Chapter 2

Enumeration of All Quasi-Ordered Relations

2.1 Introduction

In this chapter, we make use of the implication algorithm given by Ohuchi et. al. to find a reachability matrix on a finite crisp set \( N \). This is equivalent to finding a quasi ordered relation on \( N \). Ohuchi et. al. described the implication algorithm in the context of a type of structural modelling known as interpretive structural modelling (ISM). ISM is an important tool for learning large scale and complex systems and the method develops a reachability matrix and derive a hierarchical directed graph. The following important references related to ISM ([41], [42], [43], [46], [47]) may be noted. We modify the implication algorithm to generate the family of all quasi ordered relations on \( N \) both labelled and unlabelled. For better understanding, the background of the implication algorithm, the algorithm itself is described first. The parallel version of the implication algorithm is also presented. We attempt to address the following problems.

Given a finite set \( N = \{1, 2, ..., n\} \), a quasi-ordered relation \( R \) is defined on \( N \), that is structure \( N \) so that \( (N,R) \) is a quasi-ordered set. Also all quasi-ordered relations (labelled and unlabelled) on \( N \) is determined. This is equivalent to finding all topologies on \( N \), because there is a one-to-one correspondence be-
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tween quasi-orders on $N$ and topologies on $N$. Given a finite set $N$, a partially-ordered relation $R$ is defined on $N$, that is, structure $N$ so that $(N, R)$ is a partially ordered set and also all possible partially ordered relations on $N$ is determined. This is equivalent to finding all $T_0$-topologies on $N$, because there is a one-one correspondence between quasi-orders on $N$ and $T_0$-topologies on $N$. The algorithms to find all the quasi-orders and partial orders are parallelized using a shared memory model.

Let $N = \{1, 2, ..., n\}$ be a finite set of $n$ elements. Let $R$ be a reflexive, transitive (quasi-order) relation on $N$. The digraph $D(R)$ of $R$ is a transitive digraph in the sense that its transitive closure is itself. We can easily verify that the 'reachability' relation defined on the vertex set of $D(R)$, namely $N$ is a quasi-ordered relation (say, $Re$) on $N$. It can be further verified that $R$ is isomorphic to $Re$ on $N$. Suppose that $uv \in D(R)$, then $uv \in Re$, since $Re$ is the reachability relation defined on the vertex set of $D(R)$. Therefore if there is an edge $uv$ in $D(R)$, there is an edge $uv$ in $D(Re)$ also. Therefore $uv \in D(R)$ if and only if $uv \in D(Re)$. Therefore $R$ is isomorphic to $Re$.

Therefore there exists a $1 - 1$ correspondence between quasi-ordered relations on $N$ and reachability relations of the digraph $D(R)$ of $R$. Therefore we use the reachability matrices of $D(R)$ in our further discussion. Formally the reachability matrix, namely $R$-matrix of a digraph $D$ is defined as follows.

The $ij^{th}$ element $r_{ij} = 1$, if vertex $j$ is reachable from vertex $i$ in $D$ (that is, there is a directed path from vertex $i$ to vertex $j$ in $D$) and $r_{ij} = 0$, otherwise.

We have a theorem characterising the reachability matrix of a digraph.

**Theorem 1** A square binary matrix $M$ of order $n$ is the reachability matrix of some digraph, if it satisfies the following conditions:

- $M + I = M$ (reflexive)
- $M^2 = M$ (transitive), where the matrix addition and multiplication are both boolean operations.
Proof. If $M$ is the reachability matrix of some digraph, then it is easy to verify that $M$ satisfies the two conditions.

To prove the converse, we have to show that if $M$ is a square binary matrix of order $n$ which satisfies the two given conditions, then $M$ is the reachability matrix of some digraph. A digraph $D$ with $n$ vertices is constructed as follows. Let $N = \{1, 2, ..., n\}$ be the vertex set of $D$. Now for distinct $i$ and $j$, define $ij \in E(D)$ if $m_{ij} = 1$ and $ij \notin E(D)$ if $m_{ij} = 0$. Then clearly $D$ is a well defined digraph. Now we will show that the reachability matrix $R(D)$ of $D$ is exactly $M$. Suppose $r_{ij}$ and $m_{ij}$ be the $ij^{th}$ element of $R(D)$ and $M$ respectively. Now $m_{ij} = 1 \implies r_{ij} = 1$, by the definition of $D$. If $m_{ij} = 0$, then by the property of $M$, it follows that $m_{ik} = 0$ for all $k \neq i$ and $m_{kj} = 0$ for all $k \neq j$. Therefore it follows that there is no $i - j$ path in $D$ and hence $r_{ij} = 0$. Suppose $r_{ij} = 0$. Therefore there is no $i - j$ path in $D$, in particular $ij \notin E(D)$, which implies that $m_{ij} = 0$. Now if $r_{ij} = 1$, then there is an $i - j$ path in $D$. We use induction on the length of paths in $D$ to prove this part of the proof. If the length of the $i - j$ path is one, then we are done. Assume that the result is true if the length of the $i - j$-path in $D$ is of length $m > 1$. Now if the length of the $i - j$-path is $m + 1$, then there is an index $k$ such that there is an $i - k$-path in $D$ of length $m$ and $kj$ is an edge of $D$. Therefore by induction we have $m_{ik} = 1$ and by definition of $D$, $m_{kj} = 1$. Therefore, by the property of $M$, $m_{ij} = \sum (m_{ik} \cdot m_{kj})$ and hence $m_{ij} = 1$ and which completes the proof of the theorem. 

Remark 1 The digraph $D(R)$ corresponding to a reachability matrix $M$ is a transitive digraph.

A partially filled matrix is a square matrix $M$ whose elements are either one, zero, or an unknown $x \in \{0,1\}$, and all main diagonal elements are one.

Definition 1 Partially filled matrix $M$ is called a Partial Reachability matrix (PR-matrix), if it satisfies the following conditions.

Consistency Property
No $m_{ij}$ exists such that $m_{ij} = 0 \land m_{ik} = m_{kj} = 1$.

**Maximality Property**

No $m_{ij}$ exists such that

\[
\begin{align*}
    m_{ij} &= x \land m_{kj} = 0 \land m_{ki} = 1 \\
    m_{ij} &= x \land m_{ik} = 0 \land m_{jk} = 1 \\
    m_{ij} &= x \land m_{ik} = 1 \land m_{kj} = 1
\end{align*}
\]

In particular, we have two extreme cases of $PR$-matrices, namely the $PR$-matrix whose entries are all unknown, except the diagonals, denoted as $PR_0$ and the reachability matrix itself (that is a $PR$-matrix whose entries are all known), denoted as $R$. A $PR$-matrix which is not a reachability matrix is called a non-trivial $PR$-matrix.

**Lemma 1** If all the unknowns of a $PR$-matrix are set to zero, the resultant matrix is a reachability matrix.

**Proof.** Let $M$ be the $PR$-matrix and $M'$ be the matrix which is obtained by setting all unknown entries of $M$ to zero. Obviously, $M' + I = M'$.

Then we have to prove that $M'^2 = M'$

Because $M$ is a $PR$-matrix

**Case.1**

If $m_{ij} = 0, i \neq j$, then every term $m_{ik}m_{kj}$ is either zero or $x$. Hence $m_{ij} = 0 \land \sum_k m_{ik}m_{kj} = 0, k = 1, \ldots, n$. [Since in $M'$ all unknowns are set to zero].

**Case.2**

If $m_{ij} = 1$, then at least one term $m_{ik}m_{kj}$ is one. Specifically $m_{ij}m_{jj}$ and $m_{ii}m_{ij}$ are one. Hence $m_{ij} = 1 \land \sum_k m_{ik}m_{kj} = 1, k = 1, \ldots, n$.

**Case.3**

If $m_{ij} = x, i \neq j$, then every term $m_{ik}m_{kj}$ is either zero or $x$. Hence $m_{ij} = 0 \land \sum_k m_{ik}m_{kj} = 0, k = 1, \ldots, n$.

Thus in all cases we have
\[ m_{ij} = \sum_k m_{ik} m_{kj}, \ k = 1, \ldots, n. \]

Hence \( M \) is a reachability matrix.

We consider the collection \( \mathcal{M} \) of all \( PR \)-matrices of order \( n \). We define a partial order \( \leq \) on \( \mathcal{M} \) as follows. \( M_1 \leq M_2 \) if and only if every known entry of \( M_1 \) and \( M_2 \) are same and \( M_1 < M_2 \) if \( M_2 \) has at least one known entry which is an unknown entry in \( M_1 \). We can see that if \( M_1 \) and \( M_2 \) are two \( PR \)-matrices in \( \mathcal{M} \), then there need not be a least upper bound for \( M_1 \) and \( M_2 \), since \( M_1 \) and \( M_2 \) may not be comparable. Since there are several \( R \)-matrices in \( \mathcal{M} \) which are not comparable, it follows that \( \mathcal{M} \) has no universal upper bound, but all the \( R \)-matrices in \( \mathcal{M} \) are maximal elements of \( \mathcal{M} \). Now if \( M_1 \) and \( M_2 \) are two non-comparable elements of \( \mathcal{M} \), then there is a greatest lower bound for \( M_1 \) and \( M_2 \), namely the \( PR \)-matrix from which the two matrices \( M_1 \) and \( M_2 \) are first branched out. We can see that \( PR_0 \) acts as the universal lower bound in \( \mathcal{M} \). We have a lemma

**Lemma 2** The pair \( (\mathcal{M}, \leq) \) is a meet semi lattice with \( PR_0 \) as the universal lower bound.

We have a theorem

**Theorem 2** There exists a chain (totally ordered collection) of \( PR \)-matrices in \( \mathcal{M} \), namely \( M_1, M_2, \ldots, M_r \) with \( M_1 = PR_0 \) and \( M_r = R \) such that there exists no \( M \in \mathcal{M} \) with \( M_i < M < M_{i+1} \), for \( i = 1, 2, \ldots, r-1 \).

**Proof.**

We begin with the \( PR \)-matrix \( PR_0 \) denoted as \( M_1 \). We give a value 1 or 0 to any unknown entry \( x \) in \( M_1 \), let the resulting matrix be \( M_2 \). Clearly we can show that \( M_2 \) is a \( PR \)-matrix and \( M_1 \leq M_2 \) and there is no \( M \in \mathcal{M} \) with \( M_1 < M < M_2 \). Again we give a value 1 or 0 for an unknown entry \( x \) in \( M_2 \). Now the resulting partially filled matrix may not be a \( PR \)-matrix, but by giving some unknown values \( x \) in the resulting matrix either 0 or 1 so
that the consistency and maximality conditions of the PR-matrix are satisfied, it can be made a PR-matrix. Let the resulting matrix be denoted as $M_3$. Thus we obtained a nested sequence $M_1, M_2, M_3$ in $M$ with no $M \in M$ with $M_i < M < M_i + 1$, for $i = 1, 2$. Continuing this procedure successively, by induction we can prove the theorem. 

Now we discuss the algorithms which will finally produce the nested sequence of PR-matrices in the Theorem 2.

### 2.2 Algorithms for Partial Reachability Matrices

We first define certain index sets which are used in the algorithms.

Let $M$ be a PR-matrix. For each index $i$, the following index sets are defined.

- $L_M(i) = \{k; m_{ik} = 1\}$
- $D_M(i) = \{k; m_{ki} = 1\}$
- $L_m(i) = \{k; m_{ik} = 0\}$
- $D_m(i) = \{k; m_{ki} = 0\}$
- $K1(M) = \{(p, q); m_{pq} = 1\}$
- $K0(M) = \{(p, q); m_{pq} = 0\}$

Let $M$ be a PR-matrix. For the element $m_{ij}$, the following index sets are defined:

- $Y11(m_{ij}) = \{(p, q); p \in D_M(i) \text{ and } q \in L_M(j)\}$
- $Y10(m_{ij}) = \{(p, q); p \in L_M(i) \text{ and } q \in L_m(j) \text{ or } p \in D_m(j) \text{ and } q \in D_M(i)\}$
- $Y00(m_{ij}) = \{(p, q); p \in L_M(i) \text{ and } q \in D_M(j)\}$
- $W11(m_{ij}) = \{(p, q); m_{pq} = x, \text{ and } (p, q) \in Y11(m_{ij})\}$
- $W10(m_{ij}) = \{(p, q); m_{pq} = x, \text{ and } (p, q) \in Y10(m_{ij})\}$
Algorithm 1 (Implication Algorithm)

\[ \begin{align*}
W_{00}(m_{ij}) &= \{(p, q); m_{pq} = x, \text{ and } (p, q) \in Y_{00}(m_{ij})\} \\
Z_{11}(m_{ij}) &= \{Y_{11}(m_{ij}) \cap K_1(M)\} \\
Z_{10}(m_{ij}) &= \{Y_{10}(m_{ij}) \cap K_0(M)\} \\
Z_{00}(m_{ij}) &= \{Y_{00}(m_{ij}) \cap K_0(M)\}
\end{align*} \]

BEGIN

supply the value one or zero to the unknown \( m_{ij} \);

IF (supplied value is one) THEN BEGIN

set \( m_{ij} = 1 \),

set \( m_{pq} = 1 \) for all \( (p, q) \in Y_{11}(m_{ij}); (1 \Rightarrow 1 \text{ implication}) \)

set \( m_{pq} = 0 \); for all \( (p, q) \in Y_{10}(m_{ij}); (1 \Rightarrow 0 \text{ implication}) \)

END

ELSE BEGIN

set \( m_{ij} = 0 \), \( i \neq j \);

set \( m_{pq} = 0 \); for all \( (p, q) \in Y_{00}(m_{ij}); (0 \Rightarrow 0 \text{ implication}) \)

END

END.

Time complexity

For a PR-matrix of order \( n \), the time complexity of the Algorithm 1 is \( O(n^2) \), since the computation of the index sets \( Y_{11}, Y_{10} \) and \( Y_{00} \) takes atmost \( O(n^2) \) time.

IF \( M \) is a PR matrix, Let \( M(k) \) be the PR matrix obtained from \( M \) by applying the implication algorithm supplying the value \( k = 0 \) or \( 1 \) for the next unknown. (the next unknown is the first unknown in the dictionary order).

Algorithm ARM (all reachability matrices) Input a PR matrix \( M \) and output all reachability matrices greater than \( M \).

Algorithm 2 (ARM)
**STEP1**: If $M$ is a reachability matrix, output $M$ and go to **STEP 4**

**STEP2**: $ARM(M(0))$

**STEP3**: $ARM(M(1))$

**STEP4**: STOP.

**Time complexity**

For a matrix of order $n$, at each stage of the algorithm there are $2n^2$ computations. Therefore in the worst case, there are atmost $(2n^2)^{n^2-n}$ computations. So the worst time complexity is $O((2n^2)^{n^2-n})$ and this is the worst bound.

The detailed procedure to find the set of all reachability matrices on a finite set can be described as follows for a PR-matrix of order $n$.

**Procedure 1:**

For ($i = 1; i \leq size; i + 1$)

For ($j = 1; j \leq size; j + 1$)

Step 1

If val = 1

Set $m_{ij} = 1$;

For ($p = 1; p \leq size; p + 1$)

For ($q = 1; q \leq size; q + 1$)

If $(m_{pi} = 1$ and $m_{iq} = 1$)

'let $m_{pq} = 1$;

Else if $(m_{ip} = 1$ and $m_{iq} = 0$ or $m_{pj} = 1$ and $m_{qi} = 0$)

Set $m_{pq} = 0$

Goto Step 3

Else

Step 2

set $m_{ij} = 0$;

For ($p = 1; p \leq size; p + 1$)

For ($q = 1; q \leq size; q + 1$)

If $(m_{ip} = 1$ and $m_{aq} = 1$)
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Set \( m_{pq} = 0 \).

Step 3
For \( i = 1; i \leq \text{size}; i + 1 \)
For \( j = 1; j \leq \text{size}; j + 1 \)
If \( m_{ij} = x \)
If \( \text{val} = 0 \)
Set \( m_{ij} = 0 \)
Goto Step 2
Else
Set \( m_{ij} = 1 \)
Goto Step 1
Else
Stop

Repeat the procedure and generate the next matrix.
Continuing like this generate all matrices by supplying the values 0 and 1 to the unknowns at each stage of the implication.

END

The algorithm can be easily parallelized using a shared memory model. There are many models of parallel computation. Some of them are P-RAM (parallel random-access machine), W-RAM, EREW P-RAM, CREW P-RAM, CRCW P-RAM and so on. P-RAM is a shared memory model. There are a number of processors working synchronously and communicating through the common random-access memory. The processors synchronously execute the same program. Although performing the same instructions, the processors can be working on different data. Hence such a model is also called a single-instruction, multiple-data-stream (SIMD) model. A stronger model than P-RAM is sometimes used in parallel-algorithm design in which many processors are allowed to write to the same location. This model is called W-RAM. There are a variety of acronyms to describe variations on the model of computation. The main ones are the EREW P-RAM (exclusive read exclusive write parallel random-access
machine which allows no concurrent reads or writes) and the CRCW P-RAM (in which both concurrent writes and concurrent reads are allowed). See [1] for the details regarding parallel computation. The algorithm ARMP can be implemented using any of the shared memory models mentioned above. We discuss below the parallel version of the algorithm, ARMP.

**Algorithm 3 Algorithm ARMP (parallel version of ARM)**

*Input* a PR matrix *M*

*Output* all reachability matrices > *M*

**STEP1**: If *M* is a reachability matrix, output *M* and go to **STEP4**

DO in parallel

**STEP2**: ARM(*M*(0))

**STEP3**: ARM(*M*(1))

End Do

**STEP4**: STOP.

**Time complexity**

If we have *k* processors, the complexity of the algorithm is \( O\left((2n^2(n^2-n)/k)\right) \), \( 1 \leq k \leq (2n^2)n^2-n \).

We need the following Lemma to prove the correctness of the implication algorithm.

**Lemma 3** If *M* is a partially filled reachability matrix, then 

\[ Y11(m_{ij}) \cap Y10(m_{ij}) = \phi \]

**Proof.** Suppose that \((p, q) \in Y11(m_{ij}) \cap Y10(m_{ij})\). Consider the case when \((p, q) \in Y11(m_{ij}) \cap Y10(m_{ij})\), then \((m_{ij} = 1) \times m_{pi} = 1, m_{jq} = 1 \text{ and } m_{jp} = 1, m_{iq} = 0 \text{ or } m_{pj} = 0, m_{qi} = 1\). The contradiction occurs when \((m_{ij} = 1) \times m_{pi} = 1, m_{jq} = 1 \text{ and } m_{jp} = 0, m_{qi} = 1\), that is, \( m_{ij} = 1, m_{pi} = 1 \Rightarrow m_{pi} = 1 \), which is a contradiction.
Therefore in both the cases \((p, q) \notin Y11(m_{ij}) \cap Y10(m_{ij})\).

If \(A\) and \(B\) are two sets of ordered pairs of elements, then the direct sum of \(A\) and \(B\) is the disjoint union of \(A\) and \(B\) and is denoted by \(A \oplus B\). That is \(A \oplus B = A \cup B\), where \(A \cap B = \emptyset\).

**Lemma 4** \(Y11(m_{ij})\) is the disjoint union of \(W11(m_{ij})\) and \(Z11(m_{ij})\), \(Y10(m_{ij})\) is the disjoint union of \(W10(m_{ij})\) and \(Z10(m_{ij})\) and \(Y00(m_{ij})\) is the disjoint union of \(W00(m_{ij})\) and \(Z00(m_{ij})\).

**Proof.**

\(W11(m_{ij}) \cap Z11(m_{ij}) = \emptyset\) and \(Y10(m_{ij}) \supset W11(m_{ij}) \oplus Z11(m_{ij})\) are obvious by the definitions. For any \((p, q) \in Y11(m_{ij})\), only one of \(m_{pq} = x\) or \(m_{pq} = 1\) or \(m_{pq} = 0\) is true. Hence \((p, q) \in W11(m_{ij}) \cup K1(M) \cup K0(M)\). If \((p, q) \in K0(M)\), then \(m_{pq} = 0\). But since \((p, q) \in Y11(m_{ij})\), \(m_{pi} = 1\), \(m_{jq} = 1\), therefore \(m_{pq} = 0\), \(m_{jq} = 1 \Rightarrow m_{pj} = 0\). So \((m_{ij} = x, m_{pi} = 1, m_{pj} = 0)\) contradicts the maximality property. So \((p, q) \notin K0(M)\).

Consequently, \(Y11(m_{ij}) \supset W11(m_{ij}) \cup K1(M)\)

\[Y11(m_{ij}) \subset Y11(m_{ij}) \cap (W11(m_{ij}) \cup K1(M))\]

\[= W11(m_{ij}) \cup Z11(m_{ij})\]

\[= W11(m_{ij}) \oplus Z11(m_{ij})\].

To prove that \(Y10(m_{ij}) = W10(m_{ij}) \oplus Z10(m_{ij})\).

\(W10(m_{ij}) \cap Z10(m_{ij}) = \emptyset\) and \(Y10(m_{ij}) \supset W10(m_{ij}) \oplus Z10(m_{ij})\) are obvious by the definitions. For any \((p, q) \in Y10(m_{ij})\), only one of \(m_{pq} = x\) or \(m_{pq} = 1\) or \(m_{pq} = 0\) is true. Hence \((p, q) \in W10(m_{ij}) \cup K1(M) \cup K0(M)\). If \((p, q) \in K1(M)\), then \(m_{pq} = 1\). But since \((p, q) \in Y10(m_{ij})\), \(m_{jp} = 1\), \(m_{iq} = 0\), or \(m_{pj} = 0\) and \(m_{qi} = 1\). Therefore \(m_{jp} = 1\), \(m_{ij} = 1 \Rightarrow m_{ip} = 1\). So \(m_{ip} = 1, m_{iq} = 0 \Rightarrow m_{pq} = 0\), which is a contradiction. So \((p, q) \notin K1(M)\). Consequently, \(Y10(m_{ij}) \supset W10(m_{ij}) \cup K0(M)\).

\[Y10(m_{ij}) \supset Y10(m_{ij}) \cap (W10(m_{ij}) \cup K0(M))\]

\[= W10(m_{ij}) \cup Z10(m_{ij})\]
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\[ = W10(m_{ij}) \oplus Z10(m_{ij}). \]

To prove that \( Y00(m_{ij}) = W00(m_{ij}) \oplus Z00(m_{ij}) \).

\( W00(m_{ij}) \cap Z00(m_{ij}) = \emptyset \) and \( Y00(m_{ij}) \supset W00(m_{ij}) \oplus Z00(m_{ij}) \) are obvious by the definitions. For any \((p, q) \in Y00(m_{ij})\), only one of \( m_{pq} = x \) or \( m_{pq} = 1 \) or \( m_{pq} = 0 \) is true. Hence \((p, q) \in W00(m_{ij}) \cup K1(M) \cup K0(M)\). If \((p, q) \in K1(M)\), then \( m_{pq} = 1 \). But since \((p, q) \in Y00(m_{ij})\), \( (m_{ip} = 1\) and \( m_{qj} = 1\)). Therefore \( m_{ip} = 1, m_{ij} = 0 \Rightarrow m_{pq} = 0 \), which is a contradiction. So \((p, q) \notin K1(M)\).

Consequently, \( Y00(m_{ij}) \supset W00(m_{ij}) \cup K0(M) \).

\[ Y00(m_{ij}) \subset Y00(m_{ij}) \cap (W00(m_{ij}) \cup K0(M)) \]
\[ = W00(m_{ij}) \cup Z00(m_{ij}) \]
\[ = W00(m_{ij}) \oplus Z00(m_{ij}) \].

\[ \]

\[ \]

Theorem 3 The implication algorithm is correct.

Proof. By Lemma 3 and Lemma 4 the theorem follows. \( \blacksquare \)

Theorem 4 The matrix obtained by applying the implication algorithm is again a PR-matrix.

Proof.
1) \( x_{ij} = 0 \) is supplied:

a) Consistency:- Assume that there exists an inconsistency index triplet \((p, q, r)\) in the resultant matrix \( M_1 \). Then \((p, q) \in K0(M)\) and \((p, r), (r, q) \in K1(M)\). Also \( K1(M) = K1(M) \) and \( K0(M) = K0(M) \cup W00(m_{ij}) \), since \( K0(M) = K0(M) \cup Y00(m_{ij}) \)
\[ = K0(M) \cup (W00(m_{ij}) \oplus Z00(m_{ij}) \]
\[ = K0(M) \cup (W00(m_{ij}) \oplus (Y00(m_{ij}) \cap K0(M)) \]
\[ = [K0(M) \cup W00(m_{ij})] \cup [K0(M) \cup (Y00(m_{ij}) \cap K0(M))] \]
\[ = [K0(M) \cup W00(m_{ij})] \cup K0(M) \]
\[ = K0(M) \cup W00(m_{ij}) \].
Hence \((p, q) \in K0(M) \cup W00(m_{ij})\). If \((p, q) \in K0(M)\) then \(m_{pq} = 0\), \(m_{pr}, m_{rq} = 1\). This contradicts the consistency of the matrix \(M\). Otherwise, \((p, q) \in W00(m_{ij})\) then \((m_{ip}, m_{pr}, m_{rq}, m_{qj}) = 1\) means that \(m_{ij} = 1\). This contradicts the maximality of \(M\). Thus \(M\) satisfies the consistency.

b) Maximality:- Assume that there exists an index triplet \((p, q, r)\) that violates the maximality in the resultant matrix \(M\) such that \((p, q) \in K0(M)\) and \((p, r) \in K1(M)\) and \(m_{rj} = x\). If \((p, q) \in K0(M)\), then the matrix \(M\) is not maximal. Otherwise \((p, q) \in W00(m_{ij})\), then \((m_{ip} = 1, m_{pr} = 1)\) means that \(m_{ir} = 1\). Hence \(m_{ir} = 1, m_{qj} = 1\) leads to \(m_{rj} = 0\) by the \(0 \Rightarrow 0\) implication. This contradicts the assumption that \(m_{rj} = x\). Thus first condition for maximality is satisfied.

To prove the second condition of maximality. Assume that there exists an index triplet \((p, q, r)\) that violates the maximality in the resultant matrix \(M\) such that \((p, q) \in K0(M)\) and \((r, q) \in K1(M)\) and \(m_{pr} = x\). If \((p, q) \in K0(M)\), then the matrix \(M\) is not maximal. Otherwise \((p, q) \in W00(m_{ij})\), then \((m_{qj} = 1, m_{rj} = 1)\) means that \(m_{rj} = 1\). Hence \(m_{rj} = 1, m_{ip} = 1\) leads to \(m_{pr} = 0\) by the \(0 \Rightarrow 0\) implication. This contradicts the assumption that \(m_{pr} = x\). Thus the second condition for maximality is satisfied.

The third condition for maximality is obvious since \(K1(M) = K1(M)\).

2) \(x_{ij} = 1\) is supplied:

a) Consistency:- Assume that there exists an inconsistency triplet \((p, q, r)\) in the resultant matrix \(M\). Then \((p, q) \in K0(M)\) and \((p, r), (r, q) \in K1(M)\). Also

\[
K1(M) = K1(M) \cup W11(m_{ij})
\]

and

\[
K0(M) = K0(M) \cup W10(m_{ij})
\]

Hence \((p, q) \in K0(M) \cup W10(m_{ij})\) and \((p, r), (r, q) \in K1(M) \cup W11(m_{ij})\). If \((p, q) \in K0(M)\) and \((p, r), (r, q) \in K1(M)\), \(m_{pq} = 0\) and \(m_{pr} = 1, m_{rq} = 1\). This contradicts the consistency of \(M\). So \((p, q) \notin K0(M)\). Therefore \((p, q) \in W10(m_{ij})\), so \(m_{pq} = x\). This contradicts the maximality of \(M\). Thus \(M\) satisfies the consistency.
b) Maximality:- Assume that there exists an index triplet \((p, q, r)\) that violates the maximality condition in the resultant matrix \(M'\) such that \((p, q) \in K0(M')\) and \((r, q) \in K1(M')\) and \(m_{rq} = x\). Then \((p, q) \in K0(M) \cup W10(m_{ij})\) and \((r, q) \in K1(M) \cup W11(m_{ij})\). If \((p, q) \in K0(M)\), then the matrix \(M\) is not maximal. Otherwise, \((p, q) \in W10(m_{ij})\), then (1) \(m_{jq} = 0\), \(m_{pr} = 1\) means that \(m_{rq} = 1\). Hence \(m_{jq} = 1\), \(m_{iq} = 0\) leads to \(m_{rq} = 0\) by the \(1 \Rightarrow 0\) implication. This contradicts the assumption that \(m_{rq} = x\) and (2) \(m_{pr} = 1\), \(m_{pj} = 0\) means that \(m_{rj} = 0\). Hence \(m_{rj} = 0\), \(m_{qi} = 1\) means that leads to \(m_{rq} = 0\) by the \(1 \Rightarrow 0\) implication. This contradicts the assumption that \(m_{rq} = x\). Thus the first condition for maximality is satisfied.

Assume that there exists an index triplet \((p, q, r)\) that violates the maximality condition in the resultant matrix \(M'\) such that \((p, q) \in K0(M')\) and \((r, q) \in K1(M')\) and \(m_{pr} = x\). Then \((p, q) \in K0(M) \cup W10(m_{ij})\) and \((r, q) \in K1(M) \cup W11(m_{ij})\). If \((p, q) \in K0(M)\), then the matrix \(M\) is not maximal. Otherwise, \((p, q) \in W10(m_{ij})\), then (1) \(m_{iq} = 0\), \(m_{rq} = 1\) means that \(m_{ir} = 0\). Hence \(m_{ir} = 0\), \(m_{jp} = 1\) leads to \(m_{pr} = 0\) by the \(1 \Rightarrow 0\) implication. This contradicts the assumption that \(m_{pr} = x\) and (2) \(m_{rq} = 1\), \(m_{qi} = 1\) means that \(m_{ri} = 1\). Hence \(m_{ri} = 1\), \(m_{pj} = 0\) leads to \(m_{pr} = 0\) by the \(1 \Rightarrow 0\) implication. This contradicts the assumption that \(m_{pr} = x\). Thus the second condition for maximality is satisfied.

Assume that there exists an index triplet \((p, q, r)\) that violates the maximality condition in the resultant matrix \(M'\) such that \((p, q) \in K1(M')\) and \((r, q) \in K1(M')\) and \(m_{pr} = x\). Then \((p, q) \in K1(M) \cup W10(m_{ij})\) and \((q, r) \in K1(M) \cup W11(m_{ij})\). If \((p, q) \in K1(M)\), then the matrix \(M\) is not maximal. Otherwise, \((p, q) \in W10(m_{ij})\), then (1) \(m_{iq} = 0\), \(m_{rq} = 1\) means that \(m_{ir} = 0\). Hence \(m_{ir} = 0\), \(m_{jp} = 1\) leads to \(m_{pr} = 0\) by the \(1 \Rightarrow 0\) implication. This contradicts the assumption that \(m_{pr} = x\) and (2) \(m_{rq} = 1\), \(m_{qi} = 1\) means that \(m_{ri} = 1\). Hence \(m_{ri} = 1\), \(m_{pj} = 0\) leads to \(m_{pr} = 0\) by the \(1 \Rightarrow 0\) implication. Thus the third condition for maximality is satisfied.

We can show that the matrix obtained by applying the implication algorithm to a \(PR\)-matrix \(M\) by supplying a value 1 or 0 to an unknown entry in \(M\) is
also a PR-matrix, say $M_1$. Then the number of unknown entries in $M_1$ is less than that in $M$, we can see that, $M < M_1$. We apply the implication algorithm to $M_1$ by supplying a value 1 or 0 to an unknown in $M_1$ and let the resulting PR-matrix be $M_2$. We can easily see that $M < M_1 < M_2$. Again applying the implication algorithm to $M_2$ and continuing this process we get a sequence of PR-matrices $M_1, M_2, \ldots, M_r$ with $M_r = R$, a reachability matrix such that $M_i < M < M_i + 1$, for $i = 1, 2, \ldots, r - 1$. This process of continuous application of the algorithm is called a successive application of the implication algorithm. Therefore we can use the implication algorithm to generate the reachability matrices. This is established by the following theorem.

**Theorem 5** The matrix obtained by successive application of the implication algorithm to the completely unknown PR-matrix $PR_0$ until there is no unknown element is a reachability matrix.

**Proof.** proof follows from Theorem 2. 

Note:- By applying implication algorithm to a PR-matrix once we get a PR-matrix with fewer unknowns than the original PR-matrix. Again by applying the implication algorithm to the resulting PR-matrix will also result in a PR-matrix with fewer unknowns. Thus by applying implication algorithm again and again to the resulting PR-matrices successively as above will finally lead to a matrix with no unknowns, which is a reachability matrix.

**Theorem 6** The collection of all matrices obtained by applying the implication algorithm on a completely unknown PR-matrix by giving both the values 0 and 1 to the unknowns at each stage of the algorithm is precisely the collection of all reachability matrices on a finite set.

**Proof.**

Let $N = \{1, 2, \ldots, n\}$ be a finite set. Let $R$ be the set of all reachability matrices on $N$. Let $M$ be a completely unknown PR-matrix whose elements are denoted by $x_{ij}$. Suppose that the first unknown with $i = 1$ is given the value 1 or 0 and then apply the implication procedure. Then we will get a PR-matrix
with some known values. Since both the implication procedure and reachability implies transitivity, there will be some element of $R$ which has values same as that of the PR-matrix obtained by applying the implication procedure. In the next step give the value 0 or 1 to the unknown of the PR-matrix when $i > 1$, and then apply the implication procedure. Then also the resulting PR-matrix obtained will have elements same as one of the elements of $R$. Proceeding like this give the value 1 or 0 for the unknown when $i = m (m > 2)$ and apply the implication procedure, we get a PR-matrix whose elements are same as one of the elements of $R$. Proceeding like we will get a completely filled matrix whose elements are same as some element of $R$. Hence the theorem.

Next, we modify the algorithm to get all reachability matrices up to isomorphism by multiplying with the permutation matrices, where

a permutation matrix of order $n$ is the matrix $P_{ij}$ obtained by permutation of the $i^{th}$ and $j^{th}$ rows of the identity matrix with $i < j$. Every row and column therefore contains precisely a single 1, and every permutation corresponds to a unique permutation matrix.

There are therefore $nC_2$ permutation matrices of order $n$.

Algorithm 4 Given $m$ the number of reachability matrices and let $M_1$ and $M_2$ be the first two reachability matrices obtained by successive application of the algorithm. Generate all permutation matrices and let them be denoted by $P_0, P_1, P_2, ...$. The number of permutation matrices of size $n$ is equal to $nC_2$.

Set $n = 1$
Set $s = 1$
Set $p = 1$
Set $r = 1$
Step 1
Compute $P_n M_{(r+1)} P_{(n-1)}$
IF $P_n M_{(r+1)} P_{(n-1)} = M_r$
Then include $M_{(r+1)}$ in the isomorphism class $I_s$
ELSE
Include it in another isomorphism class \( I_{(s+p)} \)

**Step 2**

Set \( n = n + 1 \)

Compute \( P_n M_{(r+1)} P_{(n-1)} \)

Compare the result with elements of \( I_s \) and \( I_{(s+p)} \)

If result is equal to the elements of \( I_s \) and \( I_{(s+p)} \) include it in \( I_s \) or \( I_{(s+p)} \) as the case may be

ELSE

Set \( p = p + 1 \)

Include the result in \( I_{(s+p)} \)

Goto Step 2

**WHILE** \( n = nC_2 + 1 \)

**STOP**

Set \( r = r + 1 \)

Goto Step 1

**While** \( r = m - 1 \)

Stop

**END**

Now we can modify the algorithm to generate a Partial order on the set \( N \) and obtain analogous results as in the case of pre-ordered relations on \( N \).

### 2.3 Partial Order Relations

We have the following analogous lemmas and theorems as in the case of pre-orders for partial orders also. First we define the index sets with the necessary changes.

Let \( M \) be a PR-matrix. For the element \( m_{ij} \), The following index sets are defined:

\[
Y_{11}(m_{ij}) = \{(p, q); p \in D_M(i) \text{ and } q \in L_M(j)\}
\]
Y10(m_{ij}) = \{(p, q); p \in L_M(j) \text{ and } q \in L_m(i) \text{ or } p \in D_m(j) \text{ and } q \in D_M(i)\}

Y00(m_{ij}) = \{(p, q); p \in L_M(i) \text{ and } q \in D_M(j)\}

W11(m_{ij}) = \{(p, q); m_{pq} = x, \text{ and } (p, q) \in Y11(m_{ij})\}

W10(m_{ij}) = \{(p, q); m_{pq} = x, \text{ and } (p, q) \in Y10(m_{ij})\} \cup \{(q, p); m_{qp} = x, \text{ and } (p, q) \in Y11(m_{ij})\}

W00(m_{ij}) = \{(p, q); m_{pq} = x, \text{ and } (p, q) \in Y00(m_{ij})\}

Z11(m_{ij}) = \{Y11(m_{ij}) \cap K1(M)\}

Z10(m_{ij}) = \{Y10(m_{ij}) \cap K0(M)\}

Z00(m_{ij}) = \{Y00(m_{ij}) \cap K0(M)\}

Algorithm 5 (Implication Algorithm for Partial orders)

BEGIN

supply the value one or zero to the unknown m_{ij};

IF (supplied value is one) THEN BEGIN

set m_{ij} = 1,

set m_{ji} = 0,

set m_{pq} = 1 \text{ and } m_{qp} = 0 \text{ for all } (p, q) \in Y11(m_{ij}); (1 \Rightarrow 1 \text{ implication})

set m_{pq} = 0; \text{ for all } (p, q) \in Y10(m_{ij}); (1 \Rightarrow 0 \text{ implication})

END

ELSE BEGIN

set m_{ij} = 0, i \neq j;

set m_{pq} = 0; \text{ for all } (p, q) \in Y00(m_{ij}); (0 \Rightarrow 0 \text{ implication})

END

END.

The time complexity of the algorithm is \(O(n^2)\).

IF \(M\) is a PR matrix, Let \(M(k)\) be the PR matrix obtained from \(M\) by applying the implication algorithm supplying the value \(k = 0\) or 1 for the next unknown. (the next unknown is the first unknown in the dictionary order).
Algorithm ARM (all reachability matrices) Input a PR matrix $M$ and output all reachability matrices greater than $M$.

**Algorithm 6 (ARM)**

*STEP1*: If $M$ is a reachability matrix, output $M$ and go to *STEP 4*

*STEP2*: ARM($M(0)$)

*STEP3*: ARM($M(1)$)

*STEP4*: STOP.

The worst time complexity of the algorithm is $O((2n^2)^{n^2-n})$.

The detailed procedure to find the set of all reachability matrices on a finite set can be described as follows for a PR-matrix of order $n$.

**Procedure 2**:

For($i = 1; i \leq \text{size}; i + 1$)

For($j = 1; j \leq \text{size}; j + 1$)

*Step 1*

If val=1

$m_{ij} = 1$;

set $m_{ji} = 0$;

For($p = 1; p \leq \text{size}; p + 1$)

For($q = 1; q \leq \text{size}; q + 1$)

If ($m_{pi} = 1$ and $m_{jq} = 1$)

set $m_{pq} = 1$ and $m_{qp} = 0$;

Else if ($m_{jp} = 1$ and $m_{iq} = 0$ or $m_{pj} = 0$ and $m_{qi} = 1$)

Set $m_{pq} = 0$

Goto Step 3

*Step 2*

set $m_{ij} = 0$;

For($p = 1; p \leq \text{size}; p + 1$)

For($q = 1; q \leq \text{size}; q + 1$)
Enumeration of All Quasi-Ordered Relations

If \( m_{ip} = 1 \) and \( m_{qj} = 1 \)
Set \( m_{pq} = 0 \).

Step 3
For \( i = 1; i \leq \text{size}; i + 1 \)
For \( j = 1; j \leq \text{size}; j + 1 \)
If \( m_{ij} = x \)
If \( \text{val} = 0 \)
Set \( m_{ij} = 0 \)
Goto Step 2
Else
Set \( m_{ij} = 1 \)
Goto Step 1
Else
Stop
Repeat the procedure and generate the next matrix.
Continuing like this generate all matrices by supplying the values 0 and 1 to the unknowns at each stage of the implication.
END

As in the case of quasi-orders, the algorithm can easily be parallelized thereby reducing the time complexity of the algorithm to \( O((2^n^2)(n^2-n)/k) \). The analogous lemmas and theorems in the case of partial orders are stated below.

Lemma 5 If \( M \) is a partially filled reachability matrix, then
\[
Y11(M_{ij}) \cap Y10(M_{ij}) = \phi
\]

Lemma 6 \( Y11(m_{ij}) \) is the disjoint union of \( W11(m_{ij}) \) and \( Z11(M_{ij}) \), \( Y10(M_{ij}) \) the disjoint union of \( W10(M_{ij}) \) and \( Z10(M_{ij}) \) and \( Y00(M_{ij}) \) the disjoint union of \( W00(M_{ij}) \) and \( Z00(M_{ij}) \)

Theorem 7 The implication algorithm is correct.
Proof.

By the above two lemmas the theorem is proved. ■

Theorem 8 The matrix obtained by applying the implication algorithm is again a PR-matrix.

Theorem 9 The collection of all matrices obtained by applying the implication algorithm on a completely unknown PR-matrix by giving both the values 0 and 1 to the unknowns at each stage of the algorithm is precisely the collection of all reachability matrices on a finite set.

For partial orders also we can modify the algorithm to get all reachability matrices up to isomorphism by multiplying with the permutation matrices.