4. $I_1$-PATH TRANSIT FUNCTION AND BETWEENNESS

4.1 Introduction

In this chapter, we go to another path transit function derived from the interval function $I$ of a graph $G$, namely the $I_1$ path transit function. This transit function is a special case of the more general path transit function $I_j$ transit function which we have defined in Chapter 2. This transit function is an immediate generalization of the $I^A$ transit function. Like we did in Chapter 3, we shall discuss the betweenness of $I_1$ transit function in this chapter also. Before going to the betweenness axioms, we first observe some basic facts about the $I_j$-transit function in general and $I_1$ transit function in particular. This we will do in Section 4.2. In Section 4.3, we shall consider the betweenness and in the last section, we shall make concluding remarks the way we did in Chapter 3.

Recalling the definition of $I_j$ transit function,

$$I_j(u,v) = \{w \in V(G) | w \text{ lies on some } I_j - u,v- \text{path in } G \} = \{w \in V | d(u,w) + d(w,v) \leq d(u,v) + j \}, \text{ for } j \geq 1$$

We also need a special class of graphs, named as $\Theta$-graphs. Theta graphs have been studied extensively in graph theory. They arise in problems concerning graph planarity, graph bandwidth, chromatic polynomials and in many other areas, see, e.g., [42] and references therein. For our purpose, we define a $\Theta$-graph as follows.

A $\Theta$-graph consists of a vertex $u$ and an edge $vx$, with $d(u,v) = d(u,x)$, together

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with three internally disjoint paths $P$, $Q$, and $R$ such that $P$ is a shortest $u,v$-path, $Q$ is a shortest $u,x$-path, and $R$ is a $u,x$-path of length $d(u,x) + 1$. An example of two $\Theta$-graphs with four and six vertices is depicted in the Figure 4.1. Note that for both graphs, the paths $P$, $Q$, and the edge $ux$ form an odd cycle $C$ of length $2d(u,v) + 1$. The path $R$ is depicted by dotted lines.

By long even house we mean an even cycle on which a triangle is attached as in the following figure.

4.2 The $I_j$ path transit function

In this section we observe some basic facts about the $I_j$ transit function, $I_1$ function and its relationship with $I^\Delta$ and the interval function $I$.

Since trees are the simplest connected graph and there is a unique path between any two vertices in a tree, it follows that all paths coincide in a tree and hence all path transit functions coincide. From this, we have the simple fact:

Observation 4.1: $I_j(u,v) = I(u,v)$ for all $u, v \in V(G)$ with $j \geq 2$ if and only if $G$ is a tree.

Observation 4.2: $I^\Delta (u,v) \subseteq I_1 (u,v)$, for all $u, v \in V(G)$.

Observation 4.3: Let $G$ be a connected graph, and let $j \geq 2 \times diam(G)$. Then $I_j(u,v) = V(G)$, for all distinct $u, v$ in $G$. 

Fig. 4.1: Forbidden subgraph for $I_1$ to be (b2), ($\Theta$ graphs)
Observation 4.4: If \( G \) is a complete graph, then \( I(u,v) \neq I^\Delta(u,v) = I_1(u,v) = I_2(u,v) = \ldots = I_j(u,v) = \ldots V(G) \), for all \( u,v \in V(G) \) and for \( j \geq 1 \).

Observation 4.5: For all \( j, I_j(u,v) \subseteq I_{j+1}(u,v) \).

Observation 4.6: If \( x \in I_1(u,v) \) and \( uv \in E(G) \) then \( ux, xv \in E(G) \).

Lemma 4.7: \( I(u,x) \subseteq I^\Delta(u,v) \subseteq I_1(u,v) \), for all \( x \in I_1(u,v) \).

Proof. Let \( x \in I_1(u,v) \). We distinguish two cases.

Case 1: \( x \in I(u,v) \).
Then we have \( I(u,x) \subseteq I(u,v) \subseteq I_1(u,v) \).

Case 2: \( x \notin I(u,v) \).
Then we have \( d(u,x) + d(x,v) = d(u,v) + 1 \).
Let \( y \in I(u,x) \). Then \( d(u,y) + d(y,x) = d(u,x) \leq d(u,v) + 1 \). Therefore \( y \in I_1(u,v) \).
Hence \( I(u,x) \subseteq I_1(u,v) \). \( \square \)

Lemma 4.8: \( I_1(u,v) = V(G) \) for all distinct \( u,v \in V(G) \) if and only if \( G \) is complete.

Proof. Let \( G \) be complete. Then \( I_1(u,v) = V(G) \), for all distinct \( u,v \in V(G) \).

Conversely, let \( I_1(u,v) = V(G) \), for all distinct \( u,v \) in \( V \). Suppose \( G \) is not complete. Then \( G \) contains at least two vertices \( u \) and \( x \) with \( d(u,x) = 2 \). Let \( v \) be a common neighbour of \( u \) and \( x \). Now \( d(u,v) = 1 \) and there is no \( u,v \)-path of length 2 containing \( x \). Hence \( v \notin I_1(u,x) \), whence \( I_1(u,x) \neq V(G) \). \( \square \)

Lemma 4.9: \( I_1 = I \) if and only if \( G \) is bipartite.

Proof. First let \( I_1 = I \). Assume that \( G \) contains an odd cycle. We choose \( C \) to be a shortest odd cycle, so that \( C \) is isometric (that is, distances measured in
4. $I_1$ - path transit function and betweenness

$C$ are equal to those measured in $G$). Then we can take two vertices $u, v \in C$ so that there exist two paths $P_1$ and $P_2$, where $P_1 = u \rightarrow \ldots \rightarrow v$ is of length $d(u, v)$, and $P_2$ is of length $d(u, v) + 1$ so that $u \rightarrow P_1 \rightarrow v \rightarrow P_2 \rightarrow u$ is a cycle of length $2d(u, v) + 1$, so that $I_1(u, v)$ contains $C$, in particular the path $P_2$. But $I$ will not contain the path $P_2$, so $I_1 \neq I$, a contradiction to the assumption.

Conversely, let $G$ be bipartite. We have to show that $I_1 = I$. Assume to the contrary that $I_1 \neq I$. Then for some $u, v \in V(G)$, we have $I_1(u, v) \neq I(u, v)$. Then $I_1(u, v)$ will contain at least two $u, v$-paths of length $d(u, v)$ and $d(u, v) + 1$, respectively, so that these two paths together form a closed walk of length $2d(u, v) + 1$, which contains an odd cycle, a contradiction.

\[4.3 \text{ Betweenness of the } I_1 \text{ path transit function}\]

For both $I^\Delta$ and $I_j$, if $G$ is a $K_3$, say on $u, v, w$, then we see that axiom (b3) fails. Since $w \in R(u, v)$, and $v \in R(u, w)$ but $w \notin R(v, v)$, for $R = I^\Delta$ or $I_j$. Because of this reason we consider axiom (b3') instead of axiom (b3) as is done for $I^\Delta$.

The following lemma justifies partially our consideration of the $I_1$ transit function only and not $I_j$, $j \geq 2$, as far as the betweenness is concerned.

**Lemma 4.10:** The path transit function $I_j$ satisfies axiom (b1), for $j \geq 2$, if and only if $G$ is a tree.

**Proof.** If $G$ is a tree, then $I_j = I$, so $I_j$ trivially satisfies axiom (b1). So let $G$ contains a cycle. We choose a shortest cycle $C$ in $G$, so that $C$ is isometric. Choose vertices $u$ and $v$ on $C$ such that $d(u, v)$ is as large as possible, and let $x$ be a neighbour of $v$ on $C$. Then $d(u, x) \geq d(u, v) - 1$. So the $u, x$-path on $C$ containing $v$ is of length at most $d(u, x) + 2$. Hence we have that $v \in I_j(u, x)$, which contradicts axiom (b1). \[\square\]

From Lemma 4.9, we have the following lemma regarding the betweenness (b1).
Lemma 4.11: $I_1$ satisfies axiom (b1) if and only if $G$ is bipartite.

Proof. If $G$ is bipartite, then, by Lemma 4.9, we have $I_1 = I$. Since $I$ satisfies axiom (b1), it follows that $I_1$ also satisfies axiom (b1).

Conversely, let $I_1$ satisfy axiom (b1). Then $x \in I_1(u, v)$, whence $v \neq I_1(u, x)$. Assume that $G$ contains an odd cycle, and let $C$ be a shortest odd cycle, so that $C$ is isometric. In $C$ we can find $u, v$ such that there exists a vertex $x$ with $d(u, x) = d(u, v)$ and $xv \in E(G)$. Now $x \in I_1(u, v)$ and $v \in I_1(u, x)$, which contradicts axiom (b1).

Next we consider the betweenness (b2).

Theorem 4.12: The path transit function $I_1$ satisfies axiom (b2) if and only if $G$ is a $\Theta$-free graph.

Proof. First let $G$ be $\Theta$-free. Assume that $I_1$ does not satisfy axiom (b2). Then, for some $u, v \in V(G)$ and $x \in I_1(u, v)$, we have $I_1(u, x) \not\subseteq I_1(u, v)$. Therefore there exists some $y \in I_1(u, x)$ with $y \not\in I_1(u, v)$. If $x \in I(u, v)$, then, since $I(u, x) \subsetneq I(u, v)$, we have $I_1(u, x) \subsetneq I_1(u, v)$. Hence we have $x \in I_1(u, v) \setminus I(u, v)$. Therefore the cycle $u \to \ldots \to x \to \ldots \to v \to \ldots \to u$ is an odd cycle. Since $I(u, x) \subseteq I_1(u, v)$, we have $y \not\in I(u, x)$, that is, there is a path $P_1 = u \to \ldots \to y \to \ldots \to x$ of length $d(u, x) + 1$. This path $P_1$ together with the odd cycle $u \to \ldots \to x \to \ldots \to v \to \ldots \to u$ is a $\Theta$-graph. This is impossible.

Conversely, let $I_1$ satisfy axiom (b2). Assume that $G$ contains an induced $\Theta$-subgraph. Then, for some $u, v \in V(\Theta)$, there exists a vertex $x$ on a $u, v$-path $P_1$ of length $d(u, v) + 1$, and there exists a $u, x$-path $P_2$ of length $d(u, x) + 1$. Since $d(u, x) + d(x, v) = d(u, v) + 1$ we have that $l(P_1) + l(P_2) = d(u, x) + 1 + d(u, v) - d(u, x) + 1 = d(u, v) + 2$, where $l(P_1)$ is the length of the path $P_1$. Therefore there exists a vertex $y \in I_1(u, v)$ with $y \not\in I_1(u, v)$, so that axiom (b2) does not hold. Hence $G$ is $\Theta$-free.
Corollary 4.13: $I_1$ is a betweenness if and only if $G$ is bipartite.

\[ \text{Fig. 4.2: Forbidden subgraph for } I_1 \text{ to be } (b3')(\text{odd hole with } d(x,v) = d(y,v)) \]

Before we move on to prove axiom $(b3')$, we have the following observation.

Observation 4.14: For axiom $(b3')$, if $uv \in E(G)$, and $x \in I_1(u,v)$, $y \in I_1(u,x)$, then we see that for $u, v \neq y, x \in I_1(y,v)$. So for any two adjacent vertices $u, v$ we see that axiom $(b3')$ is satisfied.

Theorem 4.15: $I_1$ satisfies axiom $(b3')$ if and only if $G$ is paw and odd hole free.

Proof. First let $G$ contain an induced odd cycle of length at least 5, and let $C$ be a shortest such cycle. Then $C$ is isometric. Choose the vertices \{u, v, x, y\} as in the Figure 4.2 of the odd hole. Then it is easy to check that for these vertices axiom $(b3')$ does not hold. Second, let $G$ contain an induced paw. Choose the vertices \{u, v, x, y\} as in the Figure 3.2 of the paw. Again it is easy to check that for these vertices axiom $(b3')$ does not hold.

Conversely, let $G$ be paw and odd hole free. Assume that $I_1$ does not satisfy axiom $(b3')$. Then, for some $u, v \in V(G)$ and $x \in I_1(u,v)$, $u, v \neq y \in I_1(u,x)$, we have $x \not\in I_1(y,v)$. We consider two cases.

Case 1 $x \in I(u,v)$.

If we would have $y \in I(u,v)$, then we have $x \in I(y,v)$. So $y \in I_1(u,x) \setminus I(u,x)$.

Suppose that $y$ is adjacent to two consecutive vertices $x_1, x_2$ on some shortest $u, x$-path $P$. If $x_2 \neq x$, then let $x_3$ be the neighbour of $x_2$ on $P$ closer to $x$. Since $y$ is not in $I(u,x)$, it follows that $y$ is not adjacent to $x_3$. But then $y, x_1, x_2, x_3$ induce a
paw in $G$, which is forbidden. So we must have $x_2 = x$. Let $x'$ be a neighbour of $x$ on a shortest $x, v$-path $P'$. To avoid a paw, it follows that $y$ is adjacent to $x'$. But now $u \to \ldots P \to x_1 \to y \to x' \to \ldots P' \to v$ is a path of length $d(u, x) + d(x, v)$, whence it must be a shortest $u, v$-path. Hence the path $y \to x \to \ldots P' \to v$ is a path of length $d(y, v) + 1$. This implies that $x \in I_1(y, v)$, which is impossible.

So $y$ is not adjacent to two consecutive vertices on any shortest $u, x$-path. Let $Q$ be a $u, x$-path of length $d(u, x) + 1$ containing $y$. Now we choose a shortest $u, x$-path $P$. Let $x_1$ be the last common vertex of $Q$ and $P$ before $y$ and let $x_2$ be the first common vertex of $Q$ and $P$ after $y$. We may choose $P$ such that $d(x_1, x_2)$ is as small as possible. Let $P'$ and $Q'$ be the subpaths between $x_1$ and $x_2$ of $P$ and $Q$, respectively. Then $P'$ is of length $d(x_1, x_2)$ and $Q'$ is of length $d(x_1, x_2) + 1$. So $Q' \to P'$ is a cycle of length $2d(x_1, x_2) + 1 \geq 5$. Hence it can not be induced, and there is a chord $y'x'$ between an internal vertex $y'$ of $Q'$ and an internal vertex $x'$ of $P'$. Assume that $y'$ is between $y$ and $x_1$ (otherwise replace $x_1$ by $x_2$ in the following argument). We may choose the chord such that $y_1$ is the vertex on $Q'$ closest to $x_1$ with a chord to $P'$, and amongst these chords we may choose the one with $x'$ closest to $x_1$. Let $P''$ be the subpath of $P$ between $x_1$ and $x'$, and $Q''$ that of $Q$ between $x_1$ and $y'$. Because of the choice of $P$ with $d(x_1, x_2)$ as small as possible, the subpaths $P''$ and $Q''$ must have the same length. But then together with the chord $y'x'$ they form an induced odd cycle. To avoid an odd hole the length of $P''$ and $Q''$ must be 1, so $x_1, x', y'$ form a triangle. Now $y'$ can not be adjacent to the neighbour $x''$ of $x'$ on $P$ closer to $x$ (for otherwise we get a conflict with the choice of $P$). So the triangle together with $x''$ induces a paw. This impossibility settles Case 1.

**Case 2.** $x \notin I(u, v)$.

Then there exists a $u, v$-path $R$ of length $d(u, v) + 1$ containing $x$. Moreover, the subpath $P$ of $R$ between $u$ and $x$ is a shortest $u, x$-path. Note that we may take any shortest $u, x$-path to be $P$. Now, if $y$ would be in $I(u, x)$, then it is easy to see that
First suppose that \( y \) is adjacent to two consecutive vertices \( x_1, x_2 \) on \( P \). If we would have \( x_2 \neq x \), then let \( x_3 \) be the neighbour of \( x_2 \) on \( P \) closer to \( x \). Then \( y, x_1, x_2, x_3 \) would induce a paw. So we must have \( x_2 = x \). If we would have \( u \neq x_1 \), then let \( x_4 \) be the neighbour of \( x_1 \) on \( P \) closer to \( u \). Then \( y, x_2, x_1, x_4 \) would induce a paw. So we have that \( u = x_1 \). Moreover, \( d(u, v) = d(x, v) \). Now \( y \to x \to \ldots R \ldots \to v \) is a path of length \( d(u, x) + 1 \). So, to avoid \( x \) being in \( I_1 (y, v) \) we must have that \( d(y, v) < d(u, x) \). Since \( y \) and \( x \) are adjacent, it follows that \( d(y, v) = d(x, v) - 1 \). Since \( y \neq v \), there is a neighbour \( y_1 \) of \( y \) with \( d(y_1, v) = d(y, v) - 1 = d(x, v) - 2 = d(u, v) - 2 \). But now \( u, x, y, y_1 \) induce a paw, which is forbidden.

So \( y \) is not adjacent to two consecutive vertices on \( P \). Note that in Case 1 when we dealt with the subcase that \( y \) was not adjacent to two consecutive vertices on \( P \), we only used the shortest \( u, x \)-path \( P \) and a path of length \( d(u, x) + 1 \) containing \( y \). Therefore, in a similar way as in this subcase of Case 1, we get to a contradiction. This settles Case 2, and the theorem.

\[ \square \]

### 4.4 Concluding remarks

We have seen in Chapter 3 that the class of graphs for which the \( I^\Delta \) transit function satisfies the \((b1)\) axiom is the same as the class for which \( I^\Delta \) satisfies \((b3)\). Here too, we can observe that both the classes of graphs are the same. This follows from the observation that if \( I_1 \) satisfies \((b1)\), then \( G \) is bipartite and we have seen that in a bipartite graph \( I_1 \) coincides with \( I \). Therefore \( I_1 \) satisfies the \((b3)\) axiom on a bipartite graph, which shows that the class of graphs for which \( I_1 \) satisfies \((b1)\) is the same as the class of graphs which satisfies the \((b3)\) axiom. From the discussions of this chapter, we have the following remarks:
Remark 4.16: The $I_1$ transit function of a graph $G$ is geometric if and only if $G$ is bipartite.

Remark 4.17: The $I_1$ transit function of a graph $G$ satisfies $(b2)$ and $(b3')$ if and only if $G$ is Θ, paw and odd hole free.

Remark 4.18: For the $I_1$ transit function of a graph $G$, the axioms $(b1), (b3)$ and $(b3')$ are equivalent if and only if $G$ is bipartite.

We conclude this chapter with a similar analysis of the implications of the four betweenness for the transit function $I_1$, as we did it for $I^\Lambda$ in the previous chapter. This is explained in the following table.

<table>
<thead>
<tr>
<th>$(b1)$</th>
<th>$(b2)$</th>
<th>$(b3')$</th>
<th>$(m)$</th>
<th>Graph</th>
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<td>No graph</td>
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<td>Figure 4.3</td>
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<td>Paw</td>
</tr>
<tr>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
<td>Bipartite not-interval monotone graph (for e.g. $K_{2,3}$)</td>
</tr>
<tr>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
<td>No graph</td>
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<tr>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
<td>Bipartite interval monotone graph (for e.g. $C_{2n}$)</td>
</tr>
</tbody>
</table>

Tab. 4.1: $I_1$ Transit function with possible betweenness relations. × denotes that the corresponding axiom is satisfied.
Fig. 4.3: A graph for which $I_1$ satisfies (b2) but not (m).