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Massive, charged scalar field around a charged black hole surrounded by quintessence

2.1 Introduction

2.1.1 The expanding Universe

During 1920s, Alexander Friedmann showed that the Einstein’s equation admits a solution with an expanding Universe. Initially Albert Einstein declined this notion because, like most of the 20th century physicists, he believed in the concept of a static Universe and five years earlier he had published a static model of the Universe. Later, Einstein admitted his mistake and conceded that his field equations do allow the possibility of an expanding universe.

In 1929, Edwin Hubble, who was analyzing the spectra of light coming from distant galaxies, made the remarkable discovery which completely revolutionized astronomy. He noticed that the spectra coming from most of the galaxies are shifted towards the red region indicating that they are moving away from us. By cataloging the distances to these galaxies Hubble formed what we now know as “Hubble’s Law”: the residing velocity of the distant galaxy is proportional to its distance from us and thus he concluded that the universe is
Massive, charged scalar field around a charged black hole surrounded by quintessence expanding. These discoveries made the building block of the modern Big Bang theory.

Later in 1980, the cosmic inflation model of the universe was proposed by Alan Guth to overcome some of the enigma of the Big Bang cosmologies. The model predicts a flat Universe and the total energy density of the Universe is equal to the critical density, the energy density required for the universe to be spatially flat. Observations point towards a spatially flat Universe and the prediction of inflation theory is consistent with current measurements of CMB anisotropy by the WMAP spacecraft. But current calculations suggest the total matter density of the Universe only amounts to about one third of the required critical density indicating the presence of a missing component.

In 1998, two groups of scientists, the High-z Team headed by Schmidt and Riess[59] and the Supernova Cosmology Project by Perlmutter[60] independently made another path breaking discovery in modern cosmology. Analyzing the type Ia supernovae (SNe Ia) at high redshifts, they reached the conclusion that the expansion of the universe is now accelerating rather than holding steady or decelerating. The discovery of cosmic acceleration opens a deep mystery because the two known constituents of the universe, ordinary matter and radiation will gravitationally attract each other and therefore should lead to a slowing down of the expansion. Since the expansion is speeding up we are forced to believe that there is some mysterious form of energy density called dark energy permeating all around the universe which pushes the galaxies each other against their gravitational attraction, thus causing the expansion of the Universe to speed up. At present we hardly know what exactly this dark energy is, how it originates or how it works.
The field equation describing the dynamics of the Universe can take the form of an equation for the second time derivative of the expansion factor,
\[ \frac{\ddot{a}}{a} = -\frac{4\pi}{3} \sum_i (\rho_i + 3p_i). \] (2.1)

Both the energy and the pressure govern the dynamics of the Universe and the algebraic combination, \( \sum_i (\rho_i + 3p_i) \) contributed by the gravitational effect of various components determines the expansion rate. It is customary to define the parameter representing each component of the mass-energy, by the ratio of their pressure to density, known as equation of state (EOS), \( \epsilon_i = p_i/\rho_i \). For ordinary gas, \( \epsilon_i \) is positive, ordinary matter \( \epsilon_i = 0 \) and for radiation \( \epsilon_i = 1/3 \). Each component contributes an amount \( -4\pi \rho_i (1 + 3\epsilon_i) \) to the expansion factor \( \ddot{a}/a \). Since the energy density, \( \rho_i \) is a positive quantity, it requires a component with sufficient negative pressure (\( \epsilon_i \) drops below \(-1/3\)) for a positive value of \( \ddot{a}/a \). Thus dark energy requires a negative pressure to drive the acceleration of the Universe. We can take this to be the defining property of dark energy.

2.1.2 The Cosmological constant

The simplest and oldest candidate for dark energy is the cosmological constant, \( \Lambda \), which was first introduced by Einstein for the purpose of constructing a static model of the Universe. When he applied GTR to cosmology Einstein could not get a stationary solution to the field equations. So he modified the original field equations by adding a positive constant term to the field equations and adjusted the value of the constant so that the gravitational attraction of matter would exactly counterbalanced by this term. Later, he regretfully dropped this idea knowing the findings of Hubble.
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However, observations including the discovery of cosmic acceleration and measurements of the CMBR have brought the cosmological constant back in to the picture. If the cosmological constant had a slightly larger value than Einstein proposed for getting a static solution, its repulsion would exceed the attraction of matter, and induces cosmic acceleration. Cosmological constant has \( \epsilon \) precisely equal to -1. It has the same value everywhere in space for all time, and is chemically inert. Even though the model with cosmological constant (\( \Lambda \)CDM) provides a reasonably good match to the observations, there are some fundamental issues.

The cosmological constant, \( \Lambda \) was later identified to be mathematically equivalent to the vacuum energy, an energy inherent to empty space itself. The principle of quantum field theory allows the quantum fluctuation in empty space and the resulting zero-point energy for empty space. An estimate of the total vacuum energy produced by all known fields predicts the vacuum energy density \( \rho^{\text{cal}}_\Lambda \approx 10^{76} \text{GeV}^4 \). But this predicts a huge amount-123 orders of magnitude more than than the present observed value \( \rho^{\text{obs}}_\Lambda \approx 10^{-47} \text{GeV}^4 \). This is called the cosmological constant problem\cite{61, 62} and a fundamental solution to this problem has not yet been found.

So it requires some cancellation mechanism which zeros out most of the vacuum energy. One proposal is that there may be some secret symmetry in fundamental physics results in a cancellation of large effects. But it is hard to conceive why the mechanism only cancels to 120 decimal places instead of making the cosmological constant exactly zero. To explain the amount of dark energy today, the value of the cosmological constant would have to be fine tuned at the creation of the universe to have the proper value.
2.1.3 Quintessence models

Fortunately, vacuum energy is not the only way to generate cosmic acceleration. Alternatively, dark energy may be a transient phenomenon and the realm of possibilities goes under the rubric of \textit{quintessence}[49–51]. The word quintessence stands for \textit{‘fifth element’} in the ancient and in medieval philosophy, earth, air, fire and water being the other four components that constitute the Universe, according to their imagination. It seems adequate to give this name to the contribution to the overall mass-energy content of the Universe, in addition to the previously known baryons, leptons, photons and dark matter.

Unlike \( \Lambda \), which has the same value everywhere in space for all time, quintessence is dynamical, which can interact with matter, vary with space and evolve in time. For quintessence, \( \epsilon \) has no fixed value, but it must be \( \leq -1/3 \), for a repulsive nature. Quintessence generates the required repulsive force from the energy resulting from the potential energy of a dynamical field, a mechanism similar to the inflationary cosmology theory, in which the inflation field drives the expansion in the early Universe. But the repulsive force excreted by quintessence is much weaker than the inflation. Quintessence may take many forms. The simplest model for quintessence is the energy density associated with a scalar field, \( \phi \) slowly rolling down in a potential \( V(\phi) \). The energy density for a homogeneous scalar field is a sum of kinetic, and potential energies and pressure is the difference of the two,

\[
\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi); \quad p = \frac{1}{2} \dot{\phi}^2 - V(\phi). \tag{2.2}
\]

The kinetic term has positive pressure and the potential term has negative pressure and the total pressure can be negative if the field rolls slowly enough that the kinetic energy density is less than
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the potential energy density. To realize the cosmic acceleration the equation of state,

$$
\epsilon_\phi \equiv \frac{p}{\rho} = \frac{1}{2} \frac{\dot{\phi}^2 - V(\phi)}{\dot{\phi}^2 + V(\phi)},
$$

(2.3)

must be less than $-1/3$. Quintessence is characterized by its equation of state, $-1/3 \geq \epsilon > -1$. The smaller the value of $\epsilon$, the greater its acceleration effect. Quintessence with $\epsilon$ near -1 may be the closest reasonable approximation. Since the value of $\epsilon$ differs from that of vacuum energy, quintessence produces a different rate of cosmic acceleration. Even though these dynamical dark energy candidates lack a concrete motivation from fundamental physics, quintessence hypothesis is found to be fit with many of the current observations. More precise measurements of supernovae over a longer span of distances and imprint on the CMB anisotropy and mass power spectrum may separate the two cases in future.

2.2 Quintessence and black holes

The exact solution for the Einstein’s equations for a static spherically symmetric charged black hole surrounded by the quintessential matter under the condition of additivity and linearity in energy momentum tensor is obtained by Kiselev[63]. The general metric of spherically symmetric static gravitational fields is given by,

$$
ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2).\tag{2.4}
$$

Now the Einstein’s equation for this spacetime have the form,
where prime denotes differentiation with respect to \( r \). The principle of linearity and additivity defines as,

\[
T^t_t = T^r_r = 0.
\]  

(2.8)

Under this condition, the energy momentum tensor for the quintessence is given by,

\[
2T^t_t = -e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right) + \frac{1}{r^2},
\]

(2.5)

\[
2T^r_r = -e^{-\lambda} \left( \frac{1}{r^2} + \nu' \right) + \frac{1}{r^2},
\]

(2.6)

\[
2T^\theta_\theta = 2T^\phi_\phi = -\frac{1}{2} e^{-\lambda} \left( \nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2} \right),
\]

(2.7)

where \( \rho \) is the quintessence EOS which is connected to the density as,

\[
\rho = -c \frac{3\epsilon}{2 r^3(1-\epsilon)}. \]  

(2.10)

Since the density of energy to be positive for quintessence, \( \rho > 0 \) and \( \epsilon \) to be negative, we demand that the normalization factor \( c \geq 0 \). Defining \( \lambda = -ln(1 + f) \), we get an equation for \( f \) as,

\[
(3\epsilon + 1)f + 3(1 + \epsilon)rf' + r^2f'' = 0,
\]

(2.11)

with the solutions of the form, \( f = 1 - \frac{r_g}{r} - \frac{c}{r_{(c+1)}} \), \( c \) and \( r_g \) being normalization factors. Thus the general form of spherically symmetric charged solutions for Einstein’s equation describing black hole with energy momentum tensor satisfies the additivity and linearity condition, so that the metric is given by,
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\[ ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2, \]  
\[ (2.12) \]

with,

\[ f(r) = \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{c}{r^{3c+1}} \right), \quad d\Omega^2 = (d\theta^2 + \sin^2 \theta d\phi^2), \]  
\[ (2.13) \]

\[ M \text{ and } Q \text{ can be identified as the mass and the charge of the black hole respectively. In the limit } c = 0, \text{ the metric reduces to the pure RN spacetime.} \]

The studies on QNMs of black holes were started in the presence of quintessence, after the spacetime for a black hole with quintessential matter were derived by Kiselev. Earlier works considered the simplest case, the Schwarzschild black hole surrounded by quintessence [64–68] and obtained the QNMs by WKB method.

But one can expect a charged black hole, formed when the matter which collapses to form a black hole have a net charge and a charged perturbations will develop outside the collapsing star. And if quintessence exists everywhere in the universe, it will surely distort the spacetime around black holes. So it is interesting to see how the perturbations of RN black hole behave in the presence of quintessence. In this work we are addressing this question by considering massive charged scalar field perturbations.

2.3 Scalar field around charged black hole surrounded by quintessence

2.3.1 Evolution of scalar perturbations

Now we consider massive charged scalar fielded perturbations around the charged black hole spacetime given by Eq.(2.12). The scalar per-
Evolution of scalar perturbations

Scalar perturbations can be described by the Klein-Gordon (KG) equation \[37, 72\],

\[
\Phi_{,\mu\nu} g^{\mu\nu} - ieA_\mu g^{\mu\nu} (2\Phi,_{\nu} - ieA_\nu \Phi) - ieA_\mu;\nu g^{\mu\nu} \Phi = m^2 \Phi, \tag{2.14}
\]

where \(A_\mu\) is the electromagnetic potential and \(e\) and \(m\) are the charge and mass of the scalar field. In a spherically symmetric space-time specified by Eq. (2.12), the KG equation can be simplified to,

\[
\frac{1}{f(r)} \frac{\partial^2 \Phi}{\partial t^2} - \frac{1}{r^2} \left( r^2 f(r) \partial_r^2 + 2rf(r)\partial_r + r^2 \partial_r f(r) \partial_r \right) \Phi + 
\left( -\frac{1}{r^2 \sin^2 \theta} \left( \sin \theta \partial_\theta \partial_\theta + \cos \theta \partial_\theta \right) - \frac{1}{r^2 \sin^2 \theta} \partial_\phi^2 \right) \Phi - 
\left( 2ieA_t g^{tt} \partial_t - e^2 A_t^2 g^{tt} \right) \Phi = 0. \tag{2.15}
\]

Expanding the charged scalar field in scalar spherical harmonics,

\[
\Phi = \frac{1}{r} \sum_{l,m} \eta^l_m(t, r) Y^m_l(\theta, \phi), \tag{2.16}
\]

the wave equation can be reduced to an equation for the scalar function, \(\eta^l_m(t, r)\) for each multipole moment:

\[
\frac{\partial^2 \eta}{rf(r)} + \frac{\ell(\ell + 1)}{r^3} - \frac{2ieA_t \partial_t \eta}{rf(r)} - \frac{e^2 A_t^2}{f(r)} - \frac{\partial_r f(r)}{r^2} - f(r) r \partial_r^2 \eta = 0, \tag{2.17}
\]

where the centrifugal term comes from the action of the angular momentum operator on the spherical harmonics,

\[
- \left( \partial_\theta^2 + \cot \theta \partial_\theta + 1/sin^2 \theta \partial_\phi^2 \right) Y^m_l(\theta, \phi) = \bar{L}^2 Y^m_l(\theta, \phi) \tag{2.18}
= \ell(\ell + 1) Y^m_l(\theta, \phi).
\]

Now using the coordinate transformation defined by,
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\[ dr_* = \frac{dr}{f(r)}, \]  

(2.19)

we can rewrite the perturbation equation as,

\[ \partial_t^2 \eta - 2ieA_t \partial_t \eta - \partial_r \eta + f(r) \left[ \frac{l(l+1)}{r^2} + \frac{\partial_r f(r)}{r} + m^2 \right] \eta - e^2 A_t^2 \eta = 0. \]  

(2.20)

The time component of the electromagnetic potential, \( A_t = C - \frac{Q}{r} \), with \( C \) being a constant and to avoid this physically unimportant quantity, we define,

\[ \eta = e^{iCt} \psi, \]  

(2.21)

The auxiliary field, \( \psi \) assumes to have a harmonic time dependence \( e^{-i\omega t} \). Now the radial perturbation equation can be written as,

\[ \frac{\partial^2 \psi}{\partial r_*^2} + \Theta \psi = 0, \]  

(2.22)

where \( \Theta = \omega^2 - V^2 \), \( V \) being the effective scattering potential, arises from the curvature of the spacetime, is given by,

\[ V = f(r) \left( \frac{l(l+1)}{r^2} + \frac{2M}{r^3} - \frac{2Q^2}{r^4} + \frac{c(3e+1)}{r^{3e+3}} + m^2 \right) + \frac{2eQ\omega}{r} - e^2 \frac{Q^2}{r^2}. \]  

(2.23)

It can be noticed that the effective potential depends not only on the parameters, \( M, Q, \ell, e \) and \( \omega \) but also on the frequency of perturbation, \( \omega \). This will make it difficult to evaluate the quasinormal mode frequencies using the WKB approximation. Figure 2.1 shows the general behavior of the potential. It can be observed that the height of the potential barrier decreases if quintessence is present and the asymptotic value, \( V(r \to \infty) \) increases with mass of the field.
2.3.2 Quasinormal modes of perturbations

The third order WKB approximation method can be used to determine the complex normal mode frequencies of black hole. This method gives a simple condition which can be used to get the discrete, complex frequencies of the normal modes,

\[ \frac{i\Theta_0}{\sqrt{2\Theta_0}} - \Lambda(n) - \Omega(n) = n + \frac{1}{2}, \quad (2.24) \]

where \( n \) is the mode number, \( \Lambda \) and \( \Omega \) are second and third order WKB correction terms,

\[ \Lambda = \frac{1}{(-2\Theta_0''')^{1/2}} \left[ \frac{1}{8} \left( \frac{\Theta_0^{(4)}}{\Theta_0''} \right) \left( \frac{1}{4} + \alpha^2 \right) - \left( \frac{\Theta_0'''}{\Theta_0''} \right)^2 \frac{(7 + 60\alpha^2)}{288} \right], \quad (2.25) \]
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\[
\Omega = \frac{1}{(-2\Theta_0^{''})} \left\{ \frac{5}{6912} \left( \frac{\Theta_0^{'''}^{4}}{\Theta_0^{''2}} \right) (77 + 188\alpha^2) - \frac{1}{384} \left( \frac{\Theta_0^{''''2}\Theta_0^{(4)}}{\Theta_0^{'''3}} \right) \\
(51 + 100\alpha^2) + \frac{1}{2304} \left( \frac{\Theta_0^{(4)}}{\Theta_0^{''}} \right)^2 (67 + 68\alpha^2) + \frac{1}{288} \left( \frac{\Theta_0^{'''}\Theta_0^{(5)}}{\Theta_0^{''2}} \right) \\
(19 + 28\alpha^2) + \frac{1}{288} \left( \frac{\Theta_0^{(6)}}{\Theta_0^{''}} \right) (5 + 4\alpha^2) \right\},
\]

(2.26)

\[
\alpha = n + \frac{1}{2}, \quad n = \begin{cases} 
0, 1, 2, \ldots \ldots Re(E) > 0 \\
-1, -2, -3, \ldots \ldots Re(E) < 0
\end{cases}
\]

(2.27)

where,

\[
\Theta_0^{(n)} = \frac{d^n\Theta}{dr_+^n} \bigg|_{r_+=r_0},
\]

(2.28)

denotes the \(n\)th derivatives of \(\Theta\) evaluated at \(r_0\), the value of \(r\) at which \(V\) attains maximum. Here a complexity arises from the fact that the potential \(V\) is a function of frequency \(\omega\). This makes difficult to calculate the numerical value of \(r_0\). We make use of the procedure suggested by Konoplya[73] to find \(r_0\) by fixing all the parameters other than \(\omega\), on which \(V\) depends and then find the value of \(r\) at which \(V\) attains maximum as a numerical function of \(\omega\). Substituting this value of \(r_0\), we have found the values of real and imaginary parts of \(\omega\) which satisfies the condition (2.24) by numerical methods.

It is a general experience that the quasinormal frequencies with lower mode number will decay slowly and are relevant to the description of fields around the black hole. So we consider frequencies of low lying modes for our study. For a charged black hole the relative error of third order WKB method is of the order of \(10^{-2}\) for \(l = 3, n = 0\) mode[74]. We put the normalization factor, \(c = 0.01\), so that
the deviation of frequencies from the pure RN case can be clearly understood and is much larger than the relative WKB error. In what follows we examine the behavior of QNMs.

First, we analyze the dependence of QNMs on the charge of the black hole, $Q$, in the presence of quintessence. Figures 2.2 and 2.3 shows the real and imaginary parts of $\omega$ as a function of $Q$ for fixed $l = 3, n = 0, e = 0.1, m = 0.1$ and for different values of $\epsilon$. The case with absence of quintessence is also plotted. The plot shows that the quasinormal frequencies for scalar field in a charged black hole is influenced by quintessence. The magnitudes of real and imaginary parts of $\omega$ is lower in the presence of quintessential field. This implies that due to the presence of quintessence, the quasinormal mode frequencies for scalar field in RN black hole damps more slowly. In the presence of quintessence, $Re(\omega)$ increases monotonically with the increase in $Q$ while the magnitude of $Im(\omega)$ first decreases, falling to

![Figure 2.2: $Re(\omega)$ as a function of $Q$ for $l = 3, n = 0, e = 0.1, m = 0.1$ and for different values of $\epsilon$ with $c = 0.01$. The dotted line represents the no quintessence case ($c = 0$).](image-url)
Figure 2.3: $Im(\omega)$ as a function of $Q$ for $l = 3, n = 0, e = 0.1, m = 0.1$ and for different values of $\epsilon$ with $c = 0.01$. The dotted line represents the no quintessence case ($c = 0$).

a minimum around $Q = 0.8$ and thereafter increases sharply. If we discard quintessence ($c = 0$) the results coincide with those obtained in [75, 76].

Figure 2.4 shows the explicit dependence of $Re(\omega)$ and $Im(\omega)$ with quintessential parameters $c$ and $\epsilon$ for fixed $l = 3, n = 0, e = 0, Q = 0.5$ and $m = 0.1$. For a fixed $c$, as the value of $\epsilon$ increases, $Re(\omega)$ increases while the magnitude of $Im(\omega)$ increases meaning damping is less for lower values of $\epsilon$. As the normalization factor $c$ increases, $Re(\omega)$ decreases and magnitude of $Im(\omega)$ decreases. In Figure 2.5 $Re(\omega)$ and $Im(\omega)$ are plotted as functions of $\epsilon$ with $l = 3, n = 0, m = 0.1$ for $Q = 0.1, 0.3$ and different values of $\epsilon$. Dotted line represents absence of quintessence ($c = 0$). The variation is almost linear. In the presence of quintessence, the magnitudes of $Re(\omega)$ and $Im(\omega)$ increase with $\epsilon$, as in the no quintessence case ($c = 0$), but with lower values of $Re(\omega)$ and $Im(\omega)$. A neutral field decays slower than a charged field.
Finally, we study the role of mass of scalar field on quasinormal frequencies. For low-lying QNMs, to occur tunneling, $\omega^2$ must be smaller than the peak value of the potential $V(r = r_{max})$ and the energies of the field are always larger than $m^2[12]$. This means that there is a maximum value for mass, $m_{max}$ beyond which quasinormal modes will not exist. The value of $m_{max}$ can be calculated from the

Figure 2.4: Variation of $\text{Re}(\omega)$ and $\text{Im}(\omega)$ with quintessence parameters $\epsilon$ and $c$, for $l = 3, n = 0, e = 0, Q = 0.5$ and $m = 0.1$.
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Figure 2.5: Variation of $Re(\omega)$ and $Im(\omega)$ with $\epsilon$, for $l = 3, n = 0, m = 0.1, Q = 0.1, 0.3$ and different values of $\epsilon$. Dotted curve is for $c = 0$.

condition for the existence of quasinormal modes,

$$V(r_{\text{max}}, \omega = m_{\text{max}}) = (m_{\text{max}})^2$$

(2.29)

These values of $m_{\text{max}}$ obtained for different values of $\epsilon$ are tabulated in Table 2.1. In the presence of quintessence, $m_{\text{max}}$ decreases because quintessence lowers the height of the potential barrier as we have seen in Figure 2.1 and when $\epsilon = -1$, it has the lowest value.

WKB approximation gives less accurate results as the mass of the field increases but we can use this method to obtain the qualitative dependence of QNMs on field mass. The dependence of QNMs on mass of scalar field is plotted in Figures 2.6 and 2.7. $Re(\omega)$ increases
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<table>
<thead>
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<th>( c )</th>
<th>( \epsilon )</th>
<th>( m_{\text{max}} )</th>
</tr>
</thead>
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<tr>
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<td>–</td>
<td>0.88516</td>
</tr>
<tr>
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</tr>
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</tr>
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<td>-1</td>
<td>0.67582</td>
</tr>
</tbody>
</table>

Table 2.1: The limit of mass of scalar field, \( m_{\text{max}} \) for the existence of quasinormal frequencies with \( l = 3, \epsilon = 0, Q = 0.1 \)

with increase in mass, while \(|\text{Im}(\omega)|\) decreases, which indicates that QNMs of massive fields damp slowly. This behavior is in agreement with numerical results obtained in [75].

But QNMs behave abnormally near \( m_{\text{max}} \). This is due to the fact that for larger values of field mass, the potential looses its barrier shape by broadening the potential peak as shown in Figure2.1 and WKB method gives less accurate results. Lower modes show less abnormality showing that WKB method is more accurate for fundamental modes. An interesting feature, we noticed is that this abnormal behavior is lower in the presence of quintessence and when the quintessential parameter \( \epsilon = -1 \), we can get a satisfactory curve because of the peak of effective potential broadens much less in the presence of quintessence comparing with the normal case\( (c=0) \) as understood from Figure2.1 and quintessence helps to retain barrier shape and give more accurate results at larger mass range.
Figure 2.6: Variation of $Re(\omega)$ with $m$, for $l = 3, e = 0, Q = 0.1, n = 0, 1, 2$ and different values of $\epsilon$. Dotted curve represents the Schwarzschild case.
Figure 2.7: Variation of $Im(\omega)$ with $m$, for $l = 3, e = 0, Q = 0.1, n = 0, 1, 2$ and different values of $\epsilon$. Dotted curve is for the Schwarzschild case.
2.4 Conclusion

The studies on the perturbations of black holes in accelerated expanding Universe with quintessence model for dark energy are presented in this chapter. Massive charged scalar field perturbations are considered around the charged black hole immersed in quintessence and WKB approximation method are used to evaluate the associated QNMs. The results show that quintessence influences the QNMs of charged black hole. It shows a decrease in the oscillation frequency and slowing up in the damping of QNMs in the quintessence present case than the asymptotically flat spacetime case. The behavior of QNMs vary as the equation of state for the quintessence changes. The damping are less for lower values of $\epsilon$. The dependence of QNMs on other parameters such as charge of black hole $Q$, charge and mass of scalar field is similar to that obtained in quintessence less case, but with lower oscillation frequency and higher damping time. The effect of quintessence is to retain the barrier shape of effective potential for larger field masses and thus to reduce the abnormal behavior of QNMs evaluated using WKB method at these mass ranges.