Chapter 4

Route Systems On Fuzzy Graphs

4.1 Introduction

The concept of route system was introduced by Nebeský [34, 36] while studying the set of all geodesics (shortest paths) in a connected graph. Innovating the idea of route systems, Nebeský was able to give a credulous and non-metric characterization of the set of all geodesics in a connected graph ([37]). Nebeský’s seminal idea of ‘route systems’ was generalised by Changat and Mulder ([10]) to capture finite non-empty set $V$. In this generalized format, a route system on $V$ has been defined as a collection of sequences of elements in $V$ which satisfy some axioms. To capture the specificities of various types of already known route systems on graphs, both ‘extension axioms’ and ‘exclusion axioms’ are employed. If a route system is given, it becomes possible to define the underlying graph of the route system. An important question arises at this juncture, that is, what do the specific routes in the route system signify in the underlying graph. If the route system satisfies a symmetry axiom, then the underlying graph is undirected, otherwise it is directed. New axioms are introduced into the route systems (directed as well as undirected) to characterize the walks, the trails, the paths, the minimal paths and the triangular minimal paths in the underlying (di)graph.
We extend the idea of the 'route system' on a finite non-empty set $V$ to a non-empty fuzzy subset $\sigma$ of $V$. This is carried out in order to provide an axiomatic characterisation of the route systems in the underlying (directed as well as undirected) fuzzy graph.

4.2 Route Systems

Before describing the fuzzy route systems, we reproduce the route systems in a crisp set as it is provided in Changat and Mulder (2001)([10]).

Let $V$ be a non-empty, finite set. By $L(V)$ we denote the set of all finite sequences of elements in $V$. For $\alpha, \beta$ in $L(V)$, the sequence $\alpha \beta$ is the concatenation of the sequences $\alpha$ and $\beta$. The empty sequence is denoted by $\omega$. Thus, for any sequence $\alpha$ in $L(V)$, we have $\omega \alpha = \alpha \omega = \alpha$. If $\alpha = v_1v_2\ldots v_k$, then $|\alpha| = k$ is the length of the sequence, and $\bar{\alpha} = v_kv_{k-1}\ldots v_2v_1$ is the sequence obtained from $\alpha$ by reversing the order of the elements in $\alpha$. We call this the inverse of $\alpha$. In particular, $|\omega| = 0$, and $\bar{\omega} = \omega$. In the sequel, we will denote elements of $V$ by roman letters, and sequences in $L(V)$ by Greek letters. Thus, the sequence $uav$ has $u$ as first element and $v$ as last element, whereas $\alpha$ is a, possibly empty, sequence in $L(V)$.

Let $\alpha$ be a non-empty sequence in $L(V)$. We denote the first element of $\alpha$ by $\Phi(\alpha)$, and the last element of $\alpha$ by $\Lambda(\alpha)$. Let $\alpha, \beta,$ and $\gamma$ be, possibly empty, sequences in $L(V)$. Then we write $\beta \in \alpha \beta \gamma$. Note that $\beta$ is a subsequence consisting of consecutive elements of $\alpha \beta \gamma$. If $\beta$ is non-empty, we write $\alpha < \alpha \beta$. Note that the $<$ symbol is only used for sequences having the same first element.

A route system on the set $V$ is a family of sequences $\mathcal{R} \subseteq L(V)$ satisfying the following axioms:

\begin{align*}
(\text{r0}) & \quad u \in \mathcal{R}, \text{ for any } u \in V, \\
(\text{r1}) & \quad uav \in \mathcal{R} \Rightarrow u\alpha, \alpha v \in \mathcal{R}.
\end{align*}

Let $\mathcal{R}$ be a route system. The sequences in $\mathcal{R}$ are called routes. The axiom (r0) serves only to exclude certain degenerate cases. Let $\alpha$ be a route in $\mathcal{R}$,
and let $\beta \in \alpha$. Then we call $\beta$ a **subroute** of $\alpha$. Axiom (r1) states that every subroute of a route is in itself a route. Hence we call (r1) the **subroute axiom**.

The underlying digraph $D_R = (V, A)$ of $R$ has vertex set $V$ and $(u, v)$ is an arc in $D_R$ if and only if $uv$ is a route in $R$. We will write $uv = (u, v)$, for short. Note that, if we would drop (r0), then we should take as vertex set of $D$ the set $W = \{v \in V | v \in R\}$.

The next two axioms are **symmetry axioms**:

$$
\begin{align*}
(s0) & \quad uv \in R \Rightarrow vu \in R, \\
(s1) & \quad \alpha \in R \Rightarrow \bar{\alpha} \in R.
\end{align*}
$$

Axiom (s0) is the **weak symmetry axiom**, and axiom (s1) is the **strong symmetry axiom**, or the **symmetry axiom**, for short. The underlying graph $G_R = (V, E)$ of a weakly symmetric route system has vertex set $V$ and $uv$ is an edge in $G_R$ if and only if $uv$ lies in $R$.

In the sequel we will provide the results on route systems of Changat and Mulder [10] without proof. The symmetric, or undirected analogues usually follow straightforward from the general, or directed case by applying (s1). It turns out that, in each case, we will only require weak symmetry, and that strong symmetry follows from weak symmetry and the other axioms. To obtain the undirected case, sometimes just a simple adaptation of the proof is necessary.

We start with our most general case, the route system of all walks, and specialize as we proceed. Basically we require two types of axioms for the special cases, namely an **extension axiom** to generate new routes from existing ones and an **exclusion axiom** to exclude the non-routes from the route system.

We list the axioms as follows:

$$
\begin{align*}
(r2) & \quad uax, axv \in R \Rightarrow uaxv \in R, \\
(r4) & \quad uax, axv \in R, u\Phi(ax) \neq xv, \Rightarrow uaxv \in R, \\
(r5) & \quad uaxv \in R \Rightarrow u\Phi(ax) \neq xv; \\
(r6) & \quad uax, axv \in R, u \neq v \Rightarrow uaxv \in R, \\
(r7) & \quad uav \in R \Rightarrow u \neq v,
\end{align*}
$$
To avoid confusion, we list few terminologies regarding to graphs and digraphs. All our graphs are considered as without multiple edges or arcs, but we allow loops. Let $D = (V, A)$ be a digraph. A directed walk in $D$ is a sequence of vertices $v_1v_2 \ldots v_k$ such that $v_iv_{i+1}$ is an arc, for $1 \leq i < k$. Similarly, a walk in a graph $G = (V, E)$ is a sequence of vertices $v_1v_2 \ldots v_k$ such that $v_iv_{i+1}$ is an edge, for $1 \leq i < k$. The length of the walk is $k - 1$. A (directed) trail is a (directed) walk without repetition of vertices. A (directed) path is a (directed) trail without repetition of arcs, resp. edges. A (directed) minimal path is a (directed) path, in which $v_iv_j$ is not an arc, resp. edge, for $1 \leq i < j - 1 \leq k$. A (directed) triangular minimal path is a (directed) path, in which $v_iv_j$ is not an arc, resp. edge, for $1 \leq i < j - 2 < k$. A (directed) geodesic is a (directed) path of minimal length between its endpoints. A (directed) triangular geodesic is a (directed) triangular minimal path, such that, if we take a minimal (directed) subpath $P$ contained in it, then it is a (directed) geodesic, and each other vertex of the path is adjacent to only two consecutive vertices of $P$.

Let $D = (V, A)$ be a digraph, and let $G = (V, E)$ be a graph. Clearly, the family of all directed walks in $D$ is a route system satisfying (r2). The family of all directed trails in $D$ is a route system satisfying (r4) and (r5). The family of all directed paths in $D$ is a route system satisfying (r6) and (r7). The family of all directed minimal paths in $D$ is a route system satisfying (r8) and
(r9). The family of directed triangular minimal paths in $D$ is a route system satisfying (r10) and (r11). Clearly, the walks, trails, paths, minimal paths, and triangular minimal paths of $G$ all satisfy axioms (s0) and (s1), and the respective extension and exclusion axiom.

An appealing characterization of the route system consisting of all geodesics of a graph was given by Nebeský in [36]. Thus he was able to produce a non-metric characterization of the set of all geodesics of a graph. The characterization of the directed route system of all directed geodesics in a digraph is still open, as is the characterization of the (directed) route system all (directed) triangular geodesics of a (di)graph.

All the lemmata below are proved by induction on the length of the routes. We state all the results on route systems on graphs in [10] without proof. The basic idea for characterizing the route system as the family of directed walks in a directed graph is given in Lemma 12. The undirected analogue is given in Lemma 13.

**Lemma 12** ([10]) Let $R$ and $S$ be route systems on $V$ with $D_R = D_S$. If $R$ and $S$ both satisfy (r2), then $R = S$.

The next lemma follows immediately from Lemma 12.

**Lemma 13** ([10]) Let $R$ and $S$ be weakly symmetric route systems on $V$ with $G_R = G_S$. If $R$ and $S$ both satisfy (r2), then $R = S$.

As observed above, the directed walks in a digraph, as well as the walks in a graph, form a route system satisfying (r2). Hence the next theorem follows immediately from Lemmata 12 and 13.

**Theorem 18** ([10]) Let $V$ be a finite nonempty set, and let $R$ be a route system on $V$. Then $R$ is the family of all directed walks of $D_R$ if and only if $R$ satisfies (r2). If $R$ satisfies (s0), then $R$ is the family of all walks of $G_R$ if and only if $R$ satisfies (r10) and (r11).
Lemma 14 ([10]) Let \( R \) and \( S \) be route systems on \( V \) with \( D_R = D_S \). If \( R \) and \( S \) both satisfy \((r_{2i})\) and \((r_{2i} + 1)\), then \( R = S \), for \( i = 2, 3, 4, 5 \).

The weakly symmetric analogue of Lemma 14 is given in the next Lemma.

Lemma 15 ([10]) Let \( R \) and \( S \) be weakly symmetric route systems on \( V \) with \( G_R = G_S \). If \( R \) and \( S \) both satisfy \((r_{2i})\) and \((r_{2i} + 1)\) then \( R = S \), for \( i = 2, 3, 4, 5 \).

Lemmata 14 and 15 provide us with the following batch of theorems, in which the route systems associated to the various types of walks in a (di)graph are characterized.

Theorem 19 ([10]) Let \( V \) be a finite nonempty set, and let \( R \) be a route system on \( V \). Then \( R \) is the family of all directed trails of \( D_R \) if and only if \( R \) satisfies \((t_{1})\) and \((t_{5})\). If \( R \) satisfies \((s_{0})\), then \( R \) is the family of all walks of \( G_R \) if and only if \( R \) satisfies \((t_{4})\) and \((t_{5})\).

Theorem 20 ([10]) Let \( V \) be a finite nonempty set, and let \( R \) be a route system on \( V \). Then \( R \) is the family of all directed paths of \( D_R \) if and only if \( R \) satisfies \((t_{6})\) and \((t_{7})\). If \( R \) satisfies \((s_{0})\), then \( R \) is the family of all paths of \( G_R \) if and only if \( R \) satisfies \((t_{6})\) and \((t_{7})\).

Theorem 21 ([10]) Let \( V \) be a finite nonempty set, and let \( R \) be a route system on \( V \). Then \( R \) is the family of all directed minimal paths of \( D_R \) if and only if \( R \) satisfies \((t_{8})\) and \((t_{9})\). If \( R \) satisfies \((s_{0})\), then \( R \) is the family of all minimal paths of \( G_R \) if and only if \( R \) satisfies \((t_{8})\) and \((t_{9})\).

Theorem 22 ([10]) Let \( V \) be a finite nonempty set, and let \( R \) be a route system on \( V \). Then \( R \) is the family of all directed triangular minimal paths of \( D_R \) if and only if \( R \) satisfies \((t_{10})\) and \((t_{11})\). If \( R \) satisfies \((s_{0})\), then \( R \) is the family of all triangular minimal paths of \( G_R \) if and only if \( R \) satisfies \((t_{10})\) and \((t_{11})\).
The next theorem is proved by Nebeský in [37] (Theorem 1).

**Theorem 23** ([37]) Let \( V \) be a finite nonempty set, and let \( \mathcal{R} \) be a symmetric route system on \( V \). Then \( \mathcal{R} \) is the family of all geodesics of \( G_\mathcal{R} \) if and only if \( \mathcal{R} \) satisfies the axioms \((r9), (r12), (r13), (r14) \) and \((r15)\).

### 4.3 Fuzzy Route Systems

Given the route system in a crisp graph, we now proceed to generalise it in a fuzzy graph.

Before introducing the route systems on a fuzzy subset \( \sigma \), to avoid ambiguity and unwarranted confusion, we list some terminologies on fuzzy graphs and digraphs. We follow Nair and Moderson [23] and Bhutani ([5], [6]) for the terminologies and definitions on Fuzzy graphs. A fuzzy sub digraph of a crisp directed graph \( G \) with node set \( S \) and arc set \( A \subseteq S \times S \) is defined as a fuzzy subset \( \sigma \) of \( S \) and fuzzy subset \( \mu \) of \( A \) such that \( \mu(u, v) \leq \sigma(u) \land \sigma(v) \) for all \( u, v \in S \). In other words, a fuzzy digraph is a fuzzy subset \( \sigma \) of \( S \) and a fuzzy relation \( \mu \) on \( \sigma \) such that \( \mu(u, v) \leq \sigma(u) \land \sigma(v) \) for all \( u, v \in S \). The fuzzy subsets \( \sigma \) and \( \mu \) are called as the node set and arc sets of the fuzzy digraph respectively and denote such a fuzzy digraph as \( D = (\sigma, \mu) \). If the fuzzy relation \( \mu \) is symmetric, then \( D \) is called as undirected or simply a fuzzy graph. We denote the undirected fuzzy graph as \( G \). For a node \( v_i \in S \), let us denote \( \sigma(v_i) \) as \( \sigma_i \) and \( \mu(v_i, v_j) \) as \( \mu(\sigma_i, \sigma_j) \) respectively. We follow these notations throughout this chapter.

All the fuzzy graphs and digraphs considered in this chapter are assumed to be without multiple arcs, but may allow loops.

A directed walk in \( D \) is a sequence of vertices \( \sigma_1 \sigma_2 \ldots \sigma_k \) in \( \sigma \) such that \( \sigma_i \sigma_{i+1} \) is an arc in \( D \), for \( 1 \leq i < k \); that is \( \mu(\sigma_i, \sigma_{i+1}) > 0 \) and for all \( \sigma_i, \sigma_{i+1} \) in \( \sigma \). Similarly, a walk in a fuzzy graph \( G = (\sigma, \mu) \) is a sequence of vertices \( \sigma_1 \sigma_2 \ldots \sigma_k \) such that \( \sigma_i \sigma_{i+1} \) is an edge; that is, \( \mu(\sigma_i, \sigma_{i+1}) > 0 \), for \( 1 \leq i < k \). The length of the walk is \( k - 1 \). A (directed) trail is a (directed) walk without repetition of arcs (edges). A (directed) path is a (directed) trail without repetition of vertices. We
define a (directed) minimal path as a (directed) path in which \( \sigma_i \sigma_j \) is not an arc (edge); that is, \( \mu(\sigma_i, \sigma_j) = 0 \), respectively, for \( 1 \leq i < j - 1 \leq k \). The minimum of \( \mu(\sigma_i, \sigma_{i+1}) \) for \( 1 \leq i < k \) is called the strength of the path \( \sigma_i \sigma_{i+1} \ldots \sigma_k \) in \( \sigma \).

That is, if \( P = \sigma_1 \sigma_2 \ldots \sigma_k \) is a path in \( D \) or \( G \), then \( \bigwedge_{1 \leq i < k} (\mu(\sigma_i, \sigma_{i+1})) \) is the strength of the path \( P \) in \( D \) or \( G \). The maximum of the strengths of all paths in \( G \) from \( \sigma_1 \) to \( \sigma_k \) is named \( \text{CONN}(\sigma_1, \sigma_k) \) and we say that \( G \) is connected if \( \text{CONN}_G(\sigma_i, \sigma_k) > 0 \) for all \( \sigma_i, \sigma_k \in G \). An arc \( (\sigma_i, \sigma_{i+1}) \) in \( G \) is strong if \( \mu(\sigma_i, \sigma_{i+1}) \geq 0 \) and \( \mu(\sigma_i, \sigma_{i+1}) \geq \text{CONN}^{-1}_G(\sigma_i, \sigma_{i+1}) \). A path is termed as strong if all arcs in the path are strong ([5], [6]). A (directed) triangular minimal path is a (directed) path, in which \( \sigma_i \sigma_j \) is not an arc, resp. edge, that is \( \mu(\sigma_i, \sigma_j) = 0 \) for \( 1 \leq i < j - 2 < k \). A (directed) geodesic is a (directed) shortest strong path between its endpoints ([7]). A (directed) triangular geodesic is a (directed) triangular minimal strong-path, such that, if we take a minimal (directed) sub strong path \( P \) contained in it, then it is a (directed) geodesic and any other vertex of the path is adjacent to only two consecutive vertices of \( P \).

We will begin with a non-empty fuzzy subset \( \sigma \) of a crisp set \( V \) by defining a route system on \( \sigma \), and introduce the axioms on the route system.

Let \( \mathcal{FS}(V) \) denote the set of all fuzzy singletons of \( \sigma \). That is, \( \mathcal{FS}(V) = \{ \sigma_i : V \to [0, 1] | \text{supp}(\sigma_i) = 1 \text{ and } \sigma_i \leq \sigma \} \) and let \( v_i \in \text{supp}(\sigma_i) \) and by \( \mathcal{FL}(V) \) we denote the set of all finite sequences of elements of \( \mathcal{FS}(V) \). Now, for an element \( F\sigma = \sigma_1 \sigma_2 \ldots \sigma_k \in \mathcal{FL}(V) \), define the membership function of \( F\sigma \) as \( W(\sigma_1 \sigma_2 \ldots \sigma_k) : \mathcal{FL}(V) \to [0, 1] | (\sigma_1 \sigma_2 \ldots \sigma_k)(v_1 v_2 \ldots v_m) \leq \bigwedge_{1 \leq i \leq k} (\sigma_i(v_i) \vee \sigma_i(v_j)) \) where \( v_i \in V \) and \( \sigma_i(v_i) \neq 0 \) and \( \sigma_i(v_j) = 0 \), \( i \neq j \). For \( F\alpha, F\beta \) in \( \mathcal{FL}(V) \), the sequence \( F\alpha F\beta \) is the concatenation of the sequences \( F\alpha \) and \( F\beta \). The empty sequence is denoted by \( F\omega \). Therefore, for any sequence \( F\alpha \) in \( \mathcal{FL}(V) \), we have \( F\omega F\alpha = F\alpha F\omega = F\alpha \). If \( F\alpha = \sigma_1 \sigma_2 \ldots \sigma_k \), \( |F\alpha| = k \) is the length of the sequence and \( F\alpha \) is the sequence obtained from \( F\alpha \) by reversing the order of the elements in \( F\alpha \). We call this sequence as the inverse of \( F\alpha \). In particular, \( |F\omega| = 0 \), and \( F\omega = F\omega \). In the sequel, we will denote the sequences in \( \mathcal{FL}(V) \) by \( F\alpha, F\beta \) etc, and use Greek letters to denote fuzzy subsets. Thus, the sequence \( \sigma_u F\alpha \sigma_v \) has \( \sigma_u \) as first element and \( \sigma_v \) as last element, whereas \( F\alpha \) is a (possibly empty) sequence in \( \mathcal{FL}(V) \).
Let $F\alpha$ be a non empty sequence in $\mathcal{F}L(V)$. We denote the first element of $F\alpha$ by $\Phi(F\alpha)$, and the last element of $F\alpha$ by $\Lambda(F\alpha)$. Let $F\alpha$, $F\beta$, and $F\gamma$ be (possibly empty) sequences in $\mathcal{F}L(V)$. Then we write $F\beta \in F\alpha F\beta F\gamma$. Note that $F\beta$ is a subsequence consisting of consecutive elements of $F\alpha F\beta F\gamma$. If $F\beta$ is non empty, we write $F\alpha < F\alpha F\beta$. Note that the $<$ symbol is only used for sequences having the same first element.

We have straightforward lemma

**Lemma 16** The pair $(\mathcal{F}L(V), \sigma_1 \sigma_2 \ldots \sigma_k)$ is a fuzzy sub family of $L(V)$.

Following Changat and Mulder(2001), we define a *route system* on the fuzzy subset $\sigma$ of $V$ as a family of sequences $\mathcal{FR} \subseteq \mathcal{F}L(V)$ satisfying the following axioms:

(F0) $\sigma_u \in \mathcal{FR}$, for any $\sigma_u \in \mathcal{FS}(V)$,
(F1) $\sigma_u F\alpha \sigma_u \in \mathcal{FR} \Rightarrow \sigma_u F\alpha, F\alpha \sigma_u \in \mathcal{FR}$.

If $\mathcal{FR}$ is a route system on $\sigma$, then the sequences in $\mathcal{FR}$ are called *routes*. The axiom $F(0)$ excludes certain degenerate cases from $\mathcal{FR}$. We call $F\beta$ a *subroute* of $F\alpha$ when $F\alpha$ is a route in $\mathcal{FR}$ and $F\beta \in F\alpha$. We call $F(1)$ the *subroute axiom* because every subroute of a route is in itself a route.

Consider the digraph $D_{\mathcal{FR}}$ with $\mathcal{FS}(V)$ as vertex set, $FE$ as arc set, where $\sigma_i \sigma_j \in FE$ if and only if $\sigma_i \sigma_j \in \mathcal{FR}$. Let fuzzy sets $(\sigma, \mu)$, be defined on $(\mathcal{FS}(V), \mathcal{FE})$ as $\sigma V (\sigma_i) = \sigma (v_i)$, where $v_i \in \text{Supp} \sigma_i$ so that $\sigma V (\sigma_i) = \sigma (v)$ for all $v \in V$ and $\mu V (\sigma_i \sigma_j) = \sigma_i (v_i) \wedge \sigma_j (v_j) \leq \sigma (v_i) \wedge \sigma (v_j) = \sigma V (\sigma_i) \wedge \sigma V (\sigma_j)$. Therefore, $(\sigma, \mu)$ is a fuzzy subgraph of $(\mathcal{FS}(V), \mathcal{FE})$. We call this fuzzy subgraph as the *underlying fuzzy digraph* of $\mathcal{FR}$.

Given a route system $\mathcal{R}$ on a non-empty set $V$ and a route system $\mathcal{FR}$ on a fuzzy subset $\sigma$ of $V$, we have the underlying digraph $D_{\mathcal{R}}$ on $V$ and the underlying fuzzy digraph $D_{\mathcal{FR}}$ on $\sigma$. We have a simple lemma relating $D_{\mathcal{R}}$ and $D_{\mathcal{FR}}$.

**Lemma 17** The fuzzy digraph $D_{\mathcal{FR}}$ is a fuzzy subdigraph of $D_{\mathcal{R}}$. 
Proof. Now \( \sigma \ast (v) = \sigma \ast (\sigma_v) = \sigma(v) \) and \( \mu \ast (xy) = \mu \ast (\sigma_x \sigma_y) \leq \sigma_x(x) \land \sigma_y(y) = \sigma \ast (\sigma_x) \land \sigma \ast (\sigma_y) \), which completes the proof.

Now we introduce the symmetry axioms to arrive at the undirected analogues of the route systems in the fuzzy graphs.

\[
(Fs0) \quad \sigma_u \sigma_v \in \mathcal{FR} \Rightarrow \sigma_v \sigma_u \in \mathcal{FR},
\]

is the weak symmetry axiom and

\[
(Fs1) \quad F \alpha \in \mathcal{FR} \Rightarrow \overline{F} \alpha \in \mathcal{FR}.
\]

is the strong symmetry or symmetry axiom.

As it is in the crisp graph, the underlying fuzzy subgraph \( G_{FR} = (FS(V), \mathcal{F}(E)) \) of a weakly symmetric route system has vertex set \( FS(V) \) and \( \sigma_u \sigma_v \) is an edge in \( G_{FR} \) only when \( \sigma_u \sigma_v \) lies in \( \mathcal{FR} \) with \( \mu^*(\sigma_u \sigma_v) = \mu^*(\sigma_v \sigma_u) \). We have the analogous lemma in the case of \( G_R \), namely,

Lemma 18 The fuzzy graph \( G_{FR} \) is a fuzzy subgraph of \( G_R \).

We will now derive few results from the forgone discussion. The symmetric or undirected analogues generally follow from the weak symmetry axiom \( (Fs0) \) and other axioms.

We begin with the most general case, that is the route system of all walks before proceeding to desegregate it. Fundamentally two types of axioms are required to grapple with special cases; they are extension axioms to generate new routes from existing ones and exclusion axioms to exclude the non-routes from the route system.

We schematically present the axioms:

\[
(Fr2) \quad \sigma_u \alpha x, \alpha \sigma_x \sigma_v \in \mathcal{FR} \Rightarrow \sigma_u \alpha \sigma_x \sigma_v \in \mathcal{FR},
\]

\[
(Fr3) \quad \sigma_u \sigma_x, \alpha \sigma_x \sigma_v \in \mathcal{FR}, \sigma_u \Phi(\alpha \sigma_x) \neq \alpha \sigma_x \sigma_v \Rightarrow \sigma_u \alpha \sigma_x \sigma_v \in \mathcal{FR},
\]

\[
(Fr5) \quad \sigma_u \sigma_x \sigma_v \in \mathcal{FR} \Rightarrow \sigma_u \Phi(\alpha \sigma_x) \neq \sigma_x \sigma_v;
\]
We can see that as in the crisp case, in the fuzzy digraph $D(\sigma, \mu)$ and fuzzy graph $G(\sigma, \mu)$, the family of all directed walks in $D$ is a route system satisfying $(Fr2)$. Similarly, the family of all directed trails, directed paths, directed minimal paths and directed triangular minimal paths form a route system satisfying the corresponding axioms $(Fr_{i})$, $(Fr(i + 1))$ for $i = 4, 6, 8$ and 10 respectively. Also the walks, trails, paths, minimal paths and triangular minimal paths of $G$ all satisfy $(Fs0)$ and $(Fs1)$ and the respective extension and exclusion axioms.

The family of all strong paths, all strong minimal paths and all strong triangular minimal paths in a fuzzy graph $G$ is a route system satisfying the axioms
(Fr0), (Fr1), (Fr16) and (Fr{i}), (Fr{i+1}) for \( i = 6, 8 \) and 10 respectively. Also the family of all geodesics in a fuzzy graph \( G \) is a route system satisfying (Fr0), (Fr1), (Fr9), (Fr12), (Fr13), (Fr14), (Fr15) and (Fr16). In the following results, we prove the converse.

**Lemma 19** Let \( FR \) and \( FS \) be route systems on \( \sigma \) with \( D_{FR} = D_{FS} \). If \( FR \) and \( FS \) both satisfy (Fr2), then \( FR = FS \).

**Proof.** By inducting the lengths of the routes in \( FR \), we can prove that \( FR \subseteq FS \). Let \( Fr_\alpha = \sigma_1\sigma_2\ldots\sigma_k \) be a route in \( FR \). Let \( k = 1 \), then \( Fr_\alpha \) will lie in \( FS \), by (Fr0). Further, if \( k = 2 \), then \( Fr_\alpha \) will lie in \( FS \). This is so because of the underlying fuzzy digraphs of \( FR \) and \( FS \) have the same arcs. Let us assume that \( k \geq 3 \). By (Fr1), it follows that \( \sigma_1\sigma_2\ldots\sigma_{k-1} \) and \( \sigma_2\ldots\sigma_{k-1}\sigma_k \) are in \( FR \).

Therefore, inductive principle permits us to state that they are in \( FS \). Hence, applying the axiom (Fr2) we can state that \( Fr_\alpha \) lies in \( FS \). In a similar manner we can prove that \( FS \subseteq FR \) and hence the lemma. 

We have an analogous Lemma to that of Lemma 13 which can be proved from Lemma 19, namely,

**Lemma 20** ([10]) Let \( FR \) and \( FS \) be weakly symmetric route systems on \( \sigma \) with \( G_{FR} = G_{FS} \). If \( FR \) and \( FS \) both satisfy (Fr2), then \( FR = FS \).

Analogous to that of the crisp case, the directed walks in a fuzzy digraph, as well as the walks in a fuzzy graph, form a route system satisfying (Fr2). Therefore, we have the following theorem which can be proved from Lemmata 19 and 20.

**Theorem 24** Let \( \sigma \) be a finite nonempty fuzzy subset and let \( FR \) be a route system on \( \sigma \). Then \( FR \) is the family of all directed walks of \( D_{FR} \) if and only if \( FR \) satisfies (Fr2). If \( FR \) satisfies (Fr0), then \( FR \) is the family of all walks of \( G_{FR} \) if and only if \( FR \) satisfies (Fr2).
Lemma 21 Let $\mathcal{F}_R$ and $\mathcal{F}_S$ be route systems on the fuzzy subset $\sigma$ of $V$ with $D_{\mathcal{F}_R} = D_{\mathcal{F}_S}$. If $\mathcal{F}_R$ and $\mathcal{F}_S$ both satisfy $(Fr_{2i})$ and $(Fr_{2i+1})$, then $\mathcal{F}_R = \mathcal{F}_S$, for $i = 2, 3, 4, 5$.

Proof. We prove the first part of the lemma by induction on the lengths of the routes in $\mathcal{F}_R$. Let $F\alpha = \sigma_1\sigma_2 \ldots \sigma_k$ be a route in $\mathcal{F}_R$. When $k = 1$, $F\alpha$ lies in $\mathcal{F}_S$, by $(Fr_0)$. Similarly, if $k = 2$, then $F\alpha$ lies in $\mathcal{F}_S$ since the underlying fuzzy digraphs of $\mathcal{F}_R$ and $\mathcal{F}_S$ are equal. Therefore, we assume that $k \geq 3$. In the instance of $i = 4, 5$ we assume that $\sigma_1\sigma_k$ is not in $D_{\mathcal{F}_R} = D_{\mathcal{F}_S}$. Since $\mathcal{F}_R$ satisfies $(Fr_{2i+1})$, we know that $\sigma_1\sigma_2 \neq \sigma_{k-1}\sigma_k$, in the case when $i = 2$, or $\sigma_1 \neq \sigma_k$, in the case when $i = 3, 4, 5$. By $(Fr_1)$, it follows that $\sigma_1\sigma_2 \ldots \sigma_{k-1}$ and $\sigma_2 \ldots \sigma_{k-1}\sigma_k$ are in $\mathcal{F}_R$. Hence, by induction, they are in $\mathcal{F}_S$. Since $\mathcal{F}_S$ satisfies $(Fr_{2i})$, it follows that $F\alpha$ is in $\mathcal{F}_S$. Similarly, the reverse inclusion follows.

The weakly symmetric analogue of Lemma 21 is given in the following Lemma. As the proof is straightforward we bypass it.

Lemma 22 Let $\mathcal{F}_R$ and $\mathcal{F}_S$ be weakly symmetric route systems on $\sigma$ with $G_{\mathcal{F}_R} = G_{\mathcal{F}_S}$. If $\mathcal{F}_R$ and $\mathcal{F}_S$ both satisfy $(Fr_{2i})$ and $(Fr_{2i+1})$ then $\mathcal{F}_R = \mathcal{F}_S$, for $i = 2, 3, 4, 5$.

Using the Lemmata 21 and 22 we prove the following characterising theorems in which the route systems associated to various types of walks in a fuzzy (di)graph is depicted.

Theorem 25 The route system $\mathcal{F}_R$ on the fuzzy subset $\sigma$ of $V$ is the family of all directed trails of $D_{\mathcal{F}_R}$ if and only if $\mathcal{F}_R$ satisfies $(Fr_4)$ and $(Fr_5)$. If $\mathcal{F}_R$ satisfies $(Fr_0)$, then $\mathcal{F}_R$ is the family of all trails of $G_{\mathcal{F}_R}$ if and only if $\mathcal{F}_R$ satisfies $(Fr_4)$ and $(Fr_5)$. 
Proof. Let $FS$ be the family of all directed trails of $D_{FR}$. Let $\sigma_1\sigma_2\ldots\sigma_{(k-1)}$ and $\sigma_2\ldots\sigma_{(k-1)}\sigma_k$ be two directed trails in $FS$. Since $FS$ is the family of all directed trails $\sigma_1\Phi(\sigma_2\ldots\sigma_{(k-1)}) \neq \sigma_{(k-1)}\sigma_k$. Since $FS$ is the family of all directed trails, $\sigma_1\sigma_2\ldots\sigma_k$ is also a trail and so it is in $FS$. So $(Fr4)$ is satisfied. Since $\sigma_1\sigma_2\ldots\sigma_k \in FR, \sigma_1\Phi(\sigma_2\ldots\sigma(k-2)\sigma(k-1)) \neq \sigma_{(k-1)}\sigma_k$. So $(Fr5)$ is satisfied.

Let $FR$ be a fuzzy route system satisfying $(Fr4)$ and $(Fr5)$ with underlying fuzzy directed graph $D_{FR}$. We have proved that the family of directed trails $FS$ on $D_{FR}$ satisfies $(Fr4)$ and $(Fr5)$. Now we will show that $D_{FR} = D_{FS}$. Clearly, the vertex set of both fuzzy digraphs are the same. Now $\sigma_u\sigma_v \in A(D_{FS})$ if and only if $\sigma_u\sigma_v \in FS \Rightarrow \sigma_u\sigma_v$ is a trail in $D_{FR}$ of length 2 implies that $\sigma_u\sigma_v$ is an arc of $D_{FR}$. Hence, every arc of $D_{FS}$ is an arc of $D_{FR}$. Now let $\sigma_u\sigma_v \in A(D_{FR}) \Rightarrow \sigma_u\sigma_v \in FR$ and $\sigma_u \neq \sigma_v$, since $FR$ satisfies $(Fr4) and (Fr5) \Rightarrow \sigma_u\sigma_v$ is a trail of $D_{FR}$ of length 2 implies that $\sigma_u\sigma_v \in A(D_{FS})$. Hence $D_{FS} = D_{FR}$. So we have two fuzzy route systems $FR$ and $FS$ on the fuzzy subset $\sigma$ of $V$ with the same underlying fuzzy digraphs. Therefore by the Lemma 21, for $i = 2, FR = FS$. So the first part of the theorem is proved.

Similarly we can prove that if $FR$ satisfies $(Fs0)$, then $FR$ is the family of all trails of $G_{FR}$ if and only if $FR$ satisfies $(Fr4)$ and $(Fr5)$. $
$ 

Theorem 26 The route system $FR$ on the fuzzy subset $\sigma$ of $V$ is the family of all directed paths of $D_{FR}$ if and only if $FR$ satisfies $(Fr6)$ and $(Fr7)$. When $FR$ satisfies $(Fs0), FR$ is the family of all paths of $G_{FR}$ if and only if $FR$ satisfies $(Fr6)$ and $(Fr7)$. 

Proof. Let $FS$ be the family of all directed paths of $D_{FR}$. Let $\sigma_1\sigma_2\ldots\sigma_{(k-1)}$ and $\sigma_2\ldots\sigma_{(k-1)}\sigma_k$ be any two directed paths in $FS$. Also $\sigma_1 \neq \sigma_k$, since $\sigma_1\sigma_2\ldots\sigma_{(k-1)}$ and $\sigma_2\ldots\sigma_{(k-1)}\sigma_k$ are directed paths. Since $FS$ is the family of all directed paths, $\sigma_1\sigma_2\ldots\sigma_k$ is in $FS$. Hence $FR$ satisfies $(Fr6)$. Also $\sigma_1\sigma_2\ldots\sigma_k \in FR \Rightarrow \sigma_1 \neq \sigma_k$. So $(Fr7)$ is satisfied.

Let $FR$ be a fuzzy route system satisfying $(Fr6)$ and $(Fr7)$ with underlying fuzzy digraph $D_{FR}$. We have proved that the family of directed paths $FS$ on
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$D_{FR}$ satisfies $(Fr6)$ and $(Fr7)$. Now we will establish that $D_{FR} = D_{FS}$. Clearly we have the vertex set of both fuzzy digraphs are the same. Now $\sigma_u \sigma_v \in A(D_{FS})$ if and only if $\sigma_u \sigma_v \in FS$ is a path in $D_{FS}$ of length 2 implies that $\sigma_u \sigma_v$ is an arc of $D_{FR}$. So every arc of $D_{FS}$ is an arc of $D_{FR}$. Now let $\sigma_u \sigma_v \in A(D_{FR}) \Rightarrow \sigma_u \sigma_v \in FR$ and $\sigma_u \neq \sigma_v$, since $FR$ satisfies $(Fr6)$ and $(Fr7)$ $\Rightarrow \sigma_u \sigma_v$ is a directed path of $D_{FR}$ of length 2 implies that $\sigma_u \sigma_v \in A(D_{FS})$. Hence $D_{FS} = D_{FR}$. So we have two fuzzy route systems $FR$ and $FS$ on the fuzzy subset $\sigma$ of $V$ with the same underlying fuzzy digraphs. Therefore by the Lemma 21, for $i = 3$, $FR = FS$. So the first part of the theorem is proved.

Similarly using Lemma 22(i=3), it can be proved that if $FR$ satisfies $(Fs0)$ then $FR$ is the family of all paths of $G_{FR}$ if and only if $FR$ satisfies $(Fr6)$ and $(Fr7)$.

**Theorem 27** Let $FR$ be a route system on the fuzzy subset $\sigma$ of $V$, then $FR$ is the family of all directed minimal paths of $D_{FR}$ if and only if $FR$ satisfies $(Fr8)$ and $(Fr9)$. If $FR$ satisfies $(Fs0)$, then $FR$ is the family of all minimal paths of $G_{FR}$ if and only if $FR$ satisfies $(Fr8)$ and $(Fr9)$.

**Proof.** Let $FS$ is the family of all directed minimal paths of $D_{FR}$. Let $\sigma_1 \sigma_2 \ldots \sigma_{(k-1)}$ and $\sigma_2 \sigma_3 \ldots \sigma_{k}$ be two directed minimal paths of $FS$. Then $\sigma_1 \neq \sigma_k$ and $\sigma_1 \sigma_k \notin FS$. Since $FS$ is the family of all directed minimal paths, $\sigma_1 \sigma_2 \ldots \sigma_k$ belongs to $FS$. So $(Fr8)$ is satisfied. Since $\sigma_1 \sigma_2 \ldots \sigma_k$ is a directed minimal path $\sigma_1 \sigma_k \notin FS$ so $(Fr9)$ is satisfied.

Let $FR$ be a route system satisfying $(Fr8)$ and $(Fr9)$ with underlying fuzzy digraph $D_{FR}$. We have proved that the family of directed minimal paths $FS$ on $D_{FR}$ satisfies $(Fr8)$ and $(Fr9)$. Now we will show that $D_{FR} = D_{FS}$. Clearly we have the vertex set of both fuzzy digraphs are the same.

Now $\sigma_u \sigma_v \in A(D_{FS})$ if and only if $\sigma_u \sigma_v \in FS$ implies that $\sigma_u \sigma_v$ is a directed minimal path in $D_{FS}$ of length 2 implies that $\sigma_u \sigma_v$ is an arc of $D_{FR}$. So $D_{FS} \subseteq D_{FR}$. Now let $\sigma_u \sigma_v \in A(D_{FR}) \Rightarrow \sigma_u \sigma_v \in FR$ and $\sigma_u \neq \sigma_v$, since $FR$ satisfies $(Fr8)$ and $(Fr9)$ $\Rightarrow \sigma_u \sigma_v$ is a directed minimal path of $D_{FR}$ of
length 2 ⇒ σ_uσ_v is a directed path of D_FR of length 2 ⇒ σ_uσ_v ∈ A(D_FS).
Hence D_FS = D_FR. So by the Lemma 21, for i = 4, FR = FS. So the first part of the theorem is proved.

Similarly using Lemma 22(i=4), it can be proved that if FR satisfies (Fs0) then FR is the family of all minimal paths of G_FR if and only if FR satisfies (Fr8) and (Fr9). ■ The next theorem can be proved in a similar way and so we omit the proof.

Theorem 28 When FR is a route system on the fuzzy subset σ of V, then FR is the family of all directed triangular minimal paths of D_FR if and only if FR satisfies (Fr10) and (Fr11). If FR satisfies (Fs0), then FR is the family of all triangular minimal paths of G_FR if and only if FR satisfies (Fr10) and (Fr11).

Remark 5 Let FR be a route system or a weakly symmetric route system on a fuzzy subset σ of V. And if FR is either the family of all (di)walks, (di)trails, (di) paths or (di) minimal paths in (G_FR)D_FR. Then, it can be easily observed that for any element Fβ = σ_uFασ_v in FR, every sub route Fβt of Fβ is also an element of FR. Therefore, we can state that the family of (di)walks, (di)trails, (di)paths or (di) minimal paths will have the hereditary property.

Lemma 23 If FP be the family of all paths of G_FR (the underlying fuzzy subgraph of a weakly symmetric route system FR on a fuzzy subset σ of V), then FP is the family of all strong paths of G_FR if and only if FP satisfies (Fr16).

Proof. Suppose Fβ = σ_uFασ_v be any path in FP, and let σ_uσ_{u+1} be any edge of G_FR lying in Fβ and H be the fuzzy subgraph of G_FR after deleting the edge σ_uσ_{u+1} in G_FR. Then there arise two cases.

Case 1: W(Fβ) ≥ W(Fγ) for any Fγ = σ_uFδσ_v.

Case 2: There exist two disjoint subpaths of Fβ; they are Fβt and Fβu with Fβ = FβtFβu and W(Fβ) = W(Fβu) = W(Fβt).
We proceed to prove the necessary part first.

Let \( \mathcal{FP} \) satisfies (Fr16). We can prove that every member of \( \mathcal{FP} \) is a strong path in \( G_{\mathcal{FR}} \). Since every non-trivial sub route \( F\beta t \) of \( F\beta \) belongs to \( \mathcal{FP} \) by Remark 5, \( F\beta t \) satisfies (Fr16). Any edge \( \sigma_i\sigma_{i+1} \) of \( F\beta \), in particular, satisfies (Fr16).

Since any path joining \( \sigma_i \) and \( \sigma_{i+1} \) in \( H \) has weight at most \( \mu^*(\sigma_i\sigma_{i+1}) \), we have \( \mu^*(\sigma_i\sigma_{i+1}) \geq W(F\beta) \geq \text{CONN}_H(\sigma_i,\sigma_{i+1}) \). Therefore, the edge \( \sigma_i\sigma_{i+1} \) is strong; this proves that in Case.1 every path in \( \mathcal{FR} \) is strong.

In Case.2 also, we can see that \( \mu^*(\sigma_i\sigma_{i+1}) \geq W(F\beta) \geq \text{CONN}_H(\sigma_i,\sigma_{i+1}) \), since any path joining \( \sigma_i \) and \( \sigma_{i+1} \) in \( H \) has weight at most \( W(F) \). This proves the necessary part of Case.2.

To prove the converse, let \( \mathcal{FP} \) be the family of all strong paths of \( G_{\mathcal{FR}} \). Suppose \( F\beta = \sigma_u F\alpha \delta \sigma_v \) be any path in \( \mathcal{FP} \) with \( W(F\beta) \) which is not maximum. Suppose \( F\gamma = \sigma_u F\delta \sigma_v \) be a path in \( \mathcal{FP} \) of maximum weight and let \( \sigma_i\sigma_{i+1} \) be an edge of \( F\beta \) with \( \mu^*(\sigma_i\sigma_{i+1}) = W(F\beta) \). Further, let \( F\beta t, F\beta u \) be the sub paths \( \sigma_i - \sigma_u \) and \( \sigma_v - \sigma_{i+1} \) of \( F\beta \) respectively. If there is no edge \( \sigma_k\sigma_{k+1} \in F\beta \) with \( \mu^*(\sigma_k\sigma_{k+1}) = W(F\beta) \), then, in \( H \) the path formed by concatenating the paths \( F\beta u, F\gamma = \sigma_u F\delta \sigma_v \) and \( F\beta t \) has weight greater than \( \mu^*(\sigma_i\sigma_{i+1}) \). This is a contradiction due to the fact that \( \sigma_i\sigma_{i+1} \) is a strong edge. Therefore, there exists one more edge \( \sigma_k\sigma_{k+1} \in F\beta \) with \( \mu^*(\sigma_k\sigma_{k+1}) = W(F\beta) \). We can form subpaths \( F\beta t, F\beta u \) of \( F\beta \) such that \( \sigma_i\sigma_{i+1} \in F\beta t \) and \( \sigma_k\sigma_{k+1} \in F\beta u \) with \( F\beta = F\beta t F\beta u \). Clearly, \( W(F\beta t) = W(F\beta u) = W(F\beta) \). We have already proved that if \( F\beta t \) does not satisfy Case.1, then \( F\beta t \) satisfies Case.2. We prove that if \( F\beta t \) does not satisfy Case.2, then it satisfies Case.1. If there is only one edge \( \sigma_i\sigma_{i+1} \) in \( F\beta \) with \( \mu^*(\sigma_i\sigma_{i+1}) = W(F\beta) \), then \( F\beta \) does not satisfy Case.2. Suppose \( F\beta \) does not satisfy Case.1 and let \( F\gamma = \sigma_u F\delta \sigma_v \) be a path in \( \mathcal{FP} \) with \( W(F\gamma) \geq W(F\beta) \). Now arguing in the same manner as in the previous case, we can prove that the edge \( \sigma_i\sigma_{i+1} \) is not strong is a contradiction. This shows that \( F\beta \) satisfies Case.1 and the proof is complete. ■
Theorem 29 The weakly symmetric route system \( FR \) on the fuzzy subset \( \sigma \) of \( V \) is the family of all strong paths of \( G_{FR} \) if and only if \( FR \) satisfies \((Fr6), (Fr7)\) and \((Fr16)\).

Proof. By Theorem 26, \( FR \) is the family of all paths of \( G_{FR} \) if and only if \( FR \) satisfies \((Fr6)\) and \((Fr7)\). The theorem follows from the Lemma 23.

We have analogous theorems on the family of all strong minimal paths and strong triangular minimal paths of \( G_{FR} \). We omit the proof as it is obvious by now.

Theorem 30 The weakly symmetric route system \( FR \) on the fuzzy subset \( \sigma \) of \( V \) is the family of all strong minimal paths of \( G_{FR} \) if and only if \( FR \) satisfies \((Fr8), (Fr9)\) and \((Fr16)\).

Theorem 31 The weakly symmetric route system \( FR \) on the fuzzy subset \( \sigma \) of \( V \) is the family of all strong triangular minimal paths of \( G_{FR} \) if and only if \( FR \) satisfies \((Fr10), (Fr11)\) and \((Fr16)\).

We have an analogoe of the Theorem 32, in the fuzzy case which characterize the family of all geodesics on the underlying fuzzy subgraph \( G_{FR} \) of the symmetric route system on the fuzzy subset \( \sigma \) of \( V \).

Theorem 32 The symmetric route system \( FR \) on the fuzzy subset \( \sigma \) of \( V \) is the family of all geodesics of \( G_{FR} \) if and only if \( FR \) satisfies \((Fr9), (Fr12), (Fr13), (Fr14), (Fr15)\) and \((Fr16)\).

Proof.

Using Theorem 32, we have that the support of \( FR \) is a symmetric route system on the crisp set \( V \) and that the the support of \( FR \) is the family of all geodesics of the support of \( G_{FR} \) if and only if \( FR \) satisfies \((Fr9), (Fr12), (Fr13), (Fr14)\) and the connectivity axiom \((Fr15)\). Therefore, \( FR \) is the family of all paths of shortest length in the fuzzy subgraph \( G_{FR} \) if and only if \( FR \) satisfies \((Fr9), (Fr12), (Fr13), (Fr14)\) and the connectivity axiom \((Fr15)\).
By definition a geodesic in a fuzzy graph is a strong path of shortest length in it. We can see that the family of all paths of shortest length in $G_{F_R}$ has the hereditary property and therefore using the same arguments that are there in the proof of Lemma 23, we can prove that $\mathcal{F}_R$ is the family of all strong paths of shortest length in the fuzzy subgraph $G_{F_R}$ if and only if $\mathcal{F}_R$ satisfies $(Fr9)$, $(Fr12)$, $(Fr13)$, $(Fr14)$, $(Fr15)$ and $(Fr16)$, which completes the proof.

\section{Transit Function}

Let $V$ be a (finite) non empty set. A \textit{transit function} on $V$ is a function $R: V \times V \to 2^V$ satisfying the following axioms (for any $u, v \in V$):

\begin{enumerate}
  \item[(1.1)] $u \in R(u, v)$;
  \item[(1.2)] $R(u, v) = R(v, u)$.
\end{enumerate}

If $V$ is the vertex set of a graph $G$, and $R$ is a transit function on $V(G)$, then we state that $R$ is a transit function on $G$. We can also state that the transit function satisfies the idempotent axiom if

\begin{enumerate}
  \item[(1.3)] $R(u, u) = \{u\}$.
\end{enumerate}

Note that Mulder in [8] included the $(1.3)$ as a defining axiom of a transit function. Here, we have excluded $(1.3)$ axiom in the definition, as we may come across functions that may not satisfy the idempotent axiom.

Examples of transit functions on graphs are provided by the geodesic interval function

$$I(u, v) = \{w \in V \mid w \text{ lies on a shortest } u,v\text{-path in } G\},$$

the induced path function

$$J(u, v) = \{w \in V \mid w \text{ lies on an induced } u,v\text{-path in } G\},$$
and the all-paths function $A_G$ of a graph $G$ defined as

$$A_G(u, v) = \{w \in V \mid w \text{ lies on some } u, v-\text{path in } G\}.$$  

For more examples we refer to [32].

For a symmetric route system $\mathcal{R}$ on a finite set $V$, we recall the connectivity axiom ($r15$)

For $u, v \in V$, if $u \neq v$, there exists some $\alpha \in \mathcal{R}$ such that $u \alpha v \in \mathcal{R}$.

It is to be noted that if $\mathcal{R}$ satisfies the connectivity axiom, then the underlying graph $G_\mathcal{R}$ of $\mathcal{R}$ is connected. Let $R_\mathcal{R} : V \times V \rightarrow 2(V)$ be a function defined as

$$R_\mathcal{R}(u, v) = \{z \in V \mid z \in u \alpha v, \text{ for some } \alpha \in \mathcal{R}\}.$$  

Then $u \in R_\mathcal{R}(u, v)$, since $u \in u \alpha v$ for any element $\alpha \in \mathcal{R}$. As $\mathcal{R}$ is a symmetric route system on $V$, it follows that $R_\mathcal{R}(u, v) = R_\mathcal{R}(v, u)$. Therefore, $R_\mathcal{R}$ satisfies both the axioms (11) and (12) of a transit function. We name $R_\mathcal{R}$ as a transit function derived from a symmetric route system $\mathcal{R}$ which satisfies the connectivity axiom.

Let $G_\mathcal{R}$ be the underlying graph of a symmetric route system satisfying the connectivity axiom. Since $R_\mathcal{R}$ is a transit function on $V(G_\mathcal{R})$, and hence is a transit function on $V$ (forgetting the structure $G_\mathcal{R}$ on $V$). A pertinent question that arises at this juncture is that, given a symmetric route system $\mathcal{R}$ on $V$ satisfying the connectivity axiom, when will an arbitrary transit function $T$ on $V$ coincides with the derived transit function $R_\mathcal{R}$. The solution of this in general appears to be very difficult. Moreover, the answer to this question, when the symmetric route system $\mathcal{R}$ on $V$ becomes the well known routes systems on $G_\mathcal{R}$, is also very difficult in general. However, we can derive suitable axioms on a general transit function $T$ on $G_\mathcal{R}$, when the route system satisfies the axioms of walks, paths and shortest paths of $G_\mathcal{R}$. We state these in the following theorems.
Theorem 33. Let $\mathcal{R}$ be a symmetric route system on $V$ satisfying the connectivity axiom and the (r2) axiom. Then an arbitrary transit function $T$ on $V$ coincides with the derived transit function $R_\mathcal{R}$ if and only if $T(u,v) = V$, for all $u, v \in V$.

Proof. Following Lemma 18, $R_\mathcal{R}(u,v)$ is the set of all vertices lying on all $u-v$ walks of $G_\mathcal{R}$. Since $G_\mathcal{R}$ is connected, we have $R_\mathcal{R}(u,v)$ consisting of the vertex set $V$ of $G_\mathcal{R}$. Therefore, the theorem follows.

Let $R$ be a transit function defined on a set $V$. Then the transit graph $G(R)$ of $R$ is defined as follows. It has $V$ as the vertex set and $uv$ is an edge of $G(R)$ if there is no $x \neq u, v$ such that $R(u, x) \cap R(x, v) = \{x\}$.

Recall that a subgraph $H$ of graph $G$ is a block of $G$ if either $H$ is a bridge (and its end vertices) or it is a maximal 2-connected subgraph of $G$. A basic property of blocks that we use in the sequel is the following: let $u, v, w$ be three distinct vertices of a block $H$, then there exists a path between $u$ and $v$ through $w$ in $H$. A tree of blocks in $G$ is a connected subgraph such that, whenever it contains two distinct vertices $u$ and $v$ of some block of $G$, it contains the whole block. For a connected graph $G$, we define block closure $\tilde{G}$ of $G$ as the graph with vertex $V$, where distinct $u$ and $v$ in $V$ are adjacent in $\tilde{G}$ if and only if $u$ and $v$ belong to the same block in $G$. Clearly $\tilde{G}$ is a block graph, and $G = \tilde{G}$ if and only if $G$ is a block graph.

The following fact is basic for the all-paths transit function of a graph.

Fact 1. Let $G$ be a connected graph with the block closure $\tilde{G}$. Then $A_G = A_{\tilde{G}}$.

We require the following theorem which is proved in [8].

Theorem 34 (Changat et al.) Let $\mathcal{A}$ be a transit function on a set $V$ satisfying the axioms (13), (14), (15) and (16), and let $G_\mathcal{A}$ be the transit graph of $\mathcal{A}$. Then $\mathcal{A}$ is the all-paths transit function of $G_\mathcal{A}$.
Theorem 35 Let $V$ be a finite non-empty set and let $R$ be a symmetric route system on $V$ satisfying the connectivity axiom and the axioms (r6) and (r7). Then an arbitrary transit function $T$ on $V$ coincides with the derived transit function $R_R$ if and only if $T$ satisfies the following axioms:

- (13) $T(u,u) = \{u\}$;
- (14) $w \in T(u,v) \Rightarrow T(w,v) \subseteq T(u,v)$;
- (15) $T(u,x) \cap T(x,v) = \{x\} \Rightarrow T(u,x) \cup T(x,v) = T(u,v)$;
- (16) $T(u,v) \nsubseteq T(u,w), u \neq v \Rightarrow \exists x \in T(u,v), x \neq u$ such that $T(u,x) \cap T(x,w) = \{x\}$ with the condition that the transit graph $G(R_R)$ of $R_R$ is isomorphic to the block closure graph $G_R$.

Proof. Since $R$ is a symmetric route system on $V$ satisfying the connectivity axiom and the axioms (r6) and (r7), $R_R(u,v)$ is the set of all vertices lying on all $u-v$ paths of $G_R$, by Theorem 20. Hence the derived transit function $R_R$ on $V$ is precisely the all paths transit function of $G_R$. The subgraph of $G_R$ induced by the all paths transit function $R_R(u,v)$ is the smallest tree of blocks containing $u$ and $v$; it follows from the basic property of blocks which we have already mentioned. We term this tree of blocks the path of blocks between $u$ and $v$. If $u$ and $v$ are distinct vertices of the same block of $G_R$, then $R_R(u,v)$ induces the block of $G_R$ containing $u$ and $v$. From the above mentioned properties of the all paths transit function $R_R$ of $G_R$, it can be easily verified that $R_R$ satisfies the axioms (t3),(t4),(t5) and (t6). Since $R_R$ is the all paths transit function of $G_R$, it follows that the transit graph $G(R_R)$ of $R_R$ is precisely the block closure of $G_R$. Hence the necessary part follows.

The proof of the converse can be obtained from the Theorem34.

Theorem 36 Let $V$ be a finite nonempty set, and let $R$ be a symmetric route system on $V$ satisfying the axioms (r9), (r12), (r13), (r14) and (r15). Then an arbitrary transit function $T$ on $V$ coincides with the derived transit function $R_R$ if and only if $T$ satisfies the following axioms:
(17): \( v \in T(u,x) \) and \( y \in T(v,x) \) \( \Rightarrow v \in T(u,y) \) and \( y \in T(u,x) \),

(18): \( T(u,u) = \{u\} \),

(19): \( T(u,x) = \{u,x\}, T(v,y) = \{v,y\}, u,v \in T(x,y) \) and \( x \in T(u,v) \) \( \Rightarrow y \in T(u,v) \), for all \( u,v,x,y \in V \),

(20): \( T(u,x) = \{u,x\}, T(v,y) = \{v,y\} \) and \( x \in T(u,v) \) \( \Rightarrow y \in T(x,y) \) or \( x \in T(u,y) \) or \( y \in T(u,v) \), for all \( u,v,x,y \in V \).

We make use of the following Theorem which is proved by Nebeský in [40] to prove the theorem stated above.

**Theorem 37** Let \( G \) be a finite connected graph, and let \( J \) be a transit function on \( G \). Then \( J \) is the interval function of \( G \) if and only if \( J \) satisfies the following axioms \((A), (B), (X)\) and \((Y)\):

\[
(A): v \in J(u,x) \text{ and } y \in J(v,x) \Rightarrow v \in J(u,y) \text{ and } y \in J(u,x),
\]

\[
(B): J(u,u) = \{u\},
\]

\[
(X): \{u,x\}, \{v,y\} \in E(G), u,v \in J(x,y) \text{ and } x \in J(u,v) \Rightarrow y \in J(u,v),
\]

\[
(Y): \{u,x\}, \{v,y\} \in E(G), \text{ and } x \in J(u,v) \Rightarrow v \in J(x,y) \text{ or } x \in J(u,y) \text{ or } y \in J(u,v),
\]

for all \( u,v,x,y \in V(G) \).

**Proof.** Since \( R \) is a symmetric route system on \( V \) satisfying the connectivity axiom and the axioms \((r9), (r12), (r13)\) and \((r14)\), \( R_R(u,v) \) is the set of all vertices lying on all \( u-v \) geodesics of \( G_R \), by Theorem 32. Hence the derived transit function \( R_R \) on \( V \) is precisely the geodesic transit function (interval function according to Mulder [33] and Nebeský in [37]) of \( G_R \). The theorem follows from Nebeský’s Theorem 37. •

We can now proceed to define transit function on a fuzzy subset \( \sigma \). The relation between the derived transit function of a symmetric route system and
an arbitrary transit function on the underlying graph in the crisp case, which are stated in this section can be generalised in a fuzzy framework.

Let $V$ be a finite set. Let $\sigma$ be a non-empty fuzzy subset of a crisp set $V$. A transit function on $\sigma$ is a function $FR : \sigma \times \sigma \rightarrow P(\sigma)$, where $P(\sigma)$ denotes the power set of $\sigma$ satisfying the following axioms (for any $u, \sigma_u, \nu \sigma_u \in \sigma$):

\((F\ell 1)\quad (u, \sigma_u) \in FR(\sigma_u, \sigma_u);\)
\((F\ell 2)\quad FR(\sigma_u, \sigma_u) = FR(\sigma_u, \sigma_u);\)

Given a route system $FR$ on a fuzzy subset $\sigma$ of the crisp set $V$, we can postulate analogous axioms on the transit functions on the underlying fuzzy subgraph $G_{FR}$ and similar theory of transit functions on fuzzy graphs can be developed. We propose this as an interesting problem to be investigated.