2. BASIC OBSERVATIONS OF TRANSIT FUNCTIONS AND BETWEENNESS

2.1 Introduction

In this chapter, the preliminary concepts and some observations of transit functions and betweenness are dealt with. In Section 2.2 we fix the notations, terminology and preliminary concepts. In Section 2.3, we discuss the implications of the betweenness axioms for arbitrary binary transit functions $R$. In Section 2.4, we generalize the definition of a transit function $R(u, v)$ to $n$-ary transit function $R(u_1, u_2, \ldots, u_n)$, introduce the analogous betweenness axioms, discuss the implications of betweenness as in the binary case, and also observe some basic results.

2.2 Preliminary concepts

All the basic notions and notations for graphs are taken from West [93]. Any mathematical objects involving points and connections between them may be called a graph. For completeness, we formally define a graph and the basic concepts in graph theory needed for our purpose.

A graph $G = (V, E)$ consists of two sets $V$ and $E$, the elements of $V$ are called vertices, the elements of $E$ are called the edges. Each edge has a set of two distinct vertices associated to it, which are called its end points. An edge is said to join its endpoints. The notations $V(G)$ and $E(G)$ are used for the vertex and edge-sets when $G$ is not the only graph under consideration. In this thesis, we consider only graphs $G$ with $V(G) \neq \emptyset$. Some authors call our graphs as simple. If the edge $e$
consists of vertices $u$ and $v$, then we say that $u$ and $v$ are adjacent or neighbours, and we write $e = uv$. The set of all neighbours of a vertex $v$ is called the neighbourhood of $v$ denoted as $N(v)$. If vertex $v$ in a graph is an endpoint of edge $e$, then, we say that $v$ is incident on $e$ or vice versa. A walk is a sequence of vertices $v_0, v_1, ..., v_l$ such that $v_i v_{i+1} \in E$, for $i = 0, ..., l - 1$; its length is $l$. If $v_i \neq v_j$, for $i \neq j$, the walk is called a path, we call it a $v_0, v_l$ path. By $u \rightarrow y \ldots v$, we mean it is a $u, v$-path where $u$ and $y$ are adjacent whereas $y$ and $v$ not adjacent. A walk $v_0 v_1 ... v_l$ is called a cycle of length $l$, if $v_0 = v_l$ and $v_i \neq v_j$ for $i, j \in \{1, 2, ..., l - 1\}$. A cycle of length $n$ is denoted by $C_n$.

A $u, v$-path $P$ is called a $u, v$-shortest path (geodesic path) or longest path (detour path) if $P$ is a path of minimum or maximum length respectively. A $u, v$-path $P$ is called a triangle path ($I^\Delta$-path) if there exists a $u, v$ shortest path $P'$ such that either any vertex of $P$ is a vertex of $P'$ or it is adjacent to two adjacent vertices on $P'$. The length of the shortest $u, v$-path in $G$ is the well known metric on $V$ called the shortest path metric denoted as $d_G(u, v)$, or $d(u, v)$ if the graph $G$ is understood from the context. Similarly the length of the longest $u, v$-path is also a metric on $V$, known as the detour metric denoted as $D_G(u, v)$ or $D(u, v)$. For a positive integer $j \geq 1$, a $u, v$ path $P$ of length less than or equal to $d(u, v) + j$ is called an $I_j - u, v$-path.

If $G$ and $H$ are two graphs, then $G$ is said to be isomorphic to $H$ if there is a bijection $\phi$ from $V(G)$ to $V(H)$ such that $v_i v_j \in E(G)$ if and only if $\phi(v_i) \phi(v_j) \in E(H)$. A graph $H$ is said to be an induced subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph $H$ of $G$ is an induced subgraph if for $u, v \in V(H)$ and $uv \in E(G)$ implies $uv \in E(H)$. A subgraph $H$ of $G$ is an isometric subgraph if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. If $G$ and $H$ are graphs, then we say that $H$ is a forbidden induced(isometric) subgraph of $G$, if $G$ does not have a subgraph isomorphic to $H$ as an induced(isometric) subgraph. We sometimes describe this as $G$ is $H$ free(if induced or isometric is evident from the context).
If $P$ is a $u,v$-path in $G$, then we can consider $P$ as a subgraph of $G$ and we denote the vertices of $P$ as $V(P)$. A graph $G$ is connected if it has a $u,v$-path whenever $u,v \in V(G)$ (otherwise, $G$ is disconnected). The components of a graph $G$ are its maximal connected subgraphs. Deletion of a set $S$ of vertices from $G$ means the subgraph obtained by deleting a set $S$ of vertices. A cut-vertex of a graph $G$ is a vertex whose deletion increases the number of components of $G$.

If $v$ is a vertex of a graph $G$, by $G\setminus v$, we mean the subgraph obtained by deleting $v$ and all edges incident on $v$. A graph $G$ is 2-connected if the minimum number of vertices whose removal disconnects the graph is at least 2. A block of a graph $G$ is a two connected subgraph of $G$.

By a complete graph we mean a graph in which every pair of vertices is adjacent. By a triangle we mean a complete graph on three vertices. As usual, a connected graph without cycles is defined as a tree. The diameter denoted as $\text{diam}(G)$ is defined with respect to the shortest path metric $d$, or the longest path metric $D$ in a connected graph $G$ as the maximum of $d(u,v)$ or $D(u,v)$ among all pairs of vertices $u,v \in V(G)$. In this thesis, we also deal with the latter diameter in Chapter 5. All the graphs considered here are finite connected graphs.

### 2.3 Transit functions, betweenness axioms and their implications

We begin this section with the formal definition of a transit function and the betweenness axioms. We denote the power set of a non-empty set $V$ as $2^V$.

**Definition 2.1:** A transit function on a non-empty set $V$ is a function $R : V \times V \to 2^V$ satisfying the following conditions, for every $u,v \in V$:

(t1) $u \in R(u,v)$

(t2) $R(u,v) = R(v,u)$

(t3) $R(u,u) = \{u\}$
The axioms (t1), (t2) and (t3) are known as the extension, the symmetry and the idempotent axioms respectively. The idempotent axiom is included in the definition of a transit function to avoid the trivial functions.

If $G$ is a graph with vertex set $V$ and $R$ a transit function defined on $V$, then we say that $R$ is a transit function on $G$. The prime example of a transit function on a graph $G$ is the interval function or geodesic transit function of $G$, namely,

$$I(u, v) = \{w \in V \mid w \text{ lies on some shortest } u, v\text{-path in } G\}$$

$$= \{w \in V \mid d(u, w) + d(w, v) = d(u, v)\}.$$

The examples of transit functions on $G$ derivable from $I$ that we consider are, the triangle path transit function $I^\Delta$ and $I_j$. They are respectively defined as:

$$I^\Delta(u, v) = \{w \in V(G) \mid w \text{ lies on some triangle } u, v\text{-path in } G\}$$

and

$$I_j(u, v) = \{w \in V(G) \mid w \text{ lies on some } I_j - u, v\text{-path in } G\}$$

$$= \{w \in V \mid d(u, w) + d(w, v) \leq d(u, v) + j\}, \text{ for } j \geq 1.$$

Another example of a transit function defined from a path metric is the longest path transit function defined as $L(u, v) = \{w \in V \mid w \text{ lies on some longest } u, v\text{-path in } G\}.$

Note that the above examples of transit functions are defined by special types of paths. A $u, v$-$\Phi$-path is defined as (for instance a $u, v$-geodesic) a $u, v$-path with property $\Phi$. Using this definition of a $\Phi$-path, it is possible to generalize the above examples of transit function by a $\Phi$-path transit function or path transit function in short, on $G$, defined by $R_\Phi(u, v) = \{x \in V \mid x \text{ is on some } u, v$-$\Phi$-path in $G\}.$ The $\Phi$-path transit functions are studied in [22]. Now the coarsest path transit function is the all paths transit function: $A(u, v) = \{w \in V \mid w \text{ lies on some } u, v$-path in $G\}.$

Let $R$ be any transit function on a non-empty set $V$. The betweenness axioms are:

(b1) $x \in R(u, v) \Rightarrow v \not\in R(u, x)$

(b2) $x \in R(u, v) \Rightarrow R(u, x) \subseteq R(u, v)$

(b3) $x \in R(u, v), y \in R(u, x) \Rightarrow x \in R(y, v)$

Another stronger betweenness axiom is the monotone axiom (m) defined as:
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\[(m) \quad x, y \in R(u, v) \Rightarrow R(x, y) \subseteq R(u, v)\]

A transit function satisfying the axioms \((b1)\) and \((b2)\) is known as a betweenness on \(G\) while that satisfying \((b2)\) and \((b3)\) is known as geometric transit function. The axiom \((b3)\) is known as the inversion axiom.

We also sometimes consider a modified \((b3)\) axiom, namely

\[(b3') \quad x \in R(u, v), u, v \neq y \in R(u, x) \Rightarrow x \in R(y, v)\]

It is easy to observe that given a nonempty set \(V\), a function \(R : V \times V \to 2^V\) satisfying the axioms \((t1), (t2)\) and \((b1)\) is a transit function in the sense that it satisfies the idempotent axiom \((t3)\).

**Definition 2.2:** For a non empty set \(V\), a family \(C \subseteq 2^V\) is said to be a convexity on \(V\) if

1. \(\emptyset, \text{ and } V \in C\)
2. If \(C_i, i = 1, 2, \ldots \in C\), then \(\cap_i C_i \in C\)
3. \(C_1 \subseteq C_2 \subseteq \ldots \subseteq C_i, i = 1, 2, \ldots \) is a nested sequence in \(C\), then \(\cup_i C_i \in C\)

**Remark 2.3:** A convexity \(C\) on \(V\) is called \(S_1\) convexity if all singletons are convex.

If \(R\) a transit function on a non empty set \(V\), a set \(W \subseteq V\) is called as \(R\)-convex if \(R(u, v) \subseteq W\), for every \(v \in W\). If \(R\) satisfies the monotone axiom \((m)\) implies that every \(R(u, v)\) is \(R\)-convex.

**Remark 2.4:** It can be easy to verify that the family of all \(R\)-convex sets \(C_R\) on \(V\) is an \(S_1\) convexity on \(V\). The singletons are convex due to the idempotent axiom \((t3)\) of \(R\).

As mentioned above, for a transit function \(R\), one can verify that all the betweenness axioms taken one at a time are independent, except when \((m)\) implies \((b2)\) since \((b2)\) is a special case of \((m)\). Now we can prove that \((b3)\) implies \((b1)\) and \((b3')\) and conversely \((b1)\) and \((b3')\) together implies \((b3)\).
Theorem 2.5: A transit function $R$ on $V$ satisfies the axiom $(b3)$ if and only if it satisfies $(b1)$ and $(b3')$.

Proof. Since the axiom $(b3')$ is a special case of the axiom $(b3)$, it follows that $R$ satisfies $(b3)$ implies that $R$ satisfies $(b3')$. Now, let $R$ satisfies $(b3)$ and suppose $R$ doesn’t satisfy $(b1)$. Then there exists $x \in V$ such that $x \in R(u, v), x \neq v$ and $v \in R(u, x)$. Since $R$ satisfies $(b3)$, we have, if $x \in R(u, v), v \in R(u, x) \Rightarrow x \in R(v, v) = \{v\}$, a contradiction.

Conversely, let $R$ satisfy the axioms $(b1)$ and $(b3')$. Let $x \in R(u, v)$ and $y \in R(u, x)$. Let $x = v$. For $y \neq v \in R(u, x)$ we have by $(b3')$, $x \in R(y, v)$. If $y = v$ then by $(t1)$, $x \in R(y, v)$. Therefore $(b3)$ holds when $x = v$. If $x \neq v$, by $(b1) v \notin R(u, x)$. So for any $y \in R(u, x)$ we have $y \neq v$. Therefore by $(b3')$ we have $x \in R(y, v)$. Hence $(b3)$ holds for $R$.

With the following examples, we show the status of the axioms $(b1)$, $(b2)$, $(b3)$, $(b4)$ and $(b3')$ and their non-implications. This is illustrated in the table below.

In these examples, we define a transit function on any non-empty set $V$, but, since we are interested in transit functions on graphs, we consider a complete graph $G$ with vertex set $V$. Since $G$ is complete, we see that all the transit functions are path functions.
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Example 2.6: The function $R$ on $V = \{a, b, c, d\}$ defined by $R(u, u) = \{u\}$ for every $u \in V$, $R(a, b) = \{a, b, c\}, R(a, c) = \{a, c, d\}, R(a, d) = \{a, b, d\}, R(b, c) = \{b, c\}, R(b, d) = \{b, d\}, R(c, d) = \{c, d\}$, is a transit function on $G$ satisfying (b1), but not (b2), (b3), (b3'), and (m).

Example 2.7: The function $R$ on $V = \{a, b, c, d, e, f, g\}$ defined by $R(u, u) = \{u\}$ for every $u \in V$, $R(a, b) = \{a, b\}, R(a, c) = \{a, c\}, R(a, d) = \{a, b, c, e, d\}, R(a, e) = \{a, e\}, R(b, c) = \{b, c\}, R(b, d) = \{b, d\}, R(b, e) = \{b, a, c, e\}, R(c, d) = \{c, d\}, R(c, e) = \{c, a, d, e\}, R(d, e) = \{d, e\}, R(a, f) = \{a, f\}, R(a, g) = \{a, g\}, R(b, f) = \{b, g, a, b\}, R(b, g) = \{b, g, a, c\}, R(d, f) = \{d, g\}, R(d, g) = \{f, g, a, b, c\}, R(e, f) = \{f, g, a, e\}$ is a transit function satisfying (b2) but not (b1), (b3), (b3'), and (m).

Example 2.8: The function $R$ on $V = \{a, b, c, d, e, f\}$ defined by $R(u, u) = \{u\}$ for every $u \in V$, $R(a, b) = \{a, b\}, R(a, c) = \{a, c\}, R(a, d) = \{a, b, c, e, d\}, R(a, e) = \{a, e\}, R(a, f) = \{a, b, c, e, d, f\}, R(b, c) = \{b, a, d, c\}, R(b, d) = \{b, d\}, R(b, e) = \{b, e\}, R(b, g) = \{b, g, a, e\}, R(c, d) = \{c, d\}, R(c, e) = \{c, a, d, e\}, R(d, e) = \{d, e\}$.
Example 2.9: The function $R$ on $V = \{a, b, c, d\}$ defined by $R(a, b) = \{a, b\}$, $R(a, c) = \{a, b, c\}$, $R(a, d) = \{a, b, d\}$, $R(b, c) = \{b, c, d\}$, $R(b, d) = \{b, d\}$, $R(c, d) = \{c, d\}$ and $R(u, u) = \{u\}$ for every $u \in V$ is a transit function on $G$ satisfying (b1), (b3), and (b3'), but not (b2), and (m).

Example 2.10: Let $V = \{a, b, c, d, e, f\}$ and the function $R$ defined on $V$ by $R(a, b) = \{a, b, c\}$, $R(a, c) = \{a, b, d\}$, $R(a, d) = \{a, b, e\}$, $R(a, e) = \{a, b, f\}$, $R(b, c) = \{b, c, d\}$, $R(b, d) = \{b, c, e\}$, $R(b, e) = \{b, c, f\}$, $R(b, f) = \{b, d, e\}$, $R(c, d) = \{c, d, e\}$, $R(c, e) = \{c, d, f\}$, $R(c, f) = \{c, e, d\}$, $R(d, e) = \{d, e, f\}$, $R(d, f) = \{d, e, f\}$, $R(e, f) = \{e, d, f\}$, $R(e, g) = \{e, d, f\}$, $R(g, f) = \{e, d, f\}$, $R(g, h) = \{e, d, f\}$, $R(h, i) = \{e, d, f\}$, $R(i, j) = \{e, d, f\}$, $R(j, k) = \{e, d, f\}$, $R(k, l) = \{e, d, f\}$, $R(l, m) = \{e, d, f\}$, $R(m, n) = \{e, d, f\}$, $R(n, o) = \{e, d, f\}$, $R(o, p) = \{e, d, f\}$, $R(p, q) = \{e, d, f\}$, $R(q, r) = \{e, d, f\}$, $R(r, s) = \{e, d, f\}$, $R(s, t) = \{e, d, f\}$, $R(t, u) = \{e, d, f\}$, $R(u, v) = \{e, d, f\}$, $R(v, w) = \{e, d, f\}$, $R(w, x) = \{e, d, f\}$, $R(x, y) = \{e, d, f\}$, $R(y, z) = \{e, d, f\}$, $R(z, a) = \{e, d, f\}$, $R(a, b) = \{a, b, c\}$, $R(a, c) = \{a, b, d\}$, $R(a, d) = \{a, b, e\}$, $R(a, e) = \{a, b, f\}$, $R(b, c) = \{b, c, d\}$, $R(b, d) = \{b, c, e\}$, $R(b, e) = \{b, c, f\}$, $R(b, f) = \{b, d, e\}$, $R(c, d) = \{c, d, e\}$, $R(c, e) = \{c, d, f\}$, $R(c, f) = \{c, e, d\}$, $R(d, e) = \{d, e, f\}$, $R(d, f) = \{d, e, f\}$, $R(e, f) = \{e, d, f\}$, $R(e, g) = \{e, d, f\}$, $R(g, f) = \{e, d, f\}$, $R(g, h) = \{e, d, f\}$, $R(h, i) = \{e, d, f\}$, $R(i, j) = \{e, d, f\}$, $R(j, k) = \{e, d, f\}$, $R(k, l) = \{e, d, f\}$, $R(l, m) = \{e, d, f\}$, $R(m, n) = \{e, d, f\}$, $R(n, o) = \{e, d, f\}$, $R(o, p) = \{e, d, f\}$, $R(p, q) = \{e, d, f\}$, $R(q, r) = \{e, d, f\}$, $R(r, s) = \{e, d, f\}$, $R(s, t) = \{e, d, f\}$, $R(t, u) = \{e, d, f\}$, $R(u, v) = \{e, d, f\}$, $R(v, w) = \{e, d, f\}$, $R(w, x) = \{e, d, f\}$, $R(x, y) = \{e, d, f\}$, $R(y, z) = \{e, d, f\}$, $R(z, a) = \{e, d, f\}$, $R(a, b) = \{a, b, c\}$, $R(a, c) = \{a, b, d\}$, $R(a, d) = \{a, b, e\}$, $R(a, e) = \{a, b, f\}$, $R(b, c) = \{b, c, d\}$, $R(b, d) = \{b, c, e\}$, $R(b, e) = \{b, c, f\}$, $R(b, f) = \{b, d, e\}$, $R(c, d) = \{c, d, e\}$, $R(c, e) = \{c, d, f\}$, $R(c, f) = \{c, e, d\}$, $R(d, e) = \{d, e, f\}$, $R(d, f) = \{d, e, f\}$, $R(e, f) = \{e, d, f\}$, $R(e, g) = \{e, d, f\}$, $R(g, f) = \{e, d, f\}$, $R(g, h) = \{e, d, f\}$, $R(h, i) = \{e, d, f\}$, $R(i, j) = \{e, d, f\}$, $R(j, k) = \{e, d, f\}$, $R(k, l) = \{e, d, f\}$, $R(l, m) = \{e, d, f\}$, $R(m, n) = \{e, d, f\}$, $R(n, o) = \{e, d, f\}$, $R(o, p) = \{e, d, f\}$, $R(p, q) = \{e, d, f\}$, $R(q, r) = \{e, d, f\}$, $R(r, s) = \{e, d, f\}$, $R(s, t) = \{e, d, f\}$, $R(t, u) = \{e, d, f\}$, $R(u, v) = \{e, d, f\}$, $R(v, w) = \{e, d, f\}$, $R(w, x) = \{e, d, f\}$, $R(x, y) = \{e, d, f\}$, $R(y, z) = \{e, d, f\}$, $R(z, a) = \{e, d, f\}$, $R(a, b) = \{a, b, c\}$, $R(a, c) = \{a, b, d\}$, $R(a, d) = \{a, b, e\}$, $R(a, e) = \{a, b, f\}$, $R(b, c) = \{b, c, d\}$, $R(b, d) = \{b, c, e\}$, $R(b, e) = \{b, c, f\}$, $R(b, f) = \{b, d, e\}$, $R(c, d) = \{c, d, e\}$, $R(c, e) = \{c, d, f\}$, $R(c, f) = \{c, e, d\}$, $R(d, e) = \{d, e, f\}$, $R(d, f) = \{d, e, f\}$, $R(e, f) = \{e, d, f\}$, $R(e, g) = \{e, d, f\}$, $R(g, f) = \{e, d, f\}$, $R(g, h) = \{e, d, f\}$, $R(h, i) = \{e, d, f\}$, $R(i, j) = \{e, d, f\}$, $R(j, k) = \{e, d, f\}$, $R(k, l) = \{e, d, f\}$, $R(l, m) = \{e, d, f\}$, $R(m, n) = \{e, d, f\}$, $R(n, o) = \{e, d, f\}$, $R(o, p) = \{e, d, f\}$, $R(p, q) = \{e, d, f\}$, $R(q, r) = \{e, d, f\}$, $R(r, s) = \{e, d, f\}$, $R(s, t) = \{e, d, f\}$, $R(t, u) = \{e, d, f\}$, $R(u, v) = \{e, d, f\}$, $R(v, w) = \{e, d, f\}$, $R(w, x) = \{e, d, f\}$, $R(x, y) = \{e, d, f\}$, $R(y, z) = \{e, d, f\}$, $R(z, a) = \{e, d, f\}$.
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\[ R(x, x) = \{x\} \] is a transit function satisfying \((b3')\) but not \((b1), (b2), (b3), \) and \((m). \)

\[ \square \]

**Example 2.14:** The function \( R \) on \( V = \{a, b, c, d\} \) defined by \( R(a, b) = \{a, b, c\}, \) 
\( R(a, c) = \{a, c\}, R(a, d) = \{a, b, c, d\}, R(b, c) = \{b, c\}, R(b, d) = \{a, b, c, d\}, R(c, d) = \{c, d\}, \) and \( R(u, u) = \{u\} \) for every \( u \in V \) is a transit function satisfying \((b2), (m), \) but not \((b1), (b3), (b3'). \)

\[ \square \]

**Example 2.15:** The function \( R \) on \( V = \{a, b, c\} \) defined by \( R(u, v) = V, \) for \( u \neq v \) and \( R(u, u) = \{u\} \) for every \( u, v \in V \) is a transit function on \( G \) satisfying \((b2), (m), \) but not \((b1), (b3). \)

\[ \square \]

**Example 2.16:** The function \( R \) on \( V = \{a, b, c\} \) defined by \( R(a, b) = \{a, b\}, \) 
\( R(a, c) = \{a, b, c\}, R(b, c) = \{b, c\} \) and \( R(u, u) = \{u\} \) for every \( u, v \in V \) is a transit function on \( G \) satisfying \((b1), (b2), (b3), \) and \((m). \)

\[ \square \]

As a corollary to Theorem 2.5, we have

**Corollary 2.17:** If \( R \) is a geometric transit function, then \( R \) is a betweenness. \[ \square \]

The examples 2.8 and 2.12 show that the converse of Corollary 2.17 is not true. Also, observe that the idempotent axiom \((t3)\) is necessary for the Lemma 2.5. For if \( R \) is a function from \( V \times V \to 2^V \) satisfying the axioms \((t1) \) and \((t2)\) only, then it is not true that \( R \) satisfies \((b3) \) implies \( R \) satisfies \((b1). \) The following example establishes this.

**Example 2.18:** The function \( R \) on \( V = \{a, b, c\} \) defined by \( R(u, v) = V \) for every \( u, v \in V \) satisfies \((b3) \) and \((b2), \) but not \((b1). \)

\[ \square \]
2.4 n-ary transit functions and betweenness

One may call the transit function \( R \) in the Definition 2.1 as “binary transit function”, since it associates with every pair of points \( u, v \) the function \( R(u, v) \). In this section, we generalize the transit function and the betweenness axioms from binary to n-ary, \( n > 2 \). In the definition of an n-ary transit function, we include the idempotent axiom as we did in the binary case. By relaxing the idempotent axiom, we may obtain non-trivial convex sets even though the singletons are not a convex set. We look at the analogous implications of the betweenness axioms as done in the previous section.

**Definition 2.19:** Let \( V \) be a non-empty set. Then a function \( R: \prod_{i=1}^{n} V \rightarrow 2^V \) is a transit function of arity \( n \) (or n-ary transit function) on \( V \) if \( R \) satisfies the following axioms.

1. \( u_i \in R(u_1, u_2, \ldots, u_n) \) for all \( u_i \in V, i = 1 \ldots n \)
2. \( R(u_1, u_2, \ldots, u_n) = R(\pi(u_1, u_2, \ldots, u_n)) \) for all \( u_i \in V \), where \( \pi(u_1, u_2, \ldots, u_n) \) is any permutation of \( (u_1, u_2, \ldots, u_n) \)
3. \( R(u, u, \ldots, u) = \{u\} \) for all \( u \in V \).

Similar to the binary case, if \( V \) is the vertex set of a graph \( G \) and \( R \) an n-ary transit function on \( V \), then \( R \) is called a n-ary transit function on \( G \). The prime example of a binary transit function on a graph is the geodesic interval function \( I \). It would be interesting to study the n-ary transit function corresponding to the interval function \( I \). We can see that the generalization of the interval function from binary to n-ary is the n-Steiner interval function \( S \). We just give the definition of the n-Steiner interval below as an example of an n-ary transit function on a graph. We study the betweenness axioms and some properties in connection with the Steiner interval \( S \) in Chapter 6.
Example 2.20: Let $G$ be a graph. A Steiner tree of a multi set $W \subseteq V(G)$, is a minimum order tree in $G$ that contains all vertices of $W$. The $n$-Steiner interval $S(u_1, u_2, \ldots, u_n)$ consists of all vertices in $G$ that lie on some Steiner tree with respect to $(u_1, u_2, \ldots, u_n)$ (Here the $n$-tuple $(u_1, u_2, \ldots, u_n)$ can be considered as a multi set).

We can easily verify that 2-Steiner intervals are precisely the geodesic intervals $I$ and thus the $n$-Steiner interval $S$ naturally generalize $I$.

2.4.1 Betweenness

We generalize the betweenness axioms $(b1), (b2)$ and $(m)$ in the case of an n-ary transit function. Since there is no good analogue of the inversion axiom $(b3)$ for $n > 2$, we consider the analogs of $(b1), (b2)$ and $(m)$ only.

The following betweenness axioms can be considered for a $n$-ary transit function $R$. For any $u_1, u_2, \ldots, u_n, x, x_1, x_2, \ldots, x_n \in V$, define

$(b1)$ \(x \in R(u_1, u_2, \ldots, u_n), x \neq u_k \Rightarrow u_k \notin R(y_1, y_2, \ldots, y_n), \) where $y_i = u_i$ if $y_i \neq u_k$ else $y_i = x$ for $i = 1, 2, \ldots, n$.

$(b2)$ \(x \in R(u_1, u_2, \ldots, u_n) \Rightarrow R(x, u_2, \ldots, u_n) \subseteq R(u_1, u_2, \ldots, u_n)$.

$(m)$ \(R(x_1, x_2, \ldots, x_n) \subseteq R(u_1, u_2, \ldots, u_n) \forall x_1, x_2, \ldots, x_n \in R(u_1, u_2, \ldots, u_n)$.

Observation 2.21: $(m) \Rightarrow (b2)$ for any $n$-ary transit function.

Analogous to the binary case, we have examples of transit functions to show that the above betweenness axioms taken one at a time are independent, except when $(m)$ implies $(b2)$. The illustration is given in the table below.
2. Basic observations of transit functions and betweenness

<table>
<thead>
<tr>
<th>(b1)</th>
<th>(b2)</th>
<th>(m)</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>×</td>
<td></td>
<td></td>
<td>2.22</td>
</tr>
<tr>
<td></td>
<td>×</td>
<td></td>
<td>2.23</td>
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<tr>
<td>×</td>
<td>×</td>
<td></td>
<td>2.24</td>
</tr>
<tr>
<td>×</td>
<td>×</td>
<td>×</td>
<td>2.25</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2.26</td>
</tr>
</tbody>
</table>

Tab. 2.2: n-ary transit functions with possible betweenness relations. × denotes that the corresponding axiom is satisfied.

Example 2.22: The function $R$ on $\prod_{i=1}^{n} V \rightarrow 2^V$, $|V| > n + 2$ defined by $R(u, u, \ldots, u) = \{u\}$, $R(v_1, v_2, \ldots v_n) = \{v_1, v_2, \ldots v_{n+1}\}$, $R(v_2, \ldots v_{n+1}) = \{v_2, v_3, \ldots v_{n+2}\}$ and $R(u_1, u_2 \ldots u_n) = \{u_1, u_2 \ldots u_n\}$ for any other $n$-tuple. We can see that $R$ satisfies (b1) but not (b2), and (m). 

Example 2.23: The function $R$ on $\prod_{i=1}^{n} V \rightarrow 2^V$, $|V| > n + 2$ defined by $R(u, u, \ldots, u) = \{u\}$, $R(v_1, v_2, \ldots v_n) = V$, $R(v_2, \ldots v_{n+1}) = \{v_2, \ldots, v_{n+1}\}$, $R(v_1, v_2 \ldots v_2) = \{v_1, v_2, v_3\}$, $R(v_1, v_3 \ldots v_3) = \{v_1, v_2, v_3\}$, $R(v_2, v_3 \ldots v_3) = \{v_1, v_2, v_3\}$ and in all the other cases $R(v_1, v_2 \ldots v_n) = \{v_1, v_2, \ldots v_n\}$. We can see that $R$ satisfies (b2) but not (b1), and (m). 

Example 2.24: The function $R$ on $\prod_{i=1}^{n} V \rightarrow 2^V$, $|V| > n + 2$ defined by $R(u, u, \ldots, u) = \{u\}$, $R(v_1, v_2, \ldots v_2) = \{v_1, v_2, v_3\}$, $R(v_1, v_3 \ldots v_3) = \{v_1, v_3\}$, $R(v_1, v_4 \ldots v_4) = \{v_1, v_2, v_3, v_4\}$, $R(v_2, v_3 \ldots v_5) = \{v_2, v_3, v_5\}$ and in all the other cases $R(v_1, v_2 \ldots v_n) = \{v_1, v_2, \ldots v_n\}$. We can see that $R$ satisfies (b2), and (b1) but not (m). 

Example 2.25: The function $R$ on $\prod_{i=1}^{n} V \rightarrow 2^V$, $|V| > n + 2$ defined by $R(u, u, \ldots, u) = \{u\}$, $R(v_1, v_2, \ldots v_n) = V$. We can see that $R$ satisfies (m), and (b2) but not (b1).
### Example 2.26

The function $R$ on $V \times V \times \cdots \times V \rightarrow 2^V$, $|V| > n + 2$ defined by $R(u, u, \ldots, u) = \{u\}$, $R(v_1, v_2, \ldots, v_n) = V$, and all other cases $R(u_1, u_2 \ldots u_n) = \{u_1, u_2 \ldots u_n\}$ for the rest. We can see that $R$ satisfies (m), (b2) and (b1).

Given a binary transit function $R$ on a non-empty set $V$, one can define an underlying graph $G_R$ with vertex set $V$ and $uv$ is an edge of $G_R$ if and only if $R(u, v) = \{u, v\}$. The underlying graphs of ordinary transit functions are discussed in [20], [61] and [23]. It is proved in [20] (also refer [21]) that if $R$ is a betweenness on $V$, then the underlying graph $G_R$ of $V$ is connected and both axioms (b1) and (b2) are necessary for the connectedness of $G_R$. Analogous to the binary case, for an $n$-ary transit function $R$, we obtain the underlying hypergraph which we denote again as $G_R$. We first define a hypergraph.

#### Definition 2.27
A hypergraph $H$ is a pair $(V, E)$ where $V$ is a nonempty finite set and $E$ is a collection of subsets of $V$. The members of $E$ are called edges of $H$.

#### Definition 2.28
Let $R$ be any transit function on $V$. Then the underlying hypergraph $G_R$ on $V$ is defined as follows. $V$ is the vertex set of $G_R$ and $E$ is an edge of $G_R$ if and only if $E = R(u_1, u_2, \ldots, u_n) = \{u_1, u_2, \ldots, u_n\}$.

#### Definition 2.29
Two vertices $u$ and $v$ of a hypergraph $H$ are connected if there exists a sequence $(E_1, E_2, \ldots, E_n)$ $n \geq 1$, of edges of $H$ such that: $u \in E_1$, $v \in E_n$ and, if $n > 1$, then $E_i \cap E_{i+1} \neq \emptyset$ ($1 \leq i \leq n - 1$). $H$ is connected if every pair of vertices $u$ and $v$ are connected.

Let $R$ be any transit function on $V$. For $n = 2$, we can easily see that the hypergraph $G_R$ becomes the underlying graph of $R$.

For $n$-ary transit functions $R$, $n > 2$, we have the following proposition, which is the analogue of the binary transit function proved in [20].
Proposition 2.30: Let $R : V \times V \times \ldots \times V \rightarrow 2^V$ a transit function on $V$ with $\lvert V \rvert > n$. If $R$ satisfies (b1) and (b2) then $G_R$ is connected.

**Proof.** Let $V = \{v_1, v_2, \ldots, v_n\}$. Consider $u, v \in V(G)$. Let us assume that $u = v_1$ and $v = v_n$. Consider $R(u, u, u, \ldots, v)$. Suppose $\lvert R(u, u, u, \ldots, v) \rvert > 2$, then there exists $x \in R(u, u, u, \ldots, v)$ such that $x \neq u, v$. Since $R$ satisfies (b2) axiom, $R(x, x, x, \ldots, v) \subseteq R(u, u, u, \ldots, v)$ and $R(x, x, x, \ldots, u) \subseteq R(u, u, u, \ldots, v)$. Also since $R$ satisfies (b1) axiom, both $R(x, x, x, \ldots, v)$ and $R(x, x, x, \ldots, u)$ are proper subset of $R(u, u, u, \ldots, v)$. If any one of these proper subsets contains more than 2 elements, by applying the construction described above, we can divide it in to two proper subsets. Since $R(u, u, u, \ldots, v)$ is finite, this process cannot continue indefinitely. Hence after a finite number of steps we get distinct vertices $u_1, u_2, \ldots, u_{n-1} \in R(u, u, u, \ldots, v)$ such that $\lvert R(u_1, u_i, u_{i+1}) \rvert = 2$ for $i = 0, 1, \ldots, n-1$, where $u_0 = u$ and $u_n = v$. Since $\lvert R(u_i, u_i, u_{i+1}) \rvert = 2$, $E_i = \{u_i, u_{i+1}\}$ is an edge, for $i = 0, 1, \ldots, n$ and also $E_i \cap E_{i+1} = \{ u_{i+1} \}$. Hence $E_0, E_1, \ldots, E_n$ is a $u, v$-path in $G_R$. Hence $G_R$ is connected. \qed

As in the binary case, we can show that both the axioms (b1) and (b2) are necessary for the connectedness of $G_R$. As an application of the celebrated Hall's matching theorem (marriage theorem), [43] for bipartite graphs, it is possible to establish the existence of an $n$-ary transit function $R$ satisfying (b1), but not (b2) with disconnected underlying hypergraph $G_R$. For completeness, we state the Hall's theorems and prove the existence of such a transit function.

Note that in a bipartite graph $G$, a matching in $G$ is a set of edges $S$ such that any two edges in $S$ are pairwise disjoint and a matching is maximal if for any edge in $e \in E \setminus S$, $S \cup \{e\}$ is not a matching. A perfect matching is a matching such that every vertex is incident to some edge in $S$.

**Theorem 2.31 (Hall's Marriage Theorem [43]):** Let $G$ be a bipartite graph with bipartition $U$ and $V$. Then there is a maximal matching from $U$ to $V$ if and
only if Hall’s condition is satisfied: That is, \(|N(A)| \geq |A|\) for all subsets \(A\) of \(U\). Here \(N(A)\) denotes the set of neighbours of the vertices in \(A\).

**Theorem 2.32 (Hall’s marriage theorem on regular bipartite graphs[43]):**

If \(G\) is a \(k\)-regular bipartite graph for \(k > 1\), then \(G\) has a perfect matching.

We construct \((n-1)\) bipartite graphs as follows. Let \(U\) and \(V\) be the set of all \(k\)-tuples and \((k+1)\)-tuples of distinct integers \(1, 2, \ldots, 2n+1\) (for \(k = 2, 3, \ldots, n\)), respectively so that \((a_1, a_2, \ldots, a_k)\) is such that \(i \leq j \Rightarrow a_i \leq a_j\). Define edges of \(G\) as follows. If \(u \in U\) and \(v \in V\), then \((u, v)\) is an edge of \(G\) if and only if \(u\) is a proper subset of \(v\), then \(G\) is a bipartite graph. Thus we have \((n-1)\)-bipartite graphs. Each of these bipartite graphs satisfies the conditions of the Hall’s marriage theorem, because each vertex in \(U\) has more than one neighbor. Therefore for any subset \(A\) of \(U\), we have \(|N(A)| \geq |A|\). Thus by Hall’s theorem, there exists a maximal matching from \(U\) to \(V\). In fact for \(k = n\), we have that both \(U\) and \(V\) have the same number of elements and the degree of a vertex in \(U\) is the same as the degree of a vertex in \(V\) which is equal to \(k + 1\). Hence when \(k = n\), we have by Hall’s theorem on regular on bipartite graphs, there exists a perfect matching in \(G\). Therefore when \(k = n\), two distinct \(n\)-tuples are mapped to two distinct \((n+1)\)-tuples.

Now we, use the edges of the maximal matching in all the bipartite graphs for \(k = 1, 2, \ldots, (n-1)\), as well as the edges of the regular bipartite graph when \(k = n\) to define the required \(n\)-ary transit function \(R\). We can consider a 2-tuple as an \(n\)-tuple with only two distinct elements, a 3-tuple as an \(n\)-tuple or \((n+1)\) tuple with only three distinct elements, etc and an \(n\)-tuple with all elements distinct. Therefore the vertex set \((U, V)\) of the bipartite graph \(G\) can be considered as \(U\) consists of all \(n\)-tuples and \(V\) consists of subsets of cardinality \(3 \leq (n+1)\) (by considering the tuples of \(V\) as subsets).

Define \(R\) as \(R(u, u, \ldots, u) = \{u\}\) and \(R(u = (u_1, u_2, \ldots, u_n)) = \{y_1, y_2, \ldots, y_{n+1}\}\),
where \( \{y_1, y_2, \ldots, y_{n+1}\} \) corresponds to the \((n + 1)\) tuple \((y_1, y_2, \ldots, y_{n+1}) = v\) such that \(uv\) is an edge of the maximal matching given by Hall's theorem, for \(k = 2, 3, \ldots, n\).

Clearly \(R\) is an \(n\)-ary transit function satisfying the \((b1)\) axiom and the underlying hypergraph \(G_R\) of \(R\) has no edges and so \(G_R\) is disconnected. It is to be noted that \(R\) doesn't satisfy the \((b2)\) axiom, as any two distinct vertices of \(U\) when \(k = n\) are mapped by \(R\) into two distinct subsets of the equal cardinality and hence one cannot be a subset of the other. Therefore \(R\) fails to satisfy the \((b2)\) axiom. Thus, we have proved the theorem given by:

**Theorem 2.33:** There exists an \(n\)-ary transit function \(R\) satisfying the \((b1)\)-axiom, but not the \((b2)\)-axiom, with disconnected underlying hypergraph \(G_R\).

We illustrate the above theorem with an example for \(n = 3\). The actual construction of an arbitrary \(n\)-ary transit function may not be easy.

**Example 2.34:** \((R\) satisfies \((b1)\) but not \((b2))\); and \(G_R\) is not connected.

Let \(V = \{1, 2, 3, 4, 5, 6, 7\}\) and let \(R\) be a 3-ary transit function, defined on \(V\) as follows. There are 35 distinct 3 tuples and 4 tuples with distinct elements. We define \(R\) on distinct 3 tuples as follows.

\[
\begin{align*}
R(1, 2, 3) &= \{1, 2, 3, 4\}, \quad R(1, 2, 4) = \{1, 2, 4, 5\}, \quad R(1, 2, 5) = \{1, 2, 3, 5\}, \\
R(1, 2, 6) &= \{1, 2, 3, 6\}, \quad R(1, 2, 7) = \{1, 2, 3, 7\}, \quad R(1, 3, 4) = \{1, 3, 4, 5\}, \\
R(1, 3, 5) &= \{1, 3, 5, 6\}, \quad R(1, 3, 6) = \{1, 3, 6, 7\}, \quad R(1, 3, 7) = \{1, 3, 5, 7\}, \\
R(1, 4, 5) &= \{1, 4, 5, 6\}, \quad R(1, 4, 6) = \{1, 4, 6, 7\}, \quad R(1, 4, 7) = \{1, 2, 4, 7\}, \\
R(1, 5, 6) &= \{1, 2, 5, 6\}, \quad R(1, 5, 7) = \{1, 2, 5, 7\}, \quad R(1, 6, 7) = \{1, 2, 6, 7\}, \\
R(2, 3, 4) &= \{2, 3, 4, 6\}, \quad R(2, 3, 5) = \{2, 3, 4, 5\}, \quad R(2, 3, 6) = \{2, 3, 5, 6\}, \\
R(2, 3, 7) &= \{2, 3, 4, 7\}, \quad R(2, 4, 5) = \{2, 4, 5, 7\}, \quad R(2, 4, 6) = \{1, 2, 4, 6\}, \\
R(2, 4, 7) &= \{2, 4, 6, 7\}, \quad R(2, 5, 6) = \{2, 5, 6, 7\}, \quad R(2, 5, 7) = \{2, 3, 5, 7\}, \\
R(2, 6, 7) &= \{2, 3, 6, 7\}, \quad R(3, 4, 5) = \{3, 4, 5, 6\}, \quad R(3, 4, 6) = \{1, 3, 4, 6\}.
\end{align*}
\]
2. Basic observations of transit functions and betweenness

\[ R(3, 4, 7) = \{1, 3, 4, 7\}, \ R(3, 5, 6) = \{3, 5, 6, 7\}, \ R(3, 5, 7) = \{3, 4, 5, 7\}, \]
\[ R(3, 6, 7) = \{3, 4, 6, 7\}, \ R(4, 5, 6) = \{2, 4, 5, 6\}, \ R(4, 5, 7) = \{1, 4, 5, 7\}, \]
\[ R(4, 6, 7) = \{4, 5, 6, 7\}, \ R(5, 6, 7) = \{1, 5, 6, 7\}. \] There are 21 distinct 3 tuples with one element repeating. \( R \) on such 3 tuples as follows.

\[ R(1, 1, 2) = R(1, 2, 2) = \{1, 2, 3\}, \ R(1, 1, 3) = R(1, 3, 3) = \{1, 3, 4\}, \]
\[ R(1, 1, 4) = R(1, 4, 4) = \{1, 4, 5\}, \ R(1, 1, 5) = R(1, 5, 5) = \{1, 5, 6\}, \]
\[ R(1, 1, 6) = R(1, 6, 6) = \{1, 6, 7\}, \ R(1, 1, 7) = R(1, 7, 7) = \{1, 2, 7\}, \]
\[ R(2, 2, 3) = R(2, 3, 3) = \{2, 3, 4\}, \ R(2, 2, 4) = R(2, 4, 4) = \{2, 4, 7\}, \]
\[ R(2, 2, 5) = R(2, 5, 5) = \{2, 5, 6\}, \ R(2, 2, 6) = R(2, 6, 6) = \{2, 6, 7\}, \]
\[ R(2, 2, 7) = R(2, 7, 7) = \{2, 3, 7\}, \ R(3, 3, 4) = R(3, 4, 4) = \{3, 4, 5\}, \]
\[ R(3, 3, 5) = R(3, 5, 5) = \{3, 5, 6\}, \ R(3, 3, 6) = R(3, 6, 6) = \{1, 3, 6\}, \]
\[ R(3, 3, 7) = R(3, 7, 7) = \{3, 4, 7\}, \ R(4, 4, 5) = R(4, 5, 5) = \{4, 5, 6\}, \]
\[ R(4, 4, 6) = R(4, 6, 6) = \{4, 6, 7\}, \ R(4, 4, 7) = R(4, 7, 7) = \{4, 5, 7\}, \]
\[ R(5, 5, 6) = R(5, 6, 6) = \{5, 6, 7\}, \ R(5, 5, 7) = R(5, 7, 7) = \{1, 5, 7\}, \]
\[ R(6, 6, 7) = R(6, 7, 7) = \{3, 6, 7\}; \text{ and } R(u, u, u) = \{u\} \text{ for all } u \in V. \]

We have a simple example of a transit function \( R \) satisfying (b2), but not (b1) with disconnected \( G_R \).

**Example 2.35:** Define \( R(u_1, u_2, \ldots, u_n) = V(G) \) for all \( u_1, u_2, \ldots, u_n \in V(G) \) and \( R(u, u, \ldots, u) = \{u\} \) with \( |V(G)| > n \). It can be verified that \( R \) satisfies (b2) but not (b1) and that \( G_R \) is totally disconnected.

We also observe some basic results on the convexity associated with an \( n \)-ary transit function.

**Definition 2.36:** If \( R \) is a transit function on \( V \), a subset \( W \) of \( V \) is \( R \)-convex if \( R(u_1, u_2, \ldots u_n) \subseteq W \) for any \( u_1, u_2, \ldots u_n \in W \).

**Definition 2.37:** Let \( C \) be a convexity on \( V \). We say that \( C \) is of arity \( \leq n \) if \( C = \{C \subseteq V \mid F \subseteq C |F| \leq n \Rightarrow \forall F > \subset C \} \).
Proposition 2.38: $C$ is a $S_1$-convexity on $V$ of arity $\leq n$ if and only if $C$ is a $R$-convexity for a $n$-ary transit function $R$.

Proof. Let $C$ be a $S_1$-convexity on $V$ of arity $\leq n$ on $V$. Define $R(u_1, u_2, \ldots u_n) = \langle\{u_1, u_2, \ldots u_n\}\rangle$. Then $R$ is a transit function on $V$.

Conversely we prove that $C_R = C$. Take $C \in C_R$. Suppose $F \subseteq C$ and $|F| \leq n$.

$\langle F \rangle_{C_R} \subseteq C$. Therefore $C \in C$. Therefore $C_R \subseteq C$.

Now suppose $C \in C$. Let $u_1, u_2, \ldots u_n \in C$. Therefore $u_1, u_2, \ldots u_n \in \langle\{u_1, u_2, \ldots u_n\}\rangle_{C_R}$.

$R(u_1, u_2, \ldots u_n) \subseteq \langle\{u_1, u_2, \ldots u_n\}\rangle_{C_R} \subseteq C$. Therefore $R(u_1, u_2, \ldots u_n) \subseteq C$. Therefore $C \in C_R$. Therefore $C \subseteq C_R$. Therefore $C = C_R$.

Since $n$-ary transit functions $R$ are generalizations of binary transit functions, it would be interesting to look at specific instances of $R$. The $n$-ary analogue of $I$, the Steiner transit function, will be discussed in Chapter 6.