a,b]\).

**Proof.** Suppose

\[
L_n(f;x) - f(x) = O(1) n^{-1} \quad (n \to \infty)
\]

uniformly on \([a,b]\). Then there exists a constant \( M > 0 \) such that

\[
|L_n(f;x) - f(x)| \leq \frac{M}{n+1} \quad (\frac{a}{n} \leq x \leq \frac{b}{n} = 1,2,...)
\]

Consider the functions

\[
g^+(x) = \frac{1+a}{a} \frac{M}{1+x} + f(x)
\]

Then

\[
L_n(g^+; x) - g^+(x)
\]

\[
= \frac{1+a}{a} M \left\{ L_n \left( \frac{1}{1+t} ; x \right) - \frac{1}{1+x} \right\} \pm \left\{ L_n (f;x) - f(x) \right\} \quad (3.3)
\]

Also

\[
L_n\left( \frac{1}{1+t} ; x \right) = \sum_{k=0}^{n} \binom{n}{k} x^k \frac{1}{(1+x)^n} \frac{n-k+1}{n+1}
\]

\[
= \sum_{k=0}^{n} \binom{n}{k+1} \binom{n-1}{k} + \frac{1}{n+1} \binom{n}{k} x^k (1+x)^{-n}
\]
\[
\frac{n}{n+1} \cdot \frac{1}{1+x} + \frac{1}{n+1}
\]

\[
= \frac{1}{1+x} + \frac{1}{n+1} \cdot \frac{x}{1+x}
\]

Hence

\[
L_n(g^\pm(x)) - g^\pm(x) = \frac{M}{n+1} \frac{1+a}{a} \frac{x}{1+x} \pm (L_n(f;x) - f(x))
\]

\[
\geq \frac{M}{n+1} \pm (L_n(f;x) - f(x)) \quad \text{for } x \geq a
\]

\[
\geq 0 \quad \text{on } [a,b]
\]

since \(|L_n(f;x) - f(x)| \leq \frac{M}{n+1} \) on \([a,b]\)

Hence

\[
\lim_{n \to \infty} \sup_{n} (L_n(g^\pm(x)) - g^\pm(x)) \geq 0
\]

Consequently by Lemma 3.3., \(g^+(x)\) and \(g^{-1}(x)\) are convex on \([a,b]\).

Hence

\[
\Delta^2_h (g^\pm;x) \geq 0 \quad (a \leq x < x+2h \leq b)
\]

It follows that

\[
\frac{1+a}{a} M \Delta^2_h \left( \frac{1}{1+t}; x \right) \pm \Delta^2_h (f;x) \geq 0
\]

whenever \( a \leq x < x+2h \leq b \). Hence for all such \(x\) and \(h\),

\[
|\Delta^2_h (f;x)| \leq \frac{1+a}{a} M \quad \left| \Delta^2_h \left( \frac{1}{1+t}; x \right) \right|
\]

since \(\frac{1}{1+t}\) has a second derivative which is continuous on \([0,\infty]\),

\[
\Delta^2_h \left( \frac{1}{1+t}; x \right) = O(1) h^2 \quad (a \leq x < x+2h \leq b)
\]
It follows that

\[ \Delta_h^2(f; x) = O(1) \, h^2 \, (a < x < x + 2h < b) \]

Hence \( f \) belongs to \( \text{Lip}^2 \) on \([a, b]\). This proves (i).

Next suppose for each \( x \) in \((a, b)\),

\[ L_n(f; x) - f(x) = o(1) \, n^{-1} \, (n + \infty) \]

Then

\[ \limsup_{n \to \infty} n(L_n(f; x) - f(x)) = 0 \]

Then by Lemma 1.3, both \( f \) and \(-f\) are convex on \([a, b]\). This happens if and only if \( f \) is linear on \([a, b]\). This proves (ii).

From Theorem 3.4., we now deduce the converse of Corollary II.1.3.

COROLLARY 3.5. Let \( f \in \{C[0, \infty)/x^A\} \) \((A \geq 0)\)

(i) Suppose

\[ L_n(f; x) - f(x) = O(1) \, \frac{x}{n} \, (1+x)^{A+2} \, x < \infty \]

Then

\[ \Delta_h^2(f; x) \, (1+x)^A = O(1) \, h^2 \]

(ii) \( L_n(f; x) - f(x) = o(1) \, \frac{x}{n} \, (1+x)^{A+2} \, x < \infty \)

if and only if \( f \) is linear on \([0, \infty)\)
Proof Suppose

\[ L_n(f; x) - f(x) = O(1) \frac{x}{n} (1+x)^{A+2} \quad (0 \leq x < \infty) \]

Then

\[ L_n(f; x) - f(x) = O(1) \frac{1}{n} \quad (n \in \mathbb{N}) \]

uniformly on compact subintervals of \((0,\infty)\). It follows, by applying Theorem 3.4., that \( f \) belongs to \( \text{Lip}^2 \) on \([a,b]\) where \([a,b]\) is any compact subinterval of \((0,\infty)\). Consequently (see [19, pp. 5, 6]) \( f' \) is absolutely continuous on compact subintervals of \((0,\infty)\) and \( f''(x) \) exists almost everywhere on \((0,\infty)\). Now at each point in \((0,\infty)\) at which \( f''(x) \) exists, by Theorem V.1.1.

\[ \lim_{n \to \infty} L_n(f; x) - f(x) = x(1+x)^2 f''(x) \quad (3.4) \]

Also, by assumption,

\[ L_n(f; x) - f(x) = O(1) \frac{x}{n} (1+x)^{A+2} \quad (0 \leq x \leq \infty) \quad (3.5) \]

From estimates (3.4) and (3.5), it follows that there exists some constant \( M > 0 \) such that

\[ |f''(x)| \leq M(1+x)^A \]

almost everywhere on \((0,\infty)\). Since \( f' \) is absolutely continuous, for any \( x > 0, h > 0, \)

\[ \Delta_h^2 f(x) = \int_0^h \int_0^h f''(x+u+v) \, du \, dv \]
Hence

\[ |\Delta_h^2(f; x)| \leq M \int_0^h \int_0^h (1+x+u+v)^A \, du \, dv \]

\[ \leq M (1+x+2h)^A \cdot h^2 \]

Hence

\[ \frac{\Delta_h^2(f; x)}{(1+x+2h)^A} = O(1) \cdot h^2 \quad (x > 0, h > 0) \]

This proves (i)

Now if

\[ L_n(f; x) = o(1) \frac{x}{n} (1+x)^A+2 \quad (0 \leq x < \infty) \]

then by (ii) of Theorem 3.4., it follows that \( f \) is linear on \([0, \infty)\)

Conversely suppose \( f \) is linear on \([0, \infty)\), say

\[ f(x) = \alpha x + \beta \quad (0 \leq x < \infty) \]

Then

\[ L_n(f; x) - f(x) = \alpha (L_n(t; x) - x) \]

since \( L_n(1; x) = 1 \)

\[ = -\alpha x \left( \frac{x}{1+x} \right)^n \]

by (ii) of Lemma 1.1.

Since

\[ x^n \leq \frac{2^{n+2}}{n^2} \left( \frac{n+2}{n} \right) x^n \leq \frac{2}{n^2} (1+x)^{n+2} \]

\[ |L_n(f; x) - f(x)| \leq 2 |\alpha| \frac{x}{n} (1+x)^2 = o(1) \frac{x}{n} (1+x)^2 (n+\infty) \]
Hence if \( f \) is linear on \([0, \infty)\), then

\[
L_n(f;x) - f(x) \sim o(1) \frac{x}{n} (1+x)^{A+2} \quad (n \in \mathbb{N}, x \geq 0)
\]

It follows that \( f \) is linear on \([0, \infty)\) if and only if

\[
L_n(f;x) - f(x) = o(1) \frac{x}{n} (1+x)^{A+2} \quad (n \in \mathbb{N}, x \geq 0)
\]

This proves (ii).

Lastly in this section we prove the converse of Theorem II.2.8. for the case \( \gamma = 1 \).

THEOREM 3.6. Assume that \( f \) belongs to \( \{ C[0, \infty)/x^A \} \quad (A \geq 0) \)

(i) Suppose

\[
||L_n f - f||_A = O(1) \quad n^{-1} \quad (n \in \mathbb{N})
\]

where

\[
||f||_A = \sup_{t \geq 0} \frac{|f(t)|}{(1+t)^A}
\]

Then

\[
\frac{\Delta_n^2(f;x)}{(1+x+2h)^A} = O(1) \frac{h^2}{x(1+x)^2} \quad (x > 0, h > 0)
\]

and

\[
\frac{f(y)-f(x)}{(1+y)^A} = O(1) \quad x^{-1} \quad (y > x > 0)
\]
Hence if $f$ is linear on $[0,\infty)$, then

$$L_n(f;x) - f(x) = o(1) \frac{x}{n} (1+x)^{A+2} \ (n \in \mathbb{N}, \ x \geq 0)$$

It follows that $f$ is linear on $[0,\infty)$ if and only if

$$L_n(f;x) - f(x) = o(1) \frac{x}{n} (1+x)^{A+2} \ (n \in \mathbb{N}, \ x \geq 0)$$

This proves (ii)

We conclude this section by giving a set of equivalent conditions for $f$ to belong to $L_A^{0,1}$ where $L_A^{0,\gamma}$ is defined as in the introduction of Chapter II. That is

$$L_A^{0,\gamma} = \{ f \in \{ C[0,\infty)/x^A \} : ||L_n f - f||_A = O(1) \ n^{-\gamma}, \ n \in \mathbb{N} \}$$

**THEOREM 3.6.** Assume that $f$ belongs to $\{ C[0,\infty)/x^A \} \ (A \geq 0)$.

Then the following statements are equivalent.

(i) $f$ belongs to $L_A^{0,1}$

(ii) $f'$ is absolutely continuous on compact subintervals of $(0,\infty)$;

$$x(1+x)^2-A f''(x) = O(1) \text{ a.e. on } (0,\infty)$$

and

$$(1+x)^{-A} x^2 f'(x) = O(1) \quad (x > 0)$$

(iii) $\frac{\Delta^2(f;x)}{h} = O(1) \frac{h^2}{x(1+x)^2} \quad (x > 0)$

and
\[
\frac{f(y) - f(x)}{(1+y)^A} = O(1) x^{-1} \quad (y > x > 0)
\]

**Proof** Suppose assumption (i) of the theorem holds. Then

\[
\|L_n f - f\|_A = O(1) n^{-1} \quad (n \in \mathbb{N})
\]

Then if \([a, b]\) is any compact subinterval of \((0, \infty)\),

\[
L_n(f; x) - f(x) = O(1) n^{-1} \quad (n \in \mathbb{N})
\]

uniformly on \([a, b]\). Hence by Theorem 3.4 \(f\) belongs to \(\text{Lip}^* 2\) on \([a, b]\).

It follows that \(f\) belongs to \(\text{Lip}^* 2\) on compact subintervals of \((0, \infty)\). Consequently \(f'\) exists on \((0, \infty)\) and is absolutely continuous on compact subintervals of \((0, \infty)\). It follows that \(f''(x)\) exists almost everywhere on \((0, \infty)\). Now by Theorem 1.1.,

\[
\lim_{n \to \infty} n(L_n(f; x) - f(x)) = \frac{1}{2} f''(x) x(1+x)^2
\]

at each point \(x\) in \((0, \infty)\) at which \(f''(x)\) exists. Hence

Also, since by assumption (i) holds,

\[
n(L_n(f; x) - f(x)) = O(1) (1+x)^A \quad (n \in \mathbb{N}, \ x > 0)
\]

It follows that

\[
f''(x) x(1+x)^2 = O(1) (1+x)^A \quad \text{a.e. on } (0, \infty)
\]

(3.6)

Now for \(t, x > 0\)
f(t)-f(x)-f'(x) (t-x) = \int_x^t \int_x^v f''(u) \ du \ dv \\
= O(1) \left| \int_x^t \int_x^v \frac{(1+u)^A}{u(1+u)^2} \ du \ dv \right| ,

using estimate (3.6)

= O(1) \left\{ \frac{(1+x)^A}{t(1+t)^2} + \frac{(1+t)^A}{x(1+x)^2} \right\} (t-x)^2 \quad (3.7)

Now let $L_n^*$ be defined as in Chapter II, Section 2 by

$L_n^*(f;x) = \sum_{k=1}^{n} P_{n,k}(x) f\left(\frac{k}{n-k+1}\right)$

where

$P_{n,k}(x) = \binom{n}{k} x^k (1+x)^{-n}$

Then for $x > 0$, using estimate (3.7),

$L_n^*(f(t)-f(x)-f'(x) (t-x); x)$

= $O(1) \left\{ \frac{(1+x)^A}{t(1+t)^2} L_n^* \left(\frac{(t-x)^2}{t(1+t)^2}; x\right) + \frac{1}{x(1+x)} L_n^*[(t-x)^2(1+t)^A;x]\right\}$

= $O(1) \frac{(1+x)^A}{n} \quad (n \in \mathbb{N}, 0 < x \leq n), \quad (3.8)$

using (ii) of Lemma II.2.6. and (v) of lemma II.1.1.

Now, since $f'$ is absolutely continuous on compact subintervals of $(0, \infty)$ for $y > x > 0$, 
\[ f'(y) - f'(x) = \int_x^y f''(t) \, dt \]

\[ = 0(1) \int_x^y t^{-1} (1+t)^{A-2} \, dt \]

\[ = 0(1) (1+y)^A \int_x^y t^{-3} \, dt \]

\[ = 0(1) (1+y)^A (x^{-2} - y^{-2}) \]

Hence

\[ f'(y) - f'(x) = 0(1) \frac{(1+y)^A}{x^2} \quad (y > x > 0) \quad (3.9) \]

Hence for \( x > 1 \),

\[ f'(x) = f'(1) + 0(1) (1+x)^A = 0(1) (1+x)^A \]

It follows that, for \( x > 1 \),

\[ (f(x) - f(o) - xf'(x)) \, p_{n, o}(x) \]

\[ = 0(1) \left\{ (1+x)^A + x (1+x)^A \right\} \, p_{n, o}(x) \]

\[ = 0(1) \frac{(1+x)^A}{(1+x)^{n-1}} \]

\[ = 0(1) \frac{(1+x)^A}{2^n} \quad , \quad \text{since} \quad x > 1 \]

\[ = 0(1) \frac{(1+x)^A}{n} \quad (3.10) \]

From estimates (3.8) and (3.10),
\[ L_n(f(t) - f(x) - f'(x)(t-x)) = O(1) \frac{(1+x)^A}{n} \quad (n \in \mathbb{N}, 1 \leq x \leq n) \quad (3.11) \]

Since, by assumption, (i) of Theorem 3.6 holds,

\[ L_n(f;x) - f(x) = O(1) \frac{(1+x)^A}{n} \quad (n \in \mathbb{N}, x > 0) \quad (3.12) \]

From estimates (3.11) and (3.12)

\[ f'(x) L_n(t-x; x) = O(1) \frac{(1+x)^A}{n} \quad (n \in \mathbb{N}, 1 \leq x \leq n) \]

Hence, using (ii) of Lemma II.1.1.

\[ f'(x) x \left(\frac{x}{1+x}\right)^n = O(1) \frac{(1+x)^A}{(x+1)^n} \quad (n \in \mathbb{N}, 1 \leq x \leq n) \]

Take \( n = [x] + 1 \), where \([x]\) is the integral part of \( x \). Then

\[ f'(x) x \left(\frac{x}{1+x}\right)^{\lfloor x \rfloor + 1} = O(1) \frac{(1+x)^A}{(x+1)^{\lfloor x \rfloor + 1}} \quad (x \geq 1) \]

Hence

\[ f'(x) x \left(\frac{x}{1+x}\right)^x = O(1) \frac{(1+x)^A}{x} \quad (x \geq 1) \]

Hence

\[ x^2 f'(x) = O(1) \left(1 + \frac{x}{x} \right)^x \left(1+x\right)^A \quad (x \geq 1) \]

\[ = O(1) \left(1 + \frac{x}{x} \right)^x \left(1+x\right)^A \quad (x \geq 1) \]
since
\[(1 + \frac{1}{x})^x = O(1) \quad (x \geq 1)\]

Also from estimate (3.9), for \(0 < x \leq 1\),
\[f'(1) - f'(x) = \frac{O(1)}{x^2}\]

Hence
\[f'(x) = \frac{O(1)}{x^2} \quad (0 < x \leq 1) \quad (3.14)\]

From estimates (3.13) and (3.14),
\[x^2 f'(x) = O(1) (1+x)^A \quad (x > 0) \quad (3.15)\]

Thus if (i) of Theorem 3.6 holds, then \(f'\) is absolutely continuous on compact subintervals of \((0, \infty)\) and \(f', f''\) satisfy estimates (3.15) and (3.6). Thus (i) implies (ii)

Now suppose (ii) holds. Then, in particular, \(f'\) is absolutely continuous on compact subintervals of \((0, \infty)\) and
\[x(1+x)^2-A f''(x) = O(1) \quad \text{a.e. on } (0, \infty)\]

Hence
\[\Delta^2_h(f; x) = f(x+2h) - 2f(x+h) + f(x)\]
\[= \int_0^h \int_0^h f''(x+u+v) \, du \, dv\]
\[ h_0(x) \sim f(1+x) (1+x) - \int_0^x f(t) \, dt \]

Also if \( (ii) \) holds, then

\[ x^2 f'(x) = O(1) (1+x)^A \quad (x > 0) \]

Hence for \( y > x > 0 \),

\[ f(y) - f(x) = \int_x^y f'(t) \, dt \]

\[ = O(1) \int_x^y t^{-2} (1+t)^A \, dt \]

\[ = O(1) (1+y)^A \int_x^y t^{-2} \, dt \]

\[ = O(1) (1+y)^A \left( \frac{1}{x} - \frac{1}{y} \right) \]

Hence for \( y > x > 0 \),

\[ f(y) - f(x) = O(1) x^{-1} (1+y)^A \quad (3.17) \]

Thus if \( (ii) \) holds, then estimates (3.16) and (3.17) holds. That is, \( (iii) \) of Theorem 3.6 holds whenever \( (ii) \) of the theorem holds.
Lastly if (iii) of the theorem hold, then by Theorem II.2.8, (i) of Theorem 3.6. also holds. Thus the statements (i), (ii) and (iii) of Theorem 3.6. are equivalent.

4. Saturation theorem associated with Uniform Approximation for the Operators $R_n^{(\beta)}$

V. Totik in [58] solved the saturation problem for the operators $R_n^{(\beta)}$ on intervals $(a, \infty)$, $a > 0$. We, in this section, solve the global saturation problem for the operators $R_n^{(\beta)}$ associated with uniform approximation. This problem was posed by Totik in [58].

We first prove the following result.

THEOREM 4.1. Let $0 < \beta \leq \frac{1}{2}$. Assume that $f$ belongs to $C_B[0, \infty)$. Suppose

$$||R_n^{(\beta)} f - f|| = O(1) \ n^{-\beta} \ (n \in N)$$

(Here $||f|| = \sup_{t \geq 0} |f(t)|$)

Then

(i) $\Delta_h^2 f(x) = O(1) \ \frac{h^2}{x+h} \ (x > 0, \ h > 0)$

and

(ii) $\Delta_h^1 f(x) = O(1) \ \frac{h^2}{x(x+h)} \ \frac{\beta}{1-\beta} \ (x > 0, \ h > 0)$
For proving the above theorem we make use of the following results obtained by Totik in [58] while proving a saturation theorem for the operators $R_n^{(\beta)}$ on intervals $(a, \infty)$, $a > 0$.

**Lemma 4.2.** [58, pp. 235, 236]. Let $0 < \beta \leq \frac{1}{2}$.

Assume that $f$ belongs to $C_B[0, \infty)$. Suppose

$$R_n^{(\beta)}(f;x) - f(x) = O(1) \ n^{-\beta} \quad (n \in \mathbb{N})$$

uniformly on $[0, \infty)$. Then the derivative $f'$ of $f$ exists on $(0, \infty)$. Further $f'$ is absolutely continuous on compact subintervals of $(0, \infty)$.

**Lemma 4.3.** [58, p. 236]. Suppose $f$ belongs to $C_B[0, \infty)$. Suppose $f'$ exists on $(0, \infty)$ and is absolutely continuous on compact subintervals of $(0, \infty)$. Further assume that almost everywhere on $(0, \infty)$,

$$-x^2 f'(x) + \frac{x}{2} f''(x) = O(1)$$

Then

$$xf''(x) = O(1) \text{ a.e. on } (0, \infty)$$

$$x^2 f'(x) = O(1) \text{ on } (0, \infty)$$

**Proof of Theorem 4.1.** By assumption,

$$R_n^{(\beta)}(f'; x) - f(x) = O(1) \ n^{-\beta} \quad (n \in \mathbb{N}, \ x \geq 0) \quad (4.1)$$

Then applying Lemma 4.2., it follows that $f'$ is absolutely continuous on compact subintervals of $(0, \infty)$. Hence $f''(x)$ exists almost everywhere on $(0, \infty)$. 
Now let $\beta_1$, $r_\beta(x)$, $A_\beta$ be defined as in Theorem 2.1. Then

$$\beta_1 = \max \{ 1, \frac{2\beta_0}{\beta} \}, \text{ where } \beta_0 = \min \{ \beta, 1-\beta \}.$$  

Since $0 < \beta \leq \frac{1}{2}$, $\beta_1$ takes the value 2. Also at each point $x$ in $(0, \infty)$ at which $f''(x)$ exists

$$r_\beta(x) = \lim_{t \to x} \frac{f(t)-f(x)-(t-x)f'(x)}{(t-x)^2}$$  

exists and coincides with $\frac{1}{2} f''(x)$. Further

$$A_\beta = \pi^{-\frac{k}{2}} 2 \Gamma(3/2) = \pi^{-\frac{k}{2}} \Gamma(\frac{1}{2}) = 1.$$  

Hence applying Theorem 2.1., at each point $x$ in $(0, \infty)$ at which $f''(x)$ exists

$$R_n(\beta)(f;x) - f(x) = n^{\beta-1} \left( \frac{x}{2} f''(x) - \frac{1}{2} x^2 f'(x) + o(1) n^{-\beta} n \to \infty \right)$$  

Hence almost everywhere on $(0, \infty)$

$$\lim_{n \to \infty} n^\beta (R_n(\beta)(f;x) - f(x))$$

$$= \begin{cases} \frac{x}{2} f''(x) - x^2 f'(x), & \text{if } \beta = \frac{1}{2} \\ \frac{x}{2} f''(x), & \text{if } 0 < \beta < \frac{1}{2} \end{cases} \quad (4.2)$$

From estimate (4.1) and the limit in (4.2) it follows that
From estimate (4.4), using Lemma 4.3 and the fact that \( f' \) is absolutely continuous, it follows that almost everywhere \((0, \infty)\)

\[
x f''(x) = O(1) \quad \text{if } \beta < \tfrac{1}{2}
\]

(4.3)

\[
\frac{x}{2} f''(x) - x^2 f'(x) = O(1) \quad \text{if } \beta = \tfrac{1}{2}
\]

(4.4)

Thus for all \( \beta \), \( 0 < \beta \leq \tfrac{1}{2} \), from the assumption of Theorem 4.1., it follows that

\[
x f''(x) = O(1) \text{ a.e. on } (0, \infty)
\]

Consequently

\[
||\phi^2 f|| < \infty \quad \text{(4.5)}
\]

where \( \phi^2(x) \equiv x \); \( ||.|| \) is the essential supremum on \((0, \infty)\)

Now for \( x > 0, \ h > 0 \)

\[
| \Delta^2_h(f;x)| = | \int_0^h \int_0^h f''(x+u+v) \, du \, dv |
\]

\[
\leq ||\phi^2 f''|| \int_0^h \int_0^h \frac{1}{x+u+v} \, du \, dv \leq ||\phi^2 f''|| \frac{h^2}{x}
\]
That is
\[ |\Delta_h^2(f; x)| \leq 6|\phi''| \frac{h^2}{x+h} \quad (x > 0, h > 0) \] (4.6)

Also, by Lemma II.3.3,
\[ |\Delta_h^2(f; x)| \leq 3|\phi''| h \quad (x > 0, h > 0) \] (4.7)

Also from estimate (4.6)
\[ |\Delta_h^2(f; x)| \leq 2|\phi''| \frac{h^2}{x+h} \quad (0 < h \leq x) \] (4.8)

From estimates (4.7) and (4.8) it follows that
\[ |\Delta_h^2(f; x)| \leq 6|\phi''| \frac{h^2}{x+h} \quad (x > 0, h > 0) \] (4.9)

Hence from estimates (4.5) and (4.9)
\[ \Delta_h^2(f; x) = O(1) \frac{h^2}{x+h} \quad (x > 0, h > 0) \] (4.10)

This proves (i). From estimate (4.10) it follows that
\[ \Delta_h^2(f; x-h) = O(1) \frac{h^2}{x} \quad (x > h > 0) \]

so that
\[ \Delta_{h/x}^2(f; x-h, x) = O(1) \frac{h^2}{x} (x > h^2 > 0) \]
It follows that
\[ \omega_2(\delta; f) = O(1) \cdot \delta^2 \quad (0 < \delta \leq 1) \quad (4.11) \]

where \( \omega_2(\delta; f) \) is defined as in Theorem II.3.5. It now follows by applying Theorem II.3.5. that for \( n \in \mathbb{N} \),
\[ || R_n^{(\beta)} f - f_n || = O(1) \cdot \omega_2(n^{-\beta/2}; f) \]
where \( f_n(x) = f \left( \frac{x}{1+a_n x} \right) \),
\[ = O(1) \cdot n^{-\beta} \quad \text{using estimate (4.11)} \]
That is,
\[ R_n^{(\beta)} (f;x) - f(\frac{x}{1+a_n x}) = O(1) \cdot n^{-\beta} \quad (n \in \mathbb{N}, x \geq 0) \quad (4.12) \]

From estimates (4.1) and (4.12), it follows that
\[ f(x) - f(\frac{x}{1+a_n x}) = O(1) \cdot n^{-\beta} \quad (n \in \mathbb{N}, x \geq 0) \quad (4.13) \]

From estimate (4.13) it now follows by proceeding exactly as in the proof of Theorem III.3.2 that
\[ \Delta^1_h(f;x) = O(1) \left\{ \frac{h}{x(x+h)} \right\}^{\frac{\beta}{1-\beta}} \quad (x > 0, h > 0) \]
This proves (ii). The proof of the theorem is now complete. 

##
Combining Theorems V.4.1., II.3.6. and III.3.2. we arrive at the following saturation theorem for the operators $R_n^{(\beta)}$.

**COROLLARY 4.4.** Suppose $0 < \beta < 1$. Let

$$\beta_0 = \min \{ \beta, 1-\beta \}$$

Assume that $f$ belongs to $C_B([0,\infty))$. Then

$$||R_n^{(\beta)} f - f|| = O(1) n^{-\beta_0} \quad (n \in \mathbb{N})$$

if and only if

$$\Delta_h^2(f;x) = O(1) \left( \frac{h^2}{x+h} \right)^{\beta_0} \quad (x > 0, h > 0)$$

and

$$\Delta_h^1(f;x) = O(1) \left( \frac{h}{x(x+h)} \right)^{\frac{\beta_0}{1-\beta}} \quad (x > 0, h > 0)$$

##