Chapter 3

Enumeration of All Fuzzy Quasi-Ordered Relations

3.1 Introduction

An attempt is made in this chapter to generalise the algorithms presented in the previous chapter to determine fuzzy quasi ordered relations on a finite crisp set \( N \). That is, we describe the algorithm for finding a fuzzy quasi ordered relation on \( N \) and, using this algorithm as a basis, we formulate an algorithm for finding the family of all fuzzy quasi ordered relations on \( N \). Subsequently we extend these algorithms on a fuzzy subset \( \sigma \) of \( N \). We follow the same approach that has been taken in chapter 2. An algorithm and strategy for the fuzzy transitive embedding are proposed by Mitamura and Ohuchi. ([27],[26]).

Among the few references for enumerating the fuzzy quasi ordered relations on a crisp as well as fuzzy subset; three notable references in the context of fuzzy structural modelling are that of Mitamura et.al. (See [28],[30],[29]).

Given a finite crisp set a fuzzy quasi-ordered relation is defined on it, that is to structure the set so that it is a fuzzy quasi-ordered set and also given a finite fuzzy set, a fuzzy quasi-ordered relation is defined on it. Also all distinct fuzzy quasi-ordered relations on a finite crisp set as well as fuzzy set is determined. Given a finite crisp set and a fuzzy set, a fuzzy partially-ordered relation is defined on both of them, that is, structuring the sets so that it is fuzzy partially
ordered and also all distinct possible fuzzy partially ordered relations on them are determined.

3.2 Fuzzy Partial Reachability Matrices

We follow Rosenfeld, Moderson et al., Zadeh and Zimmerman ([2], [23], [48], [49]) for the definitions and terminologies needed for the later sections. The membership space of the fuzzy sets can be any lattice $L$, with the meet and join operations denoted as '$\land$' and '$\lor$' respectively. When $L = [0, 1]$, with its usual order, '$\land$' and '$\lor$', for $x, y \in [0, 1]$, $x \land y = \min(x, y)$ and $x \lor y = \max(x, y)$. The range of values in $[0, 1]$ is most commonly used for representing membership grades, but any arbitrary set with some natural full or partial ordering can be used. Elements of this set are not required to be numbers as long as the ordering among them can be interpreted as representing various strengths of membership degree. This generalised membership has the form

$$\mu_A: X \rightarrow L,$$

where $L$ denotes any set that is at least partially ordered. Since $L$ is a lattice, fuzzy sets defined by this generalised membership grade function are called $L$-fuzzy sets. (See [3], [4]). A partial ordering on a set that contains a greatest lower bound and least upper bound for all subsets of the set is called a lattice. In this chapter we take a finite lattice $L \subseteq [0, 1]$ as the membership space.

Let $S = \{1, 2, ..., n\}$ be a crisp set of elements. A fuzzy subset of $S$ is a mapping $\sigma : S \rightarrow [0,1]$ which assigns to each element $x \in S$ a degree of membership, $0 \leq \sigma(x) \leq 1$. A fuzzy relation on $S$ is a fuzzy subset of $S \times S$, that is, it is a mapping $\mu : S \times S \rightarrow [0,1]$ which associates to each pair of elements $(x, y)$ in $S \times S$ a degree of membership, $\mu(x, y) \in [0,1]$. Then $\mu$ is a fuzzy relation on $\sigma$ if $\mu(x, y) \leq \sigma(x) \land \sigma(y)$, where $\sigma$ is a fuzzy subset of $S$ and $\mu$ is a fuzzy relation on $S$. When $\sigma = S$, $\mu$ becomes a fuzzy relation on $S$.

If $\mu$ and $\nu$ are fuzzy relations on $\sigma$, then $\mu \circ \nu$ is defined as $\forall y \in S[\mu(x, y) \land \nu(y, z)]$ for all $x, y \in S$. Let $\mu$ be a fuzzy relation on $\sigma$, then $\mu$ is reflexive
if $\mu(x, x) = \sigma(x)$ for all $x \in S$, $\mu$ is transitive if $(\mu \circ \mu) \leq \mu$, $\mu$ is symmetric if $\mu(x, y) = \mu(y, x)$ for all $x, y \in S$. A relation is called antisymmetric if for $x \neq y$ whenever $\mu(x, y) > 0$ then $\mu(y, x) = 0$ for all $x, y \in S$. The transitivity, symmetry and antisymmetry properties do not depend on $\sigma$.

The following results directly follow from the above.

a. If $\mu$ is reflexive and transitive, then $\mu \circ \mu = \mu$.
b. If $\mu$ and $\nu$ are transitive and $\mu \circ \nu = \nu \circ \mu$, then $\mu \circ \nu$ is transitive.
c. If $\mu$ is reflexive then $\mu \subseteq (\mu \circ \mu)$.
d. If $\mu$ is transitive then $\mu \supseteq (\mu \circ \mu)$.
d. If $\mu$ and $\nu$ are reflexive relations, so is $\mu \circ \nu$.

A fuzzy relation which is transitive and reflexive is called a fuzzy quasi-order relation.

A fuzzy relation that is transitive, reflexive and anti-symmetric is called a fuzzy partial order relation.

A fuzzy digraph $D$ is a pair $(\sigma, \mu)$, where $\sigma$ is a fuzzy subset and $\mu$ a fuzzy relation on $\sigma$. If the fuzzy relation $\mu$ is symmetric, then $D$ is an undirected graph, otherwise it is a directed fuzzy graph. We call the elements of $\sigma$ as fuzzy nodes, and the elements of $\mu$ as arcs of $D$. Let $\mu$ be a fuzzy reflexive, transitive (fuzzy quasi-order) relation on $\sigma$. We denote an element $(s, \sigma(s))$ of $\sigma$ as $s$, and an element $\mu(s, \sigma_l)$ of $\mu$ as $s_\sigma \sigma_l$. Then we can see that $D(\sigma, \mu)$ is a transitive fuzzy digraph in the sense that if $s_i s_j \in \mu$ and $s_j s_k \in \mu$ then $s_i s_k \in \mu$ with $\mu(s_i, s_k) \geq \mu(s_i, s_j) \wedge \mu(s_j, s_k)$.

Consider a fuzzy digraph $D(\sigma, \mu)$. A path $P$ in $D$ is a sequence of distinct nodes $\sigma_x = \sigma_0, \sigma_1, ..., \sigma_n = \sigma_y$ such that $\sigma_{i-1} \sigma_i \in \mu$, $1 \leq i \leq n$ and $n \geq 0$ is called the length of the path. The consecutive pairs $(\sigma_{i-1}, \sigma_i)$ are called the arcs of the path. The strength of the path is defined as $\omega(P) = \wedge_{i=1}^n \mu(\sigma_{i-1}, \sigma_i)$. The maximum of the strengths of all paths in $D$ from $\sigma_x$ to $\sigma_y$ is called $\text{CONN}_D(\sigma_x, \sigma_y)$ and we call $D$ connected if $\text{CONN}_D(\sigma_x, \sigma_y) > 0$ for all $\sigma_x, \sigma_y \in \sigma$. Let $\sigma_s, \sigma_l \in \sigma$. We say that $\sigma_l$ is reachable from $\sigma_s$, if there exists a path in $D(\sigma, \mu)$ from $\sigma_s$ to $\sigma_l$. The relation $\beta$ on $\sigma$ defined as $\beta(\sigma_s, \sigma_l) =$
CONN_D(σ, σ) is called the reachability relation on the node set σ of D(σ, μ).
Then β is a reflexive relation, since β(σ, σ) = σ and β is transitive, since β ∘ β ≤ β. Hence the digraph D(σ, β) is a transitive fuzzy digraph.

**Theorem 10** Let σ be a fuzzy subset of a crisp set S, and μ be a reflexive and transitive fuzzy relation on σ, then D(σ, μ) is precisely D(σ, β).

**Proof.** We have to prove that if σ_i σ_j ∈ μ, then σ_i σ_j ∈ β.

Suppose σ_i σ_j ∈ μ. If there is no path from σ_i to σ_j in D(σ, μ), other than σ_i σ_j, then β(σ_i, σ_j) = μ(σ_i, σ_j) ⇒ σ_i σ_j ∈ β. Suppose that there is a path P in D from σ_i to σ_j. Then since μ is transitive, it follows that μ(σ_i, σ_j) ≥ CONN(σ_i, σ_j) = β(σ_i, σ_j) for any path from σ_i to σ_j. Therefore μ(σ_i, σ_j) ≥ β(σ_i, σ_j). Also if σ_i σ_j ∈ μ for any two nodes σ_i, σ_j ∈ σ, it is obvious that β(σ_i, σ_j) ≥ μ(σ_i, σ_j).

Suppose that σ_i σ_j ∈ β. Since D = (σ, μ) is transitive, we have σ_i σ_j ∈ μ and so β(σ_i, σ_j) ≤ μ(σ_i, σ_j) . Also by the previous argument, β(σ_i, σ_j) ≥ μ(σ_i, σ_j). Therefore β(σ_i, σ_j) = μ(σ_i, σ_j). Hence the theorem.

It is evident that a fuzzy relation σ on μ can be represented as a matrix M whose elements m_{ij}s are precisely, m_{ij} = μ(σ_i, σ_j) for σ_i, σ_j ∈ σ.

The elementwise addition and multiplication of M becomes the \lor and \land operations of σ. That is,

\[
\begin{align*}
m_{ij}m_{lm} &= \land(m_{ij}, m_{lm}) = \min(m_{ij}, m_{lm}) \\
m_{ij} + m_{lm} &= \lor(m_{ij}, m_{lm}) = \max(m_{ij}, m_{lm}).
\end{align*}
\]

Then we call μ a fuzzy relational matrix. If μ is a fuzzy quasi order on σ, then the relational matrix M is a transitive matrix, in the sense that M^2 = M.

Using theorem(10), the matrix M becomes the reachability matrix of D(σ, μ) called the Fuzzy Reachability matrix (FR-matrix). Therefore, the Fuzzy Reachability matrix represents a fuzzy quasi-order relation. Hence we deal with FR-matrices instead of fuzzy quasi-orders. We have a straightforward Theorem...
characterising the fuzzy reachability matrix of a fuzzy digraph with crisp node set $S$.

**Theorem 11** A square matrix $M$ with entries in $[0, 1]$, of order $n$ is the fuzzy reachability matrix of some fuzzy digraph, $D(\sigma, \mu)$ if and only if $M^2 = M$ (transitive), where $\sigma$ is a fuzzy subset of a crisp set $S$ with $|S| = n$.

When we consider a fuzzy subset of $\sigma$, in theorem 11, the conditions is changed as follows. $\mu(x, x) = \sigma(x)$ (reflexive) $M^2 = M$ (transitive).

**Remark 2** The fuzzy digraph $D(R_F)$ corresponding to a reachability matrix $M$ is a transitive fuzzy digraph

A Partially Filled fuzzy Reflexive Matrix (PFRM) is a matrix $M$ whose elements $m_{ij}$ are either a known $\beta(0 \leq \beta \leq 1)$ or an unknown $x(0 \leq x \leq 1)$ and all main diagonal elements are one.

**Notations** Let $M$ be a fuzzy relational matrix. Then for any $\alpha \in [0, 1]$, we denote the entry $m_{ij}$ as $\alpha_M$ if $m_{ij} \geq \alpha$ and as $\alpha_m$ if $m_{ij} < \alpha$. We use this notations in the rest of this chapter.

**Remark 3** Suppose there exists entries $m_{ij}, m_{ik}, m_{kj}$ in a PFRM, $M$ such that $m_{ij} < \min(m_{ik}, m_{kj})$, then $M$ cannot be transitive since there is an entry which violates transitivity. Therefore we say that the matrix $M$ is consistent if there exists no index triplet $(i, j, k)$ satisfying $m_{ij} < \min(m_{ik}, m_{kj})$.

Similarly, if $m_{ij}$ is an unknown element $x$ in $M$ satisfying any one of the conditions for a given $\alpha$

\[
\begin{align*}
    m_{ij} &= x & m_{ki} &= \alpha_M & m_{kj} &= \alpha_m \\
    m_{ij} &= x & m_{ik} &= \alpha_m & m_{jk} &= \alpha_M \\
    m_{ij} &= x & m_{ik} &= \alpha_M & m_{kj} &= \alpha_M \\
    m_{ij} &= x & m_{ik} &= \alpha_m & m_{kj} &= \alpha_m \\
    m_{ij} &= x & m_{ik} &= \alpha_M & m_{kj} &= \alpha_m \\
    m_{ij} &= x & m_{ik} &= \alpha_m & m_{kj} &= \alpha_M.
\end{align*}
\]
(where the $\alpha_M$'s and $\alpha_m$'s may not be the same) then $M$ cannot be transitive since there exists an unknown which can be inferred from transitivity. Therefore we say that the matrix $M$ is **maximal** if there exists no index triplet $(i, j, k)$ satisfying the above conditions.

By Remark 3, we define a Fuzzy Partial Reachability matrix as follows.

**Definition 2** A Fuzzy Partially Filled Matrix $M$ is called a Fuzzy Partial Reachability Matrix (FPR-matrix), if $M$ is consistent and maximal.

We have, in particular, two extreme cases of FPR-matrices, they are the FPR-matrix whose entries are all unknown, except the diagonals, denoted as $FPR_0$ and the fuzzy reachability matrix itself (that is a FPR-matrix whose entries are all known), denoted as $FR$. A FPR matrix which is not a fuzzy reachability matrix is called a non-trivial FPR-matrix.

**Lemma 7** If all the unknowns of a FPR-matrix are set to the minimum of the known values, the resultant matrix is a fuzzy reachability matrix.

**Proof.** Let $M$ be the FPR-matrix and $Mt$ be the fuzzy matrix which is obtained by setting all unknown entries of $M$ to the minimum of the known values. Obviously, $Mt + I = Mt$. To prove that $(Mt)^2 = Mt$.

Because $M$ is an FPR-matrix

- If $m_{ij} = \alpha$, Then $m_{tij} = \alpha$. Since $M$ is an FPR-matrix, $m_{ij} \geq \min(m_{ik}, m_{kj})$.
  - When $m_{ij} = \min(m_{ik}, m_{kj})$, then it is obvious that $\max[\min_k(m_{ik}, m_{kj})] = m_{ij}$. Therefore $m_{tij} = \alpha$, $\max[\min_k(m_{tik}, m_{tkj})] = m_{tij} = \alpha$.
  - When $m_{ij} > \min(m_{ik}, m_{kj})$ for $k \neq i, j$. But $\min(m_{ik}, m_{ij}) = m_{ij}$ and $\min(m_{ij}, m_{jj}) = m_{ij}$. Therefore $\max[\min_k(m_{ik}, m_{kj})] = m_{ij}$. Therefore $m_{tij} = \alpha$, $\max[\min_k(m_{tik}, m_{tkj})] = m_{tij}$. Hence $Mt$ is a fuzzy reachability matrix.

By theorem 11, we have that corresponding to a fuzzy reachability matrix $M$ there exists a transitive fuzzy digraph $D$ whose reachability matrix is precisely $M$. 
Theorem 12 There exists a chain (totally ordered collection) of FPR-matrices in $\mathcal{M}$, namely $M_1, M_2, \ldots, M_r$ with $M_1 = FPR_0$ and $M_r = FR$ such that there exists no $M \in \mathcal{M}$ with $M_i < M < M_i + 1$, for $i = 1, 2, \ldots, r - 1$.

Proof.

We begin with the FPR-matrix $FPR_0$. Let $FPR_0$ be denoted as $M_1$. We give a value $\alpha_M$ or $\alpha_m$ to any unknown entry $x$ in $M_1$, and let the resulting matrix be $M_2$. We can affirm that $M_2$ is a FPR-matrix and $M_1 \leq M_2$ and there is no $M \in \mathcal{M}$ with $M_1 < M < M_2$. Again we give a value $\alpha_M$ or $\alpha_m$ for an unknown entry $x$ in $M_2$. Now the resulting partially filled matrix may not be a PR-matrix, but by giving some unknown values $x$ in the resulting matrix either $\alpha_m$ or $\alpha_M$ so that the consististency and maximality conditions of the FPR-matrix are satisfied, it can be made a FR-matrix. Let the resulting matrix be denoted as $M_3$. Thus we obtained a nested sequence $M_1, M_2, M_3$ in $\mathcal{M}$ with no $M \in \mathcal{M}$ with $M_i < M < M_i + 1$, for $i = 1, 2$. Continuing this procedure successively, by induction we can prove the theorem.

Now we discuss the algorithms which will finally produce the nested sequence of FPR-matrices in the Theorem 12.

3.3 Algorithms for Fuzzy Partial Reachability Matrices

In this section we discuss algorithms to generate a fuzzy quasi-order and fuzzy partial orders on a crisp set $S$ and also algorithms to generate all fuzzy quasi-orders and fuzzy partial orders. Algorithms to generate fuzzy quasi and partial orders on a finite fuzzy set is also discussed in this section.

We first define certain index sets which are used in the algorithms.

Definition: Let $M$ be a FPR-matrix. For each index $i$, the following index sets are defined.
$L_M(i) = \{k; m_{ik} = \alpha_M\}$

$D_M(i) = \{k; m_{ki} = \alpha_M\}$

$L_M(i) = \{k; m_{ik} = \alpha_m\}$

$D_M(i) = \{k; m_{ki} = \alpha_m\}$

$KM(M) = \{(p, q); m_{pq} = \alpha_M\}$

$Km(M) = \{(p, q); m_{pq} = \alpha_m\}$

where $\alpha_m, \alpha_M \in L$, where $\alpha'_m s$ are all $\beta'_i s < \alpha$ and $\alpha'_M s$ are all $\beta'_i s \geq \alpha$, for a particular $\alpha \in [0, 1]$, where $\beta'_i s \in L$.

**Definition 4** Let $M$ be a FPR-matrix. For the element $m_{ij}$, the following index sets are defined:

$Y_{MaxMax}(m_{ij}) = Y_{MM}(m_{ij}) = X_1 \cup X_2 \cup X_3 \cup X_4$ where

$X_1 = \{(p, q); p \in D_M(i) \text{ and } q \in L_M(j)\}$

$X_2 = \{(p, q); p \in L_M(j) \text{ and } q \in L_M(i)\}$

$X_3 = \{(p, q); p \in D_M(j) \text{ and } q \in L_M(i)\}$

$X_4 = \{(p, q); p \in L_M(i) \text{ and } q \in L_M(j)\}$

$Y_{Maxmin}(m_{ij}) = Y_{Mm}(m_{ij}) = X_5 \cup X_6 \cup X_7 \cup X_8$ where

$X_5 = \{(p, q); p \in L_M(j) \text{ and } q \in L_M(i)\}$

$X_6 = \{(p, q); p \in D_M(j) \text{ and } q \in D_M(i)\}$

$X_7 = \{(p, q); p \in L_M(i) \text{ and } q \in D_M(i)\}$

$X_8 = \{(p, q); p \in D_M(i) \text{ and } q \in D_M(j)\}$

$Y_{minmin}(m_{ij}) = Y_{mm}(m_{ij}) = X_9 \cup X_{10} \cup X_{11} \cup X_{12} \cup X_{13}$ where

$X_9 = \{(p, q); p \in L_M(i) \text{ and } q \in L_M(j)\}$

$X_{10} = \{(p, q); p \in D_M(i) \text{ and } q \in L_M(j)\}$

$X_{11} = \{(p, q); p \in D_M(i) \text{ and } q \in L_M(j)\}$

$X_{12} = \{(p, q); p \in D_M(i) \text{ and } q \in L_M(j)\}$

$X_{13} = \{(p, q); p \in D_M(i) \text{ and } q \in L_M(j)\}$
\begin{align*}
W \text{MaxMax}(m_{ij}) & = W_{MM}(m_{ij}) = \{(p, q); m_{pq} = x, \text{ and } (p, q) \in Y_{MM}(m_{ij})\} \\
W \text{Maxmin}(m_{ij}) & = W_{Mm}(m_{ij}) = \{(p, q); m_{pq} = x, \text{ and } (p, q) \in Y_{Mm}(m_{ij})\} \\
W \text{minmin}(m_{ij}) & = W_{mm}(m_{ij}) = \{(p, q); m_{pq} = x, \text{ and } (p, q) \in Y_{mm}(m_{ij})\}
\end{align*}

\begin{align*}
Z_{MM}(m_{ij}) & = \{Y_{MM}(m_{ij}) \cap KM(M)\} \\
Z_{Mm}(m_{ij}) & = \{Y_{Mm}(m_{ij}) \cap Km(M)\} \\
Z_{mm}(m_{ij}) & = \{Y_{mm}(m_{ij}) \cap Km(M)\}
\end{align*}

**Algorithm 7 (Implication Algorithm for Fuzzy Reachability Matrices)**

\textit{BEGIN}

\textit{supply the value } \text{max}(\alpha_M) \text{ or } \text{min}(\alpha_m) \text{ to the unknown } m_{ij};

\textit{Step 1}

\textit{IF (supplied value is max) THEN BEGIN}

\textit{set } m_{ij} = \alpha_M \text{ (a value greater than or equal to } \alpha),

\textit{Step 2}

\textit{set } m_{pq} = \beta_2;

\text{where } \beta_2 = \min(m_{ij}, m_{pi}, m_{jq}) \text{ when } (p, q) \in X_1 \text{ of } Y_{MM}(m_{ij})

= \min(m_{ij}, m_{pj}, m_{iq}) \text{ when } (p, q) \in X_2 \text{ of } Y_{MM}(m_{ij})

= \min(m_{ij}, m_{pj}, m_{iq}) \text{ when } (p, q) \in X_3 \text{ of } Y_{MM}(m_{ij})

= \min(m_{ij}, m_{ip}, m_{jq}) \text{ when } (p, q) \in X_4 \text{ of } Y_{MM}(m_{ij})

\text{for all } (p, q) \in Y_{MM}(m_{ij}), \text{ (Max } \Rightarrow \text{ Max implication)}

\textit{set } m_{pq} = \beta_1;

\text{where } \beta_1 = m_{iq} \text{ when } (p, q) \in X_5 \text{ of } Y_{Mm}(m_{ij})

= m_{pj} \text{ when } (p, q) \in X_6 \text{ of } Y_{Mm}(m_{ij})

= m_{jq} \text{ when } (p, q) \in X_7 \text{ of } Y_{Mm}(m_{ij})

= m_{pi} \text{ when } (p, q) \in X_8 \text{ of } Y_{Mm}(m_{ij});

\text{for all } (p, q) \in Y_{Mm}(m_{ij}), \text{ (Max } \Rightarrow \text{ min implication)} \text{ END}

\textit{ELSE BEGIN}

\textit{Step 3}

\textit{set } m_{ij} = \alpha_m \text{ (a value less than } \alpha), \ i \neq j;

\textit{Step 4}
set $m_{pq} = \beta_0$;

where $\beta_0 = m_{ij}$ when $(p, q) \in X_9$ of $Y_{mm}(m_{ij})$

= $m_{ij}$ when $(p, q) \in X_{10}$ of $Y_{mm}(m_{ij})$

= $\min(m_{ij}, m_{jq}, m_{pi})$ when $(p, q) \in X_{11}$ of $Y_{mm}(m_{ij})$

= $\min(m_{ij}, m_{pi})$ when $(p, q) \in X_{12}$ of $Y_{mm}(m_{ij})$

= $\min(m_{ij}, m_{jq})$ when $(p, q) \in X_{13}$ of $Y_{mm}(m_{ij})$

for all $(p, q) \in Y_{mm}(m_{ij})$; (min => min implication)

**Time complexity**

For a given matrix of order $n$, the time complexity of the algorithm is $O(n^2)$, since the computation of the index sets $Y_{MM}$, $Y_{Mm}$ and $Y_{mm}$ takes atmost $O(n^2)$ times.

IF $M$ is a PR matrix, let $M(k)$ be the PR matrix obtained from $M$ by applying the implication algorithm supplying the value $k = \alpha_m$ or $\alpha_M$ for the next unknown from the $\beta$ set with cardinality $m$. (the next unknown is the first unknown in the dictionary order).

Algorithm AFRM (all fuzzy reachability matrices) Input a FPR matrix $M$ and output all reachability matrices greater than $M$.

**Algorithm 8 (AFRM)**

**STEP1**: If $M$ is a fuzzy reachability matrix, output $M$ and go to **STEP 4**

**STEP2**: ARM$(M(\alpha_m))$

**STEP3**: ARM$(M(\alpha_M))$

**STEP4**: STOP.

**Time complexity**

For a matrix of order $n$, at each stage of the algorithm there are $mn^2$ computations. Therefore in the worst case, there are atmost $(mn^2)n^2-n$ computations. So the worst time complexity is $O((mn^2)n^2-n)$ and this is the worst bound.
Algorithm 9 (Implication Algorithm for FPR Matrices on a fuzzy subset \( \sigma \))

Perform algorithm 7 by replacing Step 1 by 1' and step 3 by 3', where Step 1' and 3' is as follows.

Step 1'

IF (supplied value is max) THEN BEGIN

set \( m_{ij} = \alpha_M \) (a value greater than or equal to \( \alpha \) and the value supplied should be less than or equal to \( \sigma(i) \land \sigma(j) \)),

Step 3'

set \( m_{ij} = \alpha_m \) (a value less than \( \alpha \) and the value supplied should be less than or equal to \( \sigma(i) \land \sigma(j) \)), \( i \neq j \).

Lemma 8 If \( M \) is a fuzzy partially filled reachability matrix, then

\[
Y_{MM}(m_{ij}) \cap Y_{Mm}(m_{ij}) = \phi
\]

Proof.

To prove that \( Y_{MM}(m_{ij}) \cap Y_{Mm}(m_{ij}) = \phi \), it is enough to show that \( X_i \cap X_j = \phi \) for \( i = 1, 2, 3, 4 \) and \( j = 5, 6, 7, 8 \).

Suppose that \( (p, q) \in Y_{MM}(m_{ij}) \cap Y_{Mm}(m_{ij}) \), then \( m_{ij} = \alpha_M \) and \( (p, q) \in X_i \cap X_j \).

Case(1)

When \( (p, q) \in X_1 \cap X_5 \), then \( m_{pi} = \alpha_M, m_{jq} = \alpha_M \) and \( m_{jp} = \alpha_M \), \( m_{iq} = \alpha_m \). Hence \( m_{ij} = \alpha_M, m_{jq} = \alpha_M \Rightarrow m_{iq} = \min(m_{ij}, m_{jq}) \) which is a contradiction. So \( (p, q) \notin X_1 \cap X_5 \).

When \( (p, q) \in X_1 \cap X_6 \), then \( m_{pi} = \alpha_M, m_{jq} = \alpha_M \) and \( m_{pj} = \alpha_m \), \( m_{qi} = \alpha_M \). Hence \( m_{ij} = \alpha_M, m_{pi} = \alpha_M \Rightarrow m_{pj} = \min(m_{pi}, m_{ij}) \) which is a contradiction. So \( (p, q) \notin X_1 \cap X_6 \).

When \( (p, q) \in X_1 \cap X_7 \), then \( m_{pi} = \alpha_M, m_{jq} = \alpha_M \) and \( m_{ip} = \alpha_M, m_{jq} = \alpha_m \). So it is obvious that \( (p, q) \notin X_1 \cap X_7 \). So \( (p, q) \notin X_1 \cap X_7 \).

When \( (p, q) \in X_1 \cap X_8 \), then \( m_{pi} = \alpha_M, m_{jq} = \alpha_M \) and \( m_{pi} = \alpha_m, m_{qj} = \alpha_M \). So it is obvious that \( (p, q) \notin X_1 \cap X_8 \). So \( (p, q) \notin X_1 \cap X_8 \).
Case (2)

When \((p, q) \in X_2 \cap X_5\), then \(m_{jp} = \alpha_M\), \(m_{iq} = \alpha_M\) and \(m_{jp} = \alpha_M\), \(m_{iq} = \alpha_M\). So it is obvious that \((p, q) \notin X_2 \cap X_5\). So \((p, q) \notin X_2 \cap X_5\).

When \((p, q) \in X_2 \cap X_6\), then \(m_{jp} = \alpha_M\), \(m_{iq} = \alpha_M\) and \(m_{pq} = \alpha_M\), \(m_{iq} = \alpha_M\). Hence \(m_{ij} = \alpha_M\), \(m_{jp} = \alpha_M\) \(\Rightarrow m_{iq} = \min(m_{ij}, m_{jp}) = \alpha_M\). So \(m_{ip} = \alpha_M\), \(m_{pq} = \alpha_M\) \(\Rightarrow m_{ij} = \min(m_{ip}, m_{pq}) = m_{ij} = \alpha_M\), which is a contradiction. So \((p, q) \notin X_2 \cap X_6\).

When \((p, q) \in X_2 \cap X_7\), then \(m_{jp} = \alpha_M\), \(m_{iq} = \alpha_M\) and \(m_{ip} = \alpha_M\), \(m_{pq} = \alpha_M\). Hence \(m_{ij} = \alpha_M\) and \(m_{jq} = \alpha_M\) \(\Rightarrow m_{iq} = \min(m_{ij}, m_{jq}) = m_{pq} = \alpha_M\), which is a contradiction. So \((p, q) \notin X_2 \cap X_7\).

When \((p, q) \in X_2 \cap X_8\), then \(m_{jp} = \alpha_M\), \(m_{iq} = \alpha_M\) and \(m_{ip} = \alpha_M\), \(m_{pa} = \alpha_M\). Hence \(m_{ij} = \alpha_M\) and \(m_{ji} = \alpha_M\) implies \(m_{ij} = \min(m_{pi}, m_{iq}) = m_{pi} = \alpha_M\). So \(m_{pi} = \alpha_M\), \(m_{pq} = \alpha_M\) \(\Rightarrow m_{ij} = \min(m_{pi}, m_{pq}) = m_{pq} = \alpha_M\). So \(m_{pi} = \alpha_M\), \(m_{pq} = \alpha_M\) \(\Rightarrow m_{ij} = \alpha_M\) which is a contradiction, since \(m_{ij} = \alpha_M\). So \((p, q) \notin X_2 \cap X_8\).

Case (3)

When \((p, q) \in X_3 \cap X_5\), then \(m_{ij} = \alpha_M\), \(m_{iq} = \alpha_M\) and \(m_{jp} = \alpha_M\), \(m_{iq} = \alpha_M\). So it is obvious that \((p, q) \notin X_3 \cap X_5\). So \((p, q) \notin X_3 \cap X_5\).

When \((p, q) \in X_3 \cap X_6\), then \(m_{jp} = \alpha_M\), \(m_{iq} = \alpha_M\) and \(m_{pj} = \alpha_M\), \(m_{iq} = \alpha_M\). So it is obvious that \((p, q) \notin X_3 \cap X_6\). So \((p, q) \notin X_3 \cap X_6\).

When \((p, q) \in X_3 \cap X_7\), then \(m_{jp} = \alpha_M\), \(m_{iq} = \alpha_M\) and \(m_{ip} = \alpha_M\), \(m_{iq} = \alpha_M\). Hence \(m_{ij} = \alpha_M\), \(m_{jp1} = \alpha_M\) \(\Rightarrow m_{ij} = \min(m_{ij}, m_{jp}) = m_{ij} = \alpha_M\), which is a contradiction. So \((p, q) \notin X_3 \cap X_7\).

When \((p, q) \in X_3 \cap X_8\), then \(m_{jp} = \alpha_M\), \(m_{iq} = \alpha_M\) and \(m_{ip} = \alpha_M\), \(m_{pq} = \alpha_M\). Hence \(m_{pi} = \alpha_M\), \(m_{ij} = \alpha_M\) \(\Rightarrow m_{pq} = \min(m_{pi}, m_{ij}) = m_{pq} = \alpha_M\) which is a contradiction. So \((p, q) \notin X_3 \cap X_8\).

Case (4)

When \((p, q) \in X_4 \cap X_5\), then \(m_{ip} = \alpha_M\), \(m_{jq} = \alpha_M\) and \(m_{jp} = \alpha_M\), \(m_{iq} = \alpha_M\). Hence \(m_{ij} = \alpha_M\), \(m_{jq} = \alpha_M\) \(\Rightarrow m_{iq} = \min(m_{ij}, m_{jq}) = \alpha_M\) which is a contradiction. So \((p, q) \notin X_4 \cap X_5\).
When \((p, q) \in X_4 \cap X_6\), then \(m_{pi} = \alpha_M, m_{jq} = \alpha_M\) and \(m_{pj} = \alpha_m, m_{qi} = \alpha_M\). Hence \(m_{pi} = \alpha_M, m_{pq} = \alpha_m \Rightarrow m_{ij} = \min(m_{ip}, m_{pj}) = m_{pj} = \alpha_m\) which is a contradiction, since \(m_{ij} = \alpha_M\). So \((p, q) \notin X_4 \cap X_6\).

When \((p, q) \in X_4 \cap X_7\), then \(m_{ip} = \alpha_M, m_{jq} = \alpha_M\) and \(m_{ip} = \alpha_M, m_{jq} = \alpha_m\). So it is obvious that \((p, q) \notin X_4 \cap X_7\). So \((p, q) \notin X_4 \cap X_7\).

Therefore \((p, q) \notin Y_{MM}(m_{ij}) \cap Y_{MM}(m_{ij})\) and so \(Y_{MM}(m_{ij}) \cap Y_{MM}(m_{ij}) = \phi\). Hence the result. ■

Lemma 9 \(Y_{MM}(m_{ij})\) is the disjoint union of \(W_{MM}(m_{ij})\) and \(Z_{MM}(m_{ij})\), \(Y_{MM}(m_{ij})\) the disjoint union of \(W_{MM}(m_{ij})\) and \(Z_{MM}(m_{ij})\) and \(Y_{MM}(m_{ij})\) the disjoint union of \(W_{MM}(m_{ij})\) and \(Z_{MM}(m_{ij})\) respectively.

Proof.

\(W_{MM}(m_{ij}) \cap Z_{MM}(m_{ij}) = \phi\) and \(Y_{MM}(m_{ij}) \supset W_{MM}(m_{ij}) \cup Z_{MM}(m_{ij})\) are obvious by the definitions. For any \((p, q) \in Y_{MM}(m_{ij})\), only one of \(m_{pq} = x\) or \(m_{pq} = \alpha_M\) or \(m_{pq} = \alpha_m\) is true. Hence \((p, q) \in W_{MM}(m_{ij}) \cup KM(M) \cup Km(M)\).

If \((p, q) \in Km(M)\), then \(m_{pq} = \alpha_m\). But since \((p, q) \in Y_{MM}(m_{ij})\), then \(m_{ij} = \alpha_M\) and

When \((p, q) \in X_1\), \(m_{pi} = \alpha_M, m_{jq} = \alpha_M\). Then \(m_{ij} = \alpha_M, m_{pi} = \alpha_M \Rightarrow m_{pq} = \min(m_{ij}, m_{pi})\) which is a contradiction. So \((p, q) \notin Km(M)\).

When \((p, q) \in X_2\), \(m_{jp} = \alpha_M, m_{iq} = \alpha_M, m_{pq} = \alpha_m \Rightarrow m_{jq} = \min(m_{jp}, m_{pq})\). So \(m_{ij} = \alpha_M, m_{jq} = \alpha_m \Rightarrow m_{iq} = m_{jq} = \alpha_m\), which is a contradiction. So \((p, q) \notin Km(M)\).
Enumeration of All Fuzzy Quasi-Ordered Relations

When \((p, q) \in X_3\), \(m_{pq} = \alpha_M, m_{iq} = \alpha_M, m_{pq} = \alpha_m\). Hence \(m_{pi} = \min(m_{pq}, m_{iq}) = m_{pq}\). So \(m_{pi} = \alpha_m, m_{ij} = \alpha_M \Rightarrow m_{pj} = \min(m_{pi}, m_{ij}) = m_{pi} = \alpha_m\) which is a contradiction. So \((p, q) \notin K_M(M)\).

When \((p, q) \in X_4\), \(m_{ip} = \alpha_M, m_{jq} = \alpha_M\). Then \(m_{ij} = \alpha_M, m_{jq} = \alpha_M \Rightarrow m_{iq} = \min(m_{ij}, m_{jq})\). So \(m_{iq} = \alpha_M, m_{ip} = \alpha_M \Rightarrow m_{pq} = \alpha_M\) which is a contradiction. So \((p, q) \notin K_M(M)\).

Consequently, \(Y_{MM}(m_{ij}) \subseteq W_{MM}(m_{ij}) \cup K_M(M)\)
\(Y_{MM}^{-1}(m_{ij}) \subseteq Y_{MM}^{-1}(m_{ij}) \cap (W_{MM}(m_{ij}) \cup K_M(M))\)
\[= W_{MM}(m_{ij}) \cup Z_{MM}(m_{ij})\]
\[= W_{MM}(m_{ij}) \oplus Z_{MM}(m_{ij}).\]

To prove that \(Y_{MM}(m_{ij}) = W_{MM}(m_{ij}) \oplus Z_{MM}(m_{ij})\), \(W_{MM}(m_{ij}) \cap Z_{MM}(m_{ij}) = \emptyset\) and \(Y_{MM}(m_{ij}) \cap W_{MM}(m_{ij}) \cup Z_{MM}(m_{ij}) = W_{MM}(m_{ij}) \cup Z_{MM}(m_{ij})\).

When \((p, q) \in X_5\), \(m_{jp} = \alpha_M, m_{iq} = \alpha_M, m_{pq} = \alpha_M\). Therefore \(m_{pq} = \alpha_M, m_{jp} = \alpha_M \Rightarrow m_{jq} = \min(m_{pq}, m_{jp})\). So \(m_{iq} = \alpha_m, m_{jq} = \alpha_M \Rightarrow m_{ij} = \min(m_{iq}, m_{jq})\) which is a contradiction. So \((p, q) \notin K_M(M)\).

When \((p, q) \in X_6\), \(m_{pj} = \alpha_M, m_{qi} = \alpha_M, m_{pq} = \alpha_M\). Therefore \(m_{pq} = \alpha_M, m_{pq} = \alpha_M \Rightarrow m_{jq} = \min(m_{pq}, m_{jq})\). So \(m_{ij} = \alpha_m\) which is a contradiction. So \((p, q) \notin K_M(M)\).

When \((p, q) \in X_7\), \(m_{ip} = \alpha_M, m_{jq} = \alpha_M, m_{pq} = \alpha_M\). Therefore \(m_{ip} = \alpha_M, m_{ip} = \alpha_M \Rightarrow m_{ij} = \min(m_{ip}, m_{pq})\) and so \(m_{ij} = \min(m_{ip}, m_{pq}) \Rightarrow m_{ij} = \min(m_{ij}, m_{jq}) = m_{jq} = \alpha_m\), which is a contradiction. So \((p, q) \notin K_M(M)\).

When \((p, q) \in X_8\), \(m_{pi} = \alpha_m, m_{iq} = \alpha_M, m_{pq} = \alpha_M\). Therefore \(m_{pi} = \alpha_m, m_{pi} = \alpha_m \Rightarrow m_{pq} = \min(m_{pi}, m_{pq})\) and so \(m_{ij} = \alpha_m, m_{ij} = \alpha_m \Rightarrow m_{pq} = \min(m_{pq}, m_{iq}) = \alpha_M\), which is a contradiction. So \((p, q) \notin K_M(M)\).

Consequently, \(Y_{MM}(m_{ij}) \subseteq W_{MM}(m_{ij}) \cup K_M(M)\)
\(Y_{MM}^{-1}(m_{ij}) \subseteq Y_{MM}^{-1}(m_{ij}) \cap (W_{MM}(m_{ij}) \cup K_M(M))\)
\[= W_{MM}(m_{ij}) \cup Z_{MM}(m_{ij})\]
\[= W_{MM}(m_{ij}) \oplus Z_{MM}(m_{ij}).\]
To prove that \( Y_{mn}(m_{ij}) = W_{mn}(m_{ij}) \cup Z_{mn}(m_{ij}) \).

\( W_{mn}(m_{ij}) \cap Z_{mn}(m_{ij}) = \emptyset \) and \( Y_{mn}(m_{ij}) \supset W_{mn}(m_{ij}) \cup Z_{mn}(m_{ij}) \) are obvious by the definitions. For any \((p, q) \in Y_{mn}(m_{ij})\), only one of \( m_{pq} = \alpha_M \) or \( m_{pq} = \alpha_m \) is true. Hence \((p, q) \in W_{mn}(M_{ij}) \cup KM(M) \cup Km(M)\).

If \((p, q) \in KM(M)\), then \((m_{pq} = \alpha_M)\). But since \((p, q) \in Y_{mn}(m_{ij})\),

When \((p, q) \in X_9 \) (\( m_{ip} = \alpha_M \) and \( m_{iq} = \alpha_M \), \( m_{ij} = \alpha_m \)). Therefore \( m_{ip} = \alpha_M \), \( m_{ij} = \alpha_m \Rightarrow m_{pj} = \min(m_{ip}, m_{ij}) \). So \( m_{pq} = \min(m_{pj}, m_{iq}) = m_{pj} = \alpha_m \), which is a contradiction. So \((p, q) \notin KM(M)\).

When \((p, q) \in X_{10} \) (\( m_{pi} = \alpha_M \) and \( m_{jq} = \alpha_M \), \( m_{ij} = \alpha_m \)). Therefore \( m_{jq} = \alpha_M \), \( m_{ij} = \alpha_m \Rightarrow m_{iq} = \min(m_{jq}, m_{ij}) \). So \( m_{pq} = \min(m_{pi}, m_{iq}) = \alpha_m \), which is a contradiction. So \((p, q) \notin KM(M)\).

When \((p, q) \in X_{11} \) (\( m_{pi} = \alpha_m \) and \( m_{jq} = \alpha_m \), \( m_{ij} = \alpha_m \)). Therefore \( m_{ij} = \alpha_m \), \( m_{jq} = \alpha_m \Rightarrow m_{iq} = \min(m_{jq}, m_{ij}) \). So \( m_{pq} = \min(m_{pi}, m_{iq}) = \alpha_m \), which is a contradiction. So \((p, q) \notin KM(M)\).

When \((p, q) \in X_{12} \) (\( m_{pi} = \alpha_m \) and \( m_{jq} = \alpha_M \), \( m_{ij} = \alpha_m \)). Therefore \( m_{ij} = \alpha_m \), \( m_{jq} = \alpha_M \Rightarrow m_{iq} = \min(m_{jq}, m_{ij}) \). So \( m_{pq} = \min(m_{pi}, m_{iq}) = \alpha_m \), which is a contradiction. So \((p, q) \notin KM(M)\).

When \((p, q) \in X_{13} \) (\( m_{pi} = \alpha_M \) and \( m_{jq} = \alpha_m \), \( m_{ij} = \alpha_m \)). Therefore \( m_{ij} = \alpha_m \), \( m_{jq} = \alpha_m \Rightarrow m_{iq} = \min(m_{jq}, m_{ij}) \). So \( m_{pq} = \min(m_{pi}, m_{iq}) = \alpha_m \), which is a contradiction. So \((p, q) \notin KM(M)\).

Consequently, \( Y_{mn}(m_{ij}) \subset W_{mn}(m_{ij}) \cup Km(M) \)

\[ Y_{MM}(m_{ij}) \subset Y_{mn}(m_{ij}) \cap (W_{mn}(m_{ij}) \cup Km(M)) \]

\[ = W_{mm}(m_{ij}) \cup Z_{mn}(m_{ij}) \]

\[ = W_{mm}(m_{ij}) \oplus Z_{mn}(m_{ij}). \]

**Corollary 1** The implication procedure does not change the previous known values.

**Theorem 13** The foregoing implication procedure correctly obtains necessary implications.
Proof.

By the above two lemmas the theorem is proved. ■

Theorem 14 The resultant matrix of implications satisfies the conditions for a FPR-matrix.

Proof.

1) $x_{ij} = \alpha_m$ is supplied:

a) Consistency:- Assume that there exists an inconsistency index triplet $(p, q, r)$ in the resultant matrix $M_I$. Then $(p, q) \in Km(M_I)$ and $(p, r), (r, q) \in KM(M_I)$ or $(r, p), (r, q) \in KM(M_I)$. Also $KM(M_I) = KM(M)$ and $Km(M_I) = Km(M) \cup W_{mn}(m_{ij})$, since $Km(M_I)$

$$= Km(M) \cup U_{mn}(m_{ij})$$

$$= Km(M) \cup (W_{mn}(m_{ij}) \oplus Z_{mn}(m_{ij}))$$

$$= Km(M) \cup (W_{mn}(m_{ij}) \oplus (Y_{mn}(m_{ij}) \cap Km(M))$$

$$= [Km(M) \cup W_{mn}(m_{ij})] \cup [Km(M) \cup (Y_{mn}(m_{ij}) \cap Km(M))]$$

$$= [Km(M) \cup W_{mn}(m_{ij})] \cup Km(M)$$

$$= Km(M) \cup U_{mn}(m_{ij}).$$

Hence $(p, q) \in Km(M) \cup U_{mn}(m_{ij})$. If $(p, q) \in Km(M)$ then $m_{pq} = \alpha_m$, $m_{pr}, m_{rq} = \alpha_M$ or $m_{rp}, m_{rq} = \alpha_M$. This contradicts the consistency of the matrix $M$. Otherwise, $(p, q) \in W_{mn}(m_{ij})$ and so $m_{pq} = \alpha$. Then

When $(p, q) \in X_9 (m_{ip}, m_{pr}, m_{rq}, m_{ij} = \alpha_M)$ means that $m_{ip} = \alpha_M, m_{pr} = \alpha_M$ implies $m_{ri} = \min(m_{ip}, m_{pr})$. So $m_{iq} = \min(m_{ir}, m_{rq})$ and hence $m_{ij} = \min(m_{iq}, m_{aj})$ which is a value greater than or equal to $\alpha$. This contradicts the maximality of $M$. Thus $M_I$ satisfies the consistency.

When $(p, q) \in X_{10} (m_{pi}, m_{pr}, m_{rq}, m_{ij} = \alpha_M)$, then $m_{pi} = \alpha_M, m_{ij} = \alpha_m \Rightarrow m_{pj} = \min(m_{pi}, m_{ij})$. Therefore $m_{pj} = \alpha_m, m_{pr} = \alpha_M \Rightarrow m_{rj} = \min(m_{pr}, m_{pj}) = \alpha_m$, and so $m_{jq} = \alpha_M$ implies $mrq = \alpha_m$, which is a contradiction. Thus $M_I$ satisfies the consistency.

When $(p, q) \in X_{11} (m_{pi}, m_{jq} = \alpha_m, m_{pr}, m_{rq} = \alpha_M)$, then $m_{pi} = \alpha_m, m_{pr} = \alpha_M \Rightarrow m_{ri} = \min(m_{pi}, m_{pr}) = \alpha_m$. So $m_{ri} = \alpha_m, m_{ij} = \alpha_m \Rightarrow$
\[ m_{rj} = \min(m_{ri}, m_{ij}) = \alpha_m, \text{ hence } m_{jq} = \alpha_m \text{ implies } m_{rq} = \alpha_m \text{ which is a contradiction. Thus } M_I \text{ satisfies the consistency.} \]

When \((p, q) \in X_{12} \) \( (m_{pi} = \alpha_m, m_{jq} = m_{pr} = \alpha_M) \), then \( m_{ij} = \alpha_m, m_{jq} = \alpha_M \Rightarrow m_{iq} = \min(m_{ij}, m_{jq}) = m_{ij} = \alpha_m \). So \( m_{iq} = \alpha_m, m_{rq} = \alpha_M \Rightarrow m_{ir} = \min(m_{iq}, m_{rq}) = \alpha_m \text{ and hence } m_{pi} = \alpha_m \text{ implies } m_{pr} = \min(m_{pi}, m_{ir}) = \alpha_m \text{ which is a contradiction. Thus } M_I \text{ satisfies the consistency.} \]

When \((p, q) \in X_{13} \) \( (m_{pi} = m_{pr}, m_{rq} = \alpha_M, m_{jq} = \alpha_m) \), then \( m_{ij} = \alpha_m, m_{jq} = \alpha_m \Rightarrow m_{iq} = \min(m_{ij}, m_{jq}) = \alpha_m \). So \( m_{iq} = \alpha_m, m_{rq} = \alpha_M \Rightarrow m_{ir} = \min(m_{iq}, m_{rq}) = \alpha_m \). Hence \( m_{pi} = \alpha_M, m_{ir} = \alpha_m \Rightarrow m_{pr} = \min(m_{pi}, m_{ir}) = m_{ir} = \alpha_m \) which is a contradiction. Thus \( M_I \) satisfies the consistency.

b) Maximality:- Assume that there exists an index triplet \((p, q, r)\) that violates the maximality in the resultant matrix \( M_I \) such that \( (p, q) \in K_m(M_I) \) and \( (p, r) \in K_M(M) \) and \( m_{rq} = x \). If \((p, q) \in K_m(M)\), then the matrix \( M \) is not maximal.

Otherwise \((p, q) \in W_{mm}(M_{ij})\), then

When \((p, q) \in X_9 \) \( (m_{ip} = \alpha_M, m_{pr} = \alpha_M) \) means that \( m_{ir} = \min(m_{ip}, m_{pr}) \).

Hence \( m_{ir} = \min(m_{ip}, m_{pr}), m_{iq} = \alpha_M \) leads to \( m_{rq} = \alpha_m \) by the \( \text{min} \Rightarrow \text{min} \) implication. This contradicts the assumption that \( m_{rq} = x \). Thus maximality is satisfied.

When \((p, q) \in X_{10} \) \( (m_{pi} = \alpha_M, m_{pr} = \alpha_M) \) means that \( m_{ri} = \min(m_{pi}, m_{pr}) = \alpha_M \).

Hence \( m_{ri} = \min(m_{pi}, m_{pr}), m_{jq} = \alpha_M \) leads to \( m_{rq} = \alpha_m \) by the \( \text{min} \Rightarrow \text{min} \) implication. This contradicts the assumption that \( m_{rq} = x \). Thus maximality is satisfied.

When \((p, q) \in X_{11} \) \( (m_{pi} = \alpha_M, m_{pr} = \alpha_M) \) means that \( m_{ri} = \min(m_{pi}, m_{pr}) = \alpha_m \).

Hence \( m_{ri} = \min(m_{pi}, m_{pr}), m_{jq} = \alpha_m \) leads to \( m_{rq} = \alpha_m \) by the \( \text{min} \Rightarrow \text{min} \) implication. This contradicts the assumption that \( m_{rq} = x \). Thus maximality is satisfied.

When \((p, q) \in X_{12} \) \( (m_{pi} = \alpha_m, m_{pr} = \alpha_M) \) means that \( m_{ri} = \min(m_{pi}, m_{pr}) = \alpha_m \).

Hence \( m_{ri} = \min(m_{pi}, m_{pr}), m_{jq} = \alpha_M \) leads to \( m_{rq} = \alpha_m \) by the \( \text{min} \Rightarrow \text{min} \) implication. This contradicts the assumption that \( m_{rq} = x \). Thus maximality is satisfied.
Enumeration of All Fuzzy Quasi-Ordered Relations

When \((p, q) \in X_{13}\) \((m_{pi} = \alpha_M, m_{pr} = \alpha_M)\) means that \(m_{ri} = \min(m_{pi}, m_{pr}) = \alpha_M\). Hence \(m_{ri} = \min(m_{pi}, m_{pr})\), \(m_{ij} = \alpha_m\) leads to \(m_{rq} = \alpha_m\) by the \(\min \Rightarrow \min\) implication. This contradicts the assumption that \(m_{rq} = x\). Thus maximality is satisfied.

2) \(x_{ij} = \alpha_M\) is supplied:

a) Consistency:- Assume that there exists an inconsistency triplet \((p, q, r)\) in the resultant matrix \(M\). Then \((p, q) \in Km(M)\) and \((p, r), (r, q) \in KM(M)\). Also \(KM(M) = KM(M) \cup W_{MM}(m_{ij})\) and that \(KM(M) = KM(M) \cup W_{MM}(m_{ij})\). Hence \((p, q) \in Km(M) \cup W_{MM}(m_{ij})\) and \((p, r), (r, q) \in KM(M) \cup W_{MM}(m_{ij})\).

If \((p, q) \in Kg(M)\) and \((p, r), (r, q) \in Kg(M)\), \(m_{pq} = \alpha_m\) and \(m_{pr} = \alpha_M\), \(m_{rq} = \alpha_M\). This contradicts the consistency of \(M\). So \((p, q) \notin Kg(M)\). Therefore \((p, q) \in W_{MM}(m_{ij})\), so \(m_{pq} = x\). This contradicts the maximality of \(M\). Thus \(M\) satisfies the consistency.

b) Maximality:- Assume that there exists an index triplet \((p, q, r)\) that violates the maximality condition in the resultant matrix \(M\) such that \((p, q) \in Kg(M)\) and \((p, r) \in Kg(M)\) and \(m_{rq} = x\). Then \((p, q) \in Kg(M) \cup W_{MM}(m_{ij})\) and \((p, r), (r, q) \in Kg(M) \cup W_{MM}(m_{ij})\). If \((p, q) \in Kg(M)\), then the matrix \(M\) is not maximal. Otherwise, \((p, q) \in W_{MM}(m_{ij})\), then

When \((p, q) \in X_5\) \(m_{jp} = \alpha_M, m_{pr} = \alpha_M\) means that \(m_{jr} = \min(m_{jp}, m_{pr})\). Hence \(m_{jr} = \min(m_{jp}, m_{pr})\), \(m_{iq} = \alpha_m\) leads to \(m_{rq} = \alpha_m\) by the \(\max \Rightarrow \min\) implication. This contradicts the assumption that \(m_{rq} = x\). Thus maximality is proved.

When \((p, q) \in X_6\) \(m_{pj} = \alpha_m, m_{pr} = \alpha_M\) means that \(m_{rj} = \min(m_{pj}, m_{pr}) = m_{pj}\). Hence \(m_{rj} = m_{pj}, m_{qi} = \alpha_m\) leads to \(m_{rq} = \alpha_m\) by the \(\max \Rightarrow \min\) implication. This contradicts the assumption that \(m_{rq} = x\). Thus maximality is proved.

When \((p, q) \in X_7\) \(m_{ip} = \alpha_M, m_{pr} = \alpha_M\) means that \(m_{ir} = \min(m_{ip}, m_{pr})\). Hence \(m_{ir} = \min(m_{ip}, m_{pr})\), \(m_{ij} = \alpha_m\) leads to \(m_{rq} = \alpha_m\) by the \(\max \Rightarrow \min\) implication. This contradicts the assumption that \(m_{rq} = x\). Thus maximality is proved.
When \((p,q) \in X_8\) \(m_{pi} = \alpha_n\), \(m_{pr} = \alpha_M\) means that \(m_{ri} = \min(m_{pi}, m_{pr}) = m_{pi}\). Hence \(m_{ri} = m_{pis}\), \(m_{qj} = \alpha_M\) leads to \(m_{rq} = \alpha_m\) by the \(\max \Rightarrow \min\) implication. This contradicts the assumption that \(m_{rq} = x\). Thus maximality is proved. ■

Procedure to generate all fuzzy reachability (FR) matrices on a crisp set \(S\) with cardinality \(n\) for a given \(\alpha \in [0,1]\).

Procedure 3:

For \(i = 1; i \leq \text{size}; i + 1\)  
For \(j = 1; j \leq \text{size}; j + 1\)  
\(m_{ij} = \text{val}\)

Step 1

If \(\text{val} = \alpha_M\)

\(m_{ij} = \alpha_M\);

For \(p = 1; p \leq \text{size}; p + 1\)  
For \(q = 1; q \leq \text{size}; q + 1\)  
If \((p, q) \in X_1\) or \(X_2\) or \(X_3\) or \(X_4\)

set \(m_{pq} = \beta_2\);

where \(\beta_2 = \min(m_{ij}, m_{pi}, m_{jq})\) when \((p, q) \in X_1\) of \(Y_{MM}(m_{ij})\)

\(= \min(m_{ij}, m_{ip}, m_{iq})\) when \((p, q) \in X_2\) of \(Y_{MM}(m_{ij})\)

\(= \min(m_{ij}, m_{pj}, m_{iq})\) when \((p, q) \in X_3\) of \(Y_{MM}(m_{ij})\)

\(= \min(m_{ij}, m_{ip}, m_{jq})\) when \((p, q) \in X_4\) of \(Y_{MM}(m_{ij})\)

Else if \((p, q) \in X_5\) or \(X_6\) or \(X_7\) or \(X_8\)

Set \(m_{pq} = \beta_1\)

where \(\beta_1 = m_{iq}\) when \((p, q) \in X_5\) of \(Y_{MM}(m_{ij})\)

\(= m_{pj}\) when \((p, q) \in X_6\) of \(Y_{MM}(m_{ij})\)

\(= m_{jq}\) when \((p, q) \in X_7\) of \(Y_{MM}(m_{ij})\)

\(= m_{pi}\) when \((p, q) \in X_8\) of \(Y_{MM}(m_{ij})\);  

for all \((p, q) \in Y_{MM}(m_{ij})\), \((\max \Rightarrow \min\text{implication})\) Else

Set \(m_{ij} = \alpha_n\)

For \(p = 1; p \leq \text{size}; p + 1\)  
For \(q = 1; q \leq \text{size}; q + 1\)
If \((p, q) \in X_9\) or \(X_{10}\) or \(X_{11}\) or \(X_{12}\) or \(X_{13}\) set \(m_{pq} = \beta_0\);

where \(\beta_0 = m_{ij}\) when \((p, q) \in X_9\) of \(Y_{mm}(m_{ij})\)

\[m_{ij} = \min(m_{ij}, m_{jq}, m_{pi})\] when \((p, q) \in X_{10}\) of \(Y_{mm}(m_{ij})\)

\[m_{ij} = \min(m_{ij}, m_{pi})\] when \((p, q) \in X_{12}\) of \(Y_{mm}(m_{ij})\)

\[m_{ij} = \min(m_{ij}, m_{jq})\] when \((p, q) \in X_{13}\) of \(Y_{mm}(m_{ij})\)

for all \((p, q) \in Y_{mm}(m_{ij})\); \((\min \Rightarrow \min\) implication\)

For \((i = 1; i \leq \text{size}; i + 1)\)

For \((j = 1; j \leq \text{size}; j + 1)\)

If \(m_{ij} = x\)

If \(\text{val} = \alpha_M\)

Set \(\text{val} = \alpha_M\) and apply the implication rules and

Else

set \(\text{val} = \alpha_m\) and apply the implication rules

Else

STOP

Repeat the procedure and generate the next matrix.

Following like this find all possible quasi-orders by giving both values \(\alpha_M\) and \(\alpha_m\) to all the unknowns in each stage of the implication. By successive application of the implication rules by giving different values for the unknowns, where the values supplied belongs to a finite lattice \(\beta\), we get all fuzzy quasi-orders on a finite set.

Program is developed for the procedure 3 and the output is produced. The program and some sample outputs are given in Appendix II. By observing the output of the program for various values of \(\alpha\) and \(\beta\) we get the following remark.

**Remark 4** If \(N(n, m)\) denotes the number of FR-matrices of order \(n\) on a \(\beta\) set of cardinality \(m\), then \(N(2, 2)\) and \(N(n, m)\) are related by the formula

\[N(n, m) = N(2, 2)^{n-1}\]

where \(\beta = m_{ij}\) when \((p, q) \in X_9\) of \(Y_{mm}(m_{ij})\)
Using Algorithm 9 and performing the procedure 3 with necessary changes we can generate the set of all fuzzy reachability matrices on a finite fuzzy set.

**Theorem 15** For a given \( \alpha \in [0,1] \) and a finite lattice \( \beta \subseteq [0,1] \), the collection of all matrices obtained by applying the procedure (3) on a completely unknown FPR-matrix of a given order \( n \) by giving the values \( \beta = \{\beta_1, \beta_2, \ldots, \beta_m\} \) to the unknowns at each stage of the algorithm is precisely the collection of all fuzzy reachability matrices on a finite fuzzy set.

Correctness of the theorem follows from the correctness of the algorithm 7.

At this juncture, we can modify the algorithm to generate a partial order on the set \( S \) and obtain analogous results as in the case of quasi-ordered relations on \( S \).

### 3.4 Fuzzy Partial Orders

**Algorithm 10** (Implication Algorithm for Fuzzy Partial Orders)

BEGIN

supply the value \( \max(\alpha_M) \) or \( \min(\alpha_m) \) to the unknown \( m_{ij} \);

IF (supplied value is max) THEN BEGIN

set \( m_{ij} = \alpha_M \) (a value greater than or equal to \( \alpha \)),

set \( m_{ji} = 0 \)

set \( m_{pq} = \beta_2 \);

where \( \beta_2 = \min(m_{ij}, m_{pi}, m_{iq}) \) when \((p,q) \in X_1 \) of \( Y_{MM}(m_{ij}) \)

= \( \min(m_{ij}, m_{jp}, m_{iq}) \) when \((p,q) \in X_2 \) of \( Y_{MM}(m_{ij}) \)

= \( \min(m_{ij}, m_{pj}, m_{iq}) \) when \((p,q) \in X_3 \) of \( Y_{MM}(m_{ij}) \)

= \( \min(m_{ij}, m_{pi}, m_{qj}) \) when \((p,q) \in X_4 \) of \( Y_{MM}(m_{ij}) \)

set \( m_{qp} = 0 \)

for all \((p,q) \in Y_{MM}(m_{ij})\), (Max \( \Rightarrow \) Max implication)

set \( m_{pq} = \beta_1 \);

where \( \beta_1 = m_{iq} \) when \((p,q) \in X_5 \) of \( Y_{MM}(m_{ij}) \)
The analogous lemmas and theorems in the case of fuzzy partial orders are stated below.

**Lemma 10** If $M$ is a fuzzy partially filled reachability matrix, then

$$Y_{MM}(m_{ij}) \cap Y_{Mm}(m_{ij}) = \phi$$

**Lemma 11** $Y_{MM}(m_{ij})$ is the disjoint union of $W_{MM}(m_{ij})$ and $Z_{MM}(m_{ij})$, $Y_{Mm}(m_{ij})$ the disjoint union of $W_{MM}(m_{ij})$ and $Z_{Mm}(m_{ij})$ and $Y_{mm}(m_{ij})$ the disjoint union of $W_{mm}(m_{ij})$ and $Z_{mm}(m_{ij})$ respectively.

**Theorem 16** The foregoing implication procedure correctly obtains necessary implications.
Theorem 17 The resultant matrix of implications satisfies the conditions for a FPR-matrix.

All these theorems which holds in the case of fuzzy quasi-orders holds in the case of fuzzy partial orders also. As in the case of fuzzy quasi-orders, the algorithm 10 can be generalised to a fuzzy subset $\sigma$. The set of all fuzzy partial orders can also be determined as in the case of fuzzy quasi-orders by using the algorithm 10.

Route Systems On Fuzzy Graphs

4.1 Introduction

The concept of route system was introduced by Nebeský [34, 35] while studying the set of all geodesics (shortest paths) in a connected graph. Innovating the idea of route systems, Nebeský was able to give a credulous and semi-geometric characterization of the set of all geodesics in a connected graph ([37]). Nebeský's seminal idea of 'route systems' was generalised by Chang et al. and Mulder ([10]) to capture finite non-empty set $V$. In this generalized setting, a route system on $V$ has been defined as a collection of sequences of elements in $V$ which satisfy some axioms. To capture the specification of various types of already known route systems on graphs, both 'extension axioms' and 'connection axioms' are employed. If a route system is given, it becomes possible to define the underlying graph of the route system. An important question arises at this juncture, that is, what do the specific routes in the route system signify in the underlying graph. If the route system satisfies a symmetry axiom, then the underlying graph is undirected, otherwise it is directed. New axioms are introduced into the route systems (directed as well as undirected) to characterize the walks, the trails, the paths, the minimal paths and the triangular minimal paths in the underlying (di)graph.