1. BACKGROUND OF THE RESEARCH PROBLEM

1.1 Introduction

The experience of "Betweenness" is natural in our everyday life. Betweenness is an ontological entity, only when there are two points either in space or time. This is true in one's own life and in one's relationship with others. In this sense, betweenness is a universal concept, but at the same time knowledge about it warrants historical ontology.

In Mathematics too, betweenness is a natural concept which is present in several branches like Geometry, Algebra, Posets, Metric spaces, Graphs and in many other structures. For example, a point \( x \) is between two other points, say \( a \) and \( b \), if \( x \) lies in a line segment between \( a \) and \( b \) in Geometry; \( a \leq x \leq b \) in a Poset \((X, \leq)\); \( d(a, x) + d(x, b) = d(a, b) \), in a metric space \((X, d)\). Betweenness can be studied axiomatically by defining independent axioms on a non-empty set.

Betweenness as a concept was formulated in Geometry. This concept has genealogical antecedence from the geometry of Euclid. However, early 20\(^{th}\) century mathematicians recognized that Euclid’s system of axioms were incomplete. Concepts such as “between”, “inside” and “outside” were not precise. Nevertheless pictures were used for reasoning by Euclid. David Hilbert developed the modern system of axioms, which removed these limitations, in his book “Grundlagen der Geometrie” (Foundations of Geometry); which was first published in the year 1899, see [46]. The undefined terms (according to Hilbert) are points, lines, planes, “to be on”, “to be between” and “to be congruent”. The axioms of geometry, which implicitly describe the undefined terms, are grouped into five of which the second set of axioms are the
axioms of betweenness. What is being axiomatized in this group of axioms is the relation \( B \) which, intuitively speaking, says whether a point is between two other points or not. Intuitively, we may think of \( B(A, B, C) \) as the point \( B \) is between points \( A \) and \( C \). The axioms of betweenness, according to Hilbert, are the following:

- If \( B(A, B, C) \), then \( A, B, C \) are three distinct collinear points.

- If \( B(A, B, C) \), then \( B(C, B, A) \).

- If \( B(A, B, C) \), then it is not \( B(A, C, B) \).

- If \( A \) and \( B \) are two distinct points, then there exists a point \( C \) with \( B(A, B, C) \).

- If \( A, B, C \) are three distinct collinear points, then \( B(A, B, C) \) or \( B(A, C, B) \) or \( B(C, A, B) \).

- If \( A, B, C \) are three non-collinear points, and if \( p \) is a line in the plane \( ABC \) which does not contain \( A \) and which intersects the line \( BC \) at a point \( P \) with \( B(B, P, C) \), then the line \( p \) intersects the line \( AC \) at a point \( Q \) with \( B(A, Q, C) \) or it intersects the line \( AB \) at a point \( R \) with \( B(A, R, B) \).

The term "Betweenness" is found in the title of the article "The Betweenness Assumptions," authored by George Bruce Halsted and published in the American Mathematical Monthly, 1902 see [12]. Following Hilbert, the modern axiom system for "betweenness" was introduced by George Birkhoff in the early 1920's. He published the paper on his axioms, "A set of postulates for plane geometry, based on scale and protractor", see [7]. A modification of Birkhoff's axiom system was formulated by Sanders MacLane, who published his version as "Metric postulates for plane geometry" [53]. MacLane's version of the betweenness axiom was known as the "continuity axiom". The main difference between Birkhoff's axiom systems and earlier axiom systems for plane geometry is that his axioms use the real numbers and their properties. The subsequent axiom system was based on the properties of
distance, angle measure, and area in Plane Geometry. This resulted in the concept of a “metric space” which is a fundamental concept in modern geometry.

The first serious attempt to study betweenness in non-geometrical branches abstractly, was made by Huntington and Kline (1917, see [82]), who postulated a “set of independent postulates for betweenness”. The concept of betweenness can have different meanings depending upon the context. For example,

- If $K$ is the class of points on a line; $AXB$ means that point $X$ lies between the points $A$ and $B$.

- If $K$ is the class of natural numbers; $AXB$ means that number $X$ is the product of the numbers $A$ and $B$.

- If $K$ is the class of human beings; $AXB$ means that $X$ is a descendant of $A$ and an ancestor of $B$.

- If $K$ is the class of points on the circumference of a circle; $AXB$ means that the arc $A - X - B$ is less than $180^\circ$.

Huntington and Kline describe the betweenness relation as a ternary relation on a set (suitable set, depending the context) and $B(x, y, z)$ is read as “$y$ lies between $x$ and $z$”. The four basic postulates of betweenness described in [82] are:

- $B(a, b, c) \Rightarrow B(c, b, a)$.

- If $a, b, c$ are mutually distinct, then either $B(b, a, c)$ or $B(c, a, b)$ or $B(a, b, c)$ or $B(c, b, a)$ or $B(a, c, b)$ or $B(b, c, a)$.

- If $a, b, c$ are mutually distinct, then either $B(a, b, c)$ or $B(a, c, b)$, but not both.

- $B(a, b, c) \Rightarrow a, b, c$ are mutually distinct.
1.2 Betweenness in Semi-Metric Spaces and Lattices

In a linear space or vector space, the geometric idea of a line, line segment etc., is studied algebraically. The central idea of a line segment with end points $x$ and $y$ is the set of all points lying between $x$ and $y$. Thus the betweenness concept is inherent in a real vector space, which is known as "algebraic betweenness".

Betweenness in metric spaces was formally introduced by Blumenthal in his book "Distance Geometry" [10]. Given three points $x, y, z$ in a metric space $(X, d)$, they define a betweenness if $d(x; y) + d(y; z) = d(x; z)$ (say that $y$ is between $x$ and $z$). See also Banach [2]. Two notable works followed by Blumenthal are Clarkson [34] and Smiley [81]. Clarkson studied the betweenness in a metric space which is a semi-normed real vector space and characterized the coincidence of algebraic and metric betweenness [34]. Smiley investigated and unified the algebraic, metric and lattice betweenness relations and characterized the conditions of coincidence of these three types of betweenness [81]. It is proved that algebraic betweenness implies metric betweenness and lattice betweenness. The system in which the three betweenness coincide is precisely a metric lattice. The paper by R. Mendris and P. Zlatožs proves the class $M$ of all metrizable betweenness spaces [54].

Betweenness is an inherent notion in Posets and Lattices. It is introduced in Lattices by Garett Birkhoff in [6]. The paper "Transitivities of Betweenness" by E. Pitcher and M.F.Smiley appeared in 1942 [74], which follows the approach of Huntington and Kline to examine the postulates involving five elements. This paper also deals with betweenness in Lattices (defined as $B(a, b, c)$ if and only if $(a + b) + (b + c) = a + b + b + c$, $B(a, b, c)$ means $b$ is between $a$ and $c$, where $+$, $-$ are respectively, the lattice join and meet operations), and discusses betweenness in metric lattices (That is, lattices, which are also metric spaces, in which the distance function is defined using the lattice join and meet operations). The study gener-
alizes the betweenness and extends it to semi metric, metric and metric ptolemaic spaces. An important result in this paper is a characterization of modular lattices using transitivities.

The three betweenness relations of Smiley are defined as follows:

\[ R_A : (abc)_A \equiv b = \lambda a + (1 - \lambda)c, \lambda \text{ a real number, } 0 \leq \lambda \leq 1, \]
\[ R_M : (abc)_M \equiv \delta(a, b) + \delta(b, c) = \delta(a, c), \]
\[ R_L : (abc)_L \equiv (a \cap b) \cup (b \cap c) = b = (a \cup b) \cap (b \cup c) \]

Here \( R_A \) is the algebraic betweenness, \( R_M \), the metric betweenness, and \( R_L \), the lattice betweenness. The relation \( R_A \) applies to a real vector space, \( R_M \) to a semimetric space, and \( R_L \) to a lattice.

This paper continues to discuss a semimetric space which is also a lattice that has identical lattice and metric betweenness relations, provided it is a metric lattice. The theorem “If \( S \) is a real vector lattice, then lattice betweenness and algebraic betweenness coincide in \( S \) if and only if the dimension of \( S \) (as a vector space) is one” characterizes coincidence of lattice and algebraic betweenness.

Marlow Sholander developed a theory of betweenness using ternary relations to study lattices, trees and semi-lattices; in particular he introduced what is a median semi-lattice using his theory of betweenness. He published two papers. In his first paper titled “Trees, lattices, order and betweenness” see [84], he investigates betweenness for various kinds of orderings. This paper considers postulates expressed in terms of “segments”, “medians”, and “betweenness”; and characterizations are obtained for trees, lattices, and partially ordered sets. In the second paper, it is shown that lattices and trees have a common generalization, which he calls ‘median semilattice’, see [83].

Dan A. Simovici in his paper “Betweenness, Metrics and Entropies in Lattices” [85] investigates a class of metrics on lattices that are compatible with the partial order defined by the lattice, using the ternary relation of betweenness that can
be naturally defined on a metric space. The relationships between entropy-like functions and metrics defined on lattices are studied and the links between various properties of entropies and metrics shown.

The survey paper by Ivo Düntsch and Alasdair Urquhart outlines the history of axiomatizations of betweenness relations and demonstrates that the class of betweenness relations generated by a reflexive and antisymmetric binary relation is first order axiomatizable, albeit with an infinite number of variables, see [38]. The paper points out the connection of betweenness relations with comparability graphs. Such a graph may be generated by essentially different partial orders; in contrast, betweenness relations carry, in some sense, total information: If \( B \) is generated by a reflexive and antisymmetric binary relation, then this relation is determined by converse relations of the components.

Prenowitz and Jantosciak [76] introduced the concept of Join Space which is an abstract model for partially ordered sets, linear, spherical and projective geometries. A Join Space is a set \( J \) with join operation \( \circ : J \times J \rightarrow 2^J \) satisfying those known as join axioms. In the Join Space, betweenness is defined as \( b \) is between \( a \) and \( c \) if \( b \in a \circ c \). This paper has two examples of vector spaces, which are special Join Spaces, and vector spaces over an ordered field. The paper relates join spaces with ternary algebras.

Following others, Hedlíková represented the betweenness relation as a ternary relation and introduced the idea of ternary spaces. In the paper "Ternary Spaces, Media and Chebyshev Sets", [48, 49] she discusses ternary spaces. In a ternary space, a ternary relation defines a betweenness which unifies the metric, order and lattice betweenness in a vector space over a partially ordered field; and also extends betweenness to other geometric structures. This paper improves upon the existing theory of betweenness as an abstract algebraic concept. The paper also defines new concepts such as "media" and "Chebyshev sets" (ternary algebras satisfying certain special axioms of betweenness) and discusses their properties. A similar notion of
betweenness can be found in [78].

Many authors followed this approach to define theories on the algebraic betweenness of ternary algebras. For example, the characterization of weakly-median graphs considered as algebras can be found in Chepoi et al. [3], [4].

The paper "Betweenness Relations in Probabilistic Metric Spaces" [59], by R. Moynihan and B. Schweizer discusses four distinct versions of the betweenness concept for probabilistic metric spaces. Following Menger [55], they underscored the weaker condition of the probabilistic betweenness in comparison to betweenness in general metric spaces. Shortly thereafter, A. Wald [90] introduced a different definition of betweenness, based on a different triangle inequality, and showed that this relation did have all the properties of metric betweenness. Subsequently, J. F. C. Kingman [50] and F. Rhodes [77] studied betweenness in "Wald spaces" and H. Sherwood [80] considered a probabilistic version of this concept.

1.3 Betweenness in Graphs

The theory of metric betweenness in graphs is developed along with the study of betweenness in general metric spaces. The most important and well-studied metric in graphs is the "shortest path metric" (geodesic metric) on a connected graph. The geodesic betweenness give way to look at the set of all "metrically between" vertices defined between two vertices, thus resulting in the notion of "interval" or geodesic interval $I(u,v)$ in a graph. The first systematic study of the notion of the intervals in graphs was attempted by Mulder in 1980 [60]. Many significant properties of the interval on the real line can be found in the interval function $I$ of a graph. For example, If $x$ is a vertex, between $u$ and $v$, then $x$ is the only vertex between $u$ and $x$ and $x$ and $v$. This property implies the following, (b1): if $x$ is a vertex, between $u$ and $v$, different from $u$, then $u$ is not between $x$ and $v$. (b2): if $x$ is a vertex between $u$ and $v$ and $y$ is between $u$ and $x$, then $y$ is between $u$ and $v$. (b3): Similarly, if $x$
is between \( u \) and \( v \), and \( y \) is between \( x \) and \( u \), then \( x \) is between \( y \) and \( v \). These properties are described as a proposition in [60]. Mulder in his book [60] has used the concept of interval function \( I \) to study several classes of graphs, some of them are related to algebraic structures, and some are related to other discrete mathematical structures. Several authors followed Mulder and studied the interval function \( I \).

The notable among them were Nimenen [70] who considered \( I(u, v) \) from the point of view of a join space. A connected graph is a join space with \( u \circ v = I(u, v) \), if and only if it is a strong prime convex intersection graph.

Nebesky attempted to characterize the interval function \( I \) by taking the five betweenness properties of \( I \) as classical axioms, which we mentioned above as proposition stated by Mulder in [60]. Nebesky proposed additional axioms and presented the characterization of \( I \) in a series of papers starting from 1994, each time improving the proof, [63, 64, 65, 66]. Also refer [68, 69]. In [62], Mulder and Nebesky characterized \( I \) by finding the minimal set of axioms required for the characterization of \( I \) besides the five classical betweenness axioms.

The interval function is basically defined using the graph metric generated by the geodesics in graphs. So, in the literature, these intervals are also called geodesic intervals. The concept of the intervals in graphs naturally carries over to geodesic convexity. A subset \( W \) of vertices of a connected graph is geodesically convex, if it contains the intervals between pairs of vertices of \( W \). Because of the betweenness properties, the interval function \( I \) on a graph is well discernable.

Though convex sets are defined in various ways, the most useful definitions are based on the notion of betweenness. When \( X \) is a space in which such a notion is defined, a subset \( C \) of \( X \) is called convex provided that for each two points \( x \) and \( y \) of \( C \), \( C \) includes all points between \( x \) and \( y \).

We can define formally a convexity or convex structure \((X, C)\) on a finite set \( X \), using a ternary relation \( B(a, b, c) \) on \( X \) as follows:

\[
C = \{ K \subseteq X : a, c \in K, (a, b, c) \in B \Rightarrow b \in K \}.
\]
Following this approach, Calder [13] generalized the notion of the interval to include arbitrary sets using what he called as “Interval Function or Interval Operator $R$” defined on a nonempty set $X$ as a function from $X \times X$ to $2^X$ such that $R(x, y)$ contains both $x$ and $y$ (law of extension-(t1) axiom) and $R(x, y) = R(y, x)$ (law of symmetry-(t2) axiom). It may be noted that the interval operators provide a natural and frequent method of describing or constructing convex structures. Various notions of convexity structures arising from interval operators from different mathematical structures has been described by van de Vel in a thought provoking book on convexity [89], titled “Theory of convex structures”, which was published in 1993. The famous Krien-Milman Theorem on ordinary convex sets in the $n$-dimensional Euclidean space $\mathbb{R}^n$, which states that “every compact convex set is the convex hull of its extreme points”, is generalized in the abstract convexity theory as an axiom called as “anti-exchange axiom”. The convexity structures satisfying anti-exchange property are known as antimatroids. A characterization of a nested pair of classes of betweenness relations which generate a special type of antimatroids appeared in 2009, in Chvátal, [33], where it is proved that the corresponding antimatroids from ternary betweenness relations include all antimatroids arising from order convex sets of partially ordered sets and induced path convex sets of chordal graphs.

Many properties of the interval operator $R$ are precisely some strong betweenness properties found in metric spaces (as a special case in graphs with the geodesic distance), partially ordered sets, lattices, etc. Van de Vel termed them as geometric axioms. We quote the important betweenness properties in [89] as follows: An interval operator is geometric provided the following hold.

(1) Idempotent Law: $R(b, b) = \{b\}$ for all $b \in X$

(2) Monotone Law (same as the (b2) of the interval function $I$ of a graph according to Mulder): If $a, b, c \in X$ and $c \in R(a, b)$, then $R(a, c) \subseteq R(a, b)$
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(3) Inversion Law (same as the (b3)): If $a, b \in X$ and $c, d \in R(a, b)$, then
\[ c \in R(a, d) \text{ implies } d \in R(c, b) \]

The last two properties together are equivalent to the following single condition:
For all $a, b, c, d \in X$, if $c \in R(a, b)$ and $d \in R(c, b)$, then $d \in R(a, b)$ and $c \in R(a, d)$.

A set with a geometric interval operator is called a geometric interval space. Some of the propositions described in [89] are as follows:

The geodesic interval operator of a metric space is geometric.

The theorem below characterizes the modular lattices:

1. The modular interval operator $I_M$ is geometric.
2. The modular interval operator $I_M$ satisfies the monotone law,
3. For each $a, b, c \in L$, the following sets is non-empty. $M(a, b, c) = I_M(a, b) \cap I_M(b, c) \cap I_M(c, a)$
4. The lattice $L$ is modular.

Another interesting property that implies geometricity is the Peano and Pasch Property.

Peano Property: For all $a, b, c \in X$ and $y \in R(b, c)$, $z \in R(a, y)$, there is a point $z' \in R(a, b)$ such that $z \in R(c, z')$.

Pasch Property: For all $p, a, b \in X$ and $a' \in R(p, a)$, $b' \in R(p, b)$, the intervals $R(a, b')$ and $R(a', b)$ intersect.

A Pasch Peano (PP) space is an interval space satisfying the Pasch and Peano conditions known from plane geometry, and these properties imply geometricity.

1. An idempotent interval operator with the Peano Property (resp., the Pasch property) satisfies the (b2) (resp., the Inversion Law). In particular, an interval space satisfying both conditions is geometric.
2. In an idempotent interval space with the Peano Property, or in a connected graph with the geodesic interval $I$, satisfying Pasch Property, all intervals $I(u, v)$ are convex.

A class of geometric interval operators, modelled after the properties of intervals derived from a metric space and then their associated interval convexities, provide a unifying approach to many apparently diverse mathematical structures like ordered sets, partially ordered sets, Lattices, Vector Spaces, matroids, graphs, etc. See [89]. Their geometric interval properties are precisely the above mentioned betweenness properties of the geodesic interval function.

The study of geodesic interval and geodesic convexity inspired many metric graph theorists to explore the convexities generated by other intervals on graphs. The notion of induced path or minimal path or monophonic path is a subsequent generalization of the geodesics. An induced path interval between two vertices $u$ and $v$ of a connected graph is the set of all vertices lying on all induced paths connecting them. The induced path interval function $J(u, v)$ does not possess the betweenness properties of the geodesic interval $I(u, v)$. In general, the interval function $R(u, v)$ need not have the betweenness properties of $I(u, v)$. In other words, $R$ can transit from $u$ to $v$, irrespective of whether the vertices in $R(u, v)$ exist or not. Because of this general nature of the function $R$, Mulder termed it as transit functions. Even though the idea of transit function was introduced towards the last decade of the twentieth century, the paper came out black and white only very recently in 2008, see [61]. This paper proposes a comprehensive master plan on intervals, convexities and betweenness in graphs, posets, and related concepts. The framework is axiomatic. The idea of betweenness is explained using transit functions in [61] as follows. Let $R$ be a transit function on $G$. The transit function $R$ highlights the elements $w$ that might be "between" $u$ and $v$. So it may play a role in the study of betweenness. In this sense the axioms $(t1)$ and $(t2)$ may be interpreted as: $R$ sat-
isfies \((t1)\) implies that every vertex \(u\) is in between itself and any other vertex \(v\); \(R\) satisfies \((t2)\) implies that if \(x\) is between \(u\) and \(v\), then \(x\) is also between \(v\) and \(u\). In order to satisfy more non-trivial properties of betweenness for a transit function \(R\), we have to consider some additional axioms. In [61], two basic properties satisfied by the interval function \(I\), are treated as betweenness axioms for a transit function \(R\). They are, respectively,

\[
(b1) \quad x \in R(u, v), x \neq u \Rightarrow u \notin R(x, v) \text{ and } x \notin R(v, u)
\]

\[
(b2) \quad x \in R(u, v) \Rightarrow R(u, x) \subseteq R(u, v).
\]

For a transit function, besides the two basic betweenness axioms \((b1)\) and \((b2)\), the other betweenness axioms are also considered, taken from the usual interval in the real line. An important axiom is the axiom \((b3)\). The axioms \((b2)\) and \((b3)\) together constitute the geometric axioms of van de Vel. An interesting property of the interval in the set of real numbers, \(\mathbb{R}\), that need not hold for the interval function \(I(u, v)\) of a graph is that \(I(u, v)\) need not be convex. Mulder termed the graphs for which every interval is convex as interval monotone graphs [60]. In [60] Mulder observed that all connected graphs need not be interval monotone and laid down sufficient conditions for such graphs (for example, the complete bipartite graph \(K_{2,3}\) is not interval monotone). A characterization of interval monotone graphs is still an open problem. In [61], it is defined as an axiom for a transit function, which we term as the monotone axiom, denoted by \((m)\). The monotone axiom is closely related to the betweenness axiom \((b2)\), it is easy to note that a transit function satisfying \((m)\) has the \((b2)\) property. Refer also [9, 56].

In [58], Morgana and Mulder also proved that the induced path transit function \(J(u, v)\) is the betweenness on a connected graph \(G\), if and only if \(G\) has no induced subgraphs isomorphic to a house, hole or domino. It is to be noted that the induced path transit function also doesn’t satisfy the monotone axiom \((m)\). In [19], characterizations of graphs for which the monotone and the \((b2)\) axioms hold are
presented. For the monotone axiom, also refer [16] and [18]. A characterization of the convex hull and other combinatorial aspects of the induced path convexity is the subject of the paper [36] by Duchet.

Another very important transit function in graphs is the “All paths transit function”, which is the coarsest path transit function in a graph. In [23], the all paths transit function and its properties, namely, the betweenness, gatedness and convexity are described. An axiomatic characterizations of the all paths transit functions are also presented in [23]. See also [79], for all paths convexity. An analogous study of the convexity properties of the convexity associated with the triangle induced path transit function can be seen in [17]. A comprehensive survey of the convexities associated with the path transit functions, that is transit functions defined by various types of paths, is surveyed in [22].

It is interesting to note that the notions of transit functions and betweenness have found applications in the recombination theory in evolution, see for example, the papers of Stadler et. al. [86, 87, 91, 92].

Another notion of betweenness in graphs is the “betweenness centrality” of a vertex introduced by Freeman while studying social networks [39]. It provides an idea about the importance of the vertex in a social network. The betweenness centrality of a graph is the portion of shortest paths crossing through it between other vertices of the graph. See [41].

1.3.1 Steiner trees and Steiner intervals in graphs

A natural tendency in mathematics is to generalize a well studied concept to wider spectra. In this sense, it becomes meaningful to generalize the notion of a transit function defined for a pair of points on a non-empty set to any $n$-points, $n > 2$. We make an attempt in this direction. A natural question then is, “What is the analogue of the geodesic interval function $I$ for $n > 2$?” The answer for this question turns out to be the $n$-Steiner intervals.
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The Steiner trees and Steiner intervals on ordinary vertex subsets is a well studied concept in graphs and has also got many applications and several papers have appeared in the literature. We give a short survey of Steiner trees and Steiner intervals and some interesting problems related to them. The Steiner tree problem is a well-known problem with several variations and applications. It can concern points in Euclidean (or other metric) spaces, and vertices of weighted or non weighted graphs [47], and has drawn much attention due to the development of approximation algorithms, see [1, 57] and the references therein.

One of the main issues regarding Steiner intervals is related to connections between different variations of the geodetic number, see the survey paper [30]. Chartrand and Zhang proposed a natural concept of the Steiner number of a graph [31], and among several nice results “proved” an erroneous statement [31] regarding the connection between Steiner intervals of a set of vertices, and the union of geodesic intervals between pairs of the vertices from the set. This error was observed and corrected by Pelayo [73], and the development intrigued and motivated Hernando et al. [45] to raise the following problem: For which graphs does the Steiner interval of any set of vertices, whose union of geodesic intervals is the whole vertex set, also yield all vertices? Certainly this property holds for graphs in which $S(W) \subseteq \bigcup_{u,v \in W} I(u, v)$ for all $W \subseteq V(G)$, which was shown to be true in distance hereditary graphs by Hernando et al. [45], and independently also Oellermann and Puertas [72] and Eroh and Oellermann [37] extended this result in a more general family of 3-Steiner distance hereditary graphs. A characterization of these graphs (in which $S(W) \subseteq \bigcup_{u,v \in W} I(u, v)$ for all $W \subseteq V(G)$) remains open, and seems to be quite difficult.
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1.4 Approaching the problem

The studies on transit functions and the betweenness on graphs has led us to consider this subject as the topic of this thesis. We study the betweenness properties of other unexplored transit functions on graphs. In this thesis, we have followed this approach to study the betweenness of transit functions which are related to or derived from the interval function \( I(u, v) \). Further, we study the betweenness properties of the longest path transit function on a connected graph \( G \), which is also unexplored.

Finally, we extend the notion of transit functions and betweenness to \( n \)-ary transit functions. In particular, we consider the \( n \)-Steiner intervals \( S \) as a specific instance of generalization of the geodesic intervals and study its betweenness.

1.5 Synopsis

This thesis is concerned with the study of betweenness of some metric related transit functions on graphs. We study the three betweenness axioms \((b1), (b2)\) and \((m)\) and also a modified betweenness axiom denoted as \((b3')\). More precisely, we study the path transit functions for which the path property either defines a metric or is closely related to a path metric in the graph and discuss their betweenness.

The transit functions that we consider are \( I^\Delta, I_1 \), both of which are derived from the interval function \( I \) of the graph, which is well known as defining the shortest path metric in the graph. Another path metric is the longest path metric. We study the longest path transit function \( L \) and its betweenness properties. Finally, the Steiner transit function and its betweenness properties are studied. The thesis is divided into 7 chapters. The first chapter is this introductory chapter giving the background of the whole work. The first section of each chapter, except the concluding one, contain a formal introduction. There is a concluding section of similar nature in Chapters 3, 4, 5, and 6 which contain examples of graphs showing
the implications of the relevant betweenness axioms, discussed for the concerned transit function in each chapter depicted in a tabular form. This is in line with the observations of the implications of betweenness of transit functions described in Chapter 2.

Chapter 2 deals with the basic observations of transit functions and betweenness. In Section 2.2 the preliminary concepts, notations and terminologies of the thesis are discussed. Section 2.3 discusses the ordinary (binary) transit functions, the betweenness axioms and their implications. For a transit function $R$, one can verify that all the betweenness axioms taken one at a time are independent, except when $(m)$ implies $(b2)$ and $(b3)$ implies $(b1)$. We give examples of transit functions satisfying the betweenness axioms taken one at a time. In the Section 2.4 of this chapter, we generalize the definition of the ordinary transit function. We discuss some examples and preliminary results and in Subsection 2.4.1, we make a description of the implication of the betweenness axioms similar to the binary case.

In Chapter 3, we consider the $I^\Delta$ transit function. In Section 3.2 we derive the necessary and sufficient conditions in which the transit function $I^\Delta$ satisfies betweenness axioms $(b1)$, $(b2)$ and $(b3')$ axioms with forbidden induced subgraphs.

In Chapter 4, we study the betweenness of the $I_1$ path transit function. The Section 4.2 defines formally the $I_j$ path transit function and observes some properties and relationship between $I, I^\Delta, I_1$ and $I_j, j > 1$. The necessary and sufficient conditions in which the transit function $I_1$ satisfies betweenness axioms $(b1)$, $(b2)$ and $(b3')$ axioms are obtained with forbidden induced subgraphs in Section 4.3.

The title of Chapter 5 is “Betweenness and extreme sets of the longest path” where we discuss two notions, the betweenness and extremeness. In Section 5.2 the betweenness of the longest path transit function $L$ are studied. Here, we prove that the axiom $(b1)$ holds good if and only if $G$ is a tree and if $G$ is Hamiltonian then $L$ satisfies the $(b2)$ axiom if and only if $L(u, x) = L(u, v)$. We pose a conjecture that the results proved in this section for graphs whose blocks are Hamiltonian holds
for any connected graph. We briefly discuss the extremeness notion such as the
detour boundary, eccentric and peripheral sets and their relationships analogous to
that of the geodetic boundary, eccentric and peripheral sets in Section 5.3.

In Chapter 6, we study the \( n \)-Steiner intervals. In particular, we study the
relationship between geodesic intervals, Steiner intervals and the betweenness axiom
\((b2)\) and monotone axiom \((m)\). We define a property called the union property of
Steiner intervals, which relates the geodesic intervals and the Steiner interval and
consider three families of graphs (1) graphs for which the Steiner interval satisfying
the union property, (2) graphs for which the Steiner interval satisfies the monotone
axiom and (3) graphs for which the Steiner interval satisfies the \((b2)\) axiom. In
Section 6.2, we prove that for \( k > 3 \) the family of graphs in (1), (2) and (3) are
the same and they are characterized. Section 6.3 discusses the case for \( k = 3 \). We
prove that the family of graphs in (1) is properly contained in the family of graphs
in (2) which is contained in the family of graphs in (3). We characterize the family
of graphs in (1) and (2) in Subsections 6.3.1 and 6.3.2 respectively.

Chapter 7 is a conclusion by summing up all the main results and contains some
directions for future work.