5. BETWEENNESS AND EXTREME SETS OF THE LONGEST PATH

5.1 Introduction

In this chapter we discuss the longest path in $G$. Similar to the shortest path distance $d(u, v)$ from a vertex $u$ to a vertex $v$ in a graph $G$, the length $D(u, v)$ of a longest $u,v$-path in $G$, the detour distance is also a metric that has caught the attention of graph theorists in the recent years. Many analogous notions of the distance function $d(u, v)$ have appeared in the literature using the detour distance $D(u, v)$; to mention a few, namely, on detour eccentricity, center, diameter, [25, 26, 28, 29] and coloring [40].

In this chapter, we study two different notions, one, betweenness and the other, extremeness. Both notions defined with respect to the detour distance are discussed. In Section 5.2, we study betweenness. In Section 5.3, we search for the relationship between the extreme sets, periphery, boundary and eccentric and contour sets defined by the detour metric analogous to that of the corresponding sets defined by the shortest path metric.

5.2 Betweenness of the longest path transit function

In this section, we consider the betweenness axioms on the longest path transit function. A graph $G$ is called a Hamiltonian if it has a cycle $C$ which contains

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1 The results of Section 5.2 of this chapter is to appear in the paper: M. Changat, Prasanth G. Narasimha-Shenoi; I. M. Pelayo; Longest path transit function and its betweenness; in Util. Math.
all the vertices of $G$. We show that among the class of graphs in which every block is Hamiltonian, there is a subclass where the betweenness axiom $(b2)$, and the monotone axiom $(m)$ are equivalent and that this holds if and only if the longest path transit function $L$ coincides with the all paths transit function $A$.

The feasible path properties, "all", "induced", and "geodesic" are hereditary, in the sense that every subpath of these paths is again a path of the same property; namely "all", "induced", and "geodesic" respectively. Unlike these path properties, the path property "longest" is not hereditary and consequently, the longest path transit function does not share some of the properties of the other path functions. For example if $x, y, u, v$ are vertices in a cycle $C$ with $x, y$ being in the longest $u, v$-path $P$ on $C$, then the $x, y$-subpath of $P$ need not be a longest $x, y$-path.

These strange behaviour encouraged us to focus on longest path transit function (or detour transit function). We observe something interesting regarding the betweenness of the longest path function, namely certain betweenness properties coincide on the longest path transit function and this can happen, if and only if the longest path transit function coincides with the all-paths transit function.

Before considering the betweenness, we recall the definition of the longest path transit function $L$ of a graph $G$ from Chapter 2, which is as follows.

$$L(u,v) = \{w \mid w \text{ lies on a longest } u,v\text{-path in } G\}.$$ 

We use the following notation: Both $P : u \to v$ and $u \xrightarrow{P} v$ denote a $u,v$-path called $P$.

Again recalling from Chapter 2, by $D(u,v)$, we mean the length of a longest $u,v$-path $P$, i.e., $D(u,v) = |E(P)|$. We have the following lemma.

Lemma 5.1: Let $u, v, x$ be three distinct vertices in $V(G)$ such that $x \in L(u,v)$. Then $L(u,x) \cap L(x,v) = \{x\}$ if and only if $x$ is a cut vertex separating two blocks $B_1$ and $B_2$ in $G$ with $u \in V(B_1)$ and $v \in V(B_2)$. 
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Proof. Let \( x \in L(u, v) \) with \( L(u, x) \cap L(x, v) = \{ x \} \). Also, if \( P : u \xrightarrow{P_1} x \xrightarrow{P_2} v \) is a longest \( u, v \)-path, then \( P_1 \) is a longest path \( u, x \)-path and \( P_2 \) is a longest \( x, v \)-path.

Let \( \Omega \) be a \( u, v \)-path different from \( P \). We use the following notation: \( |E(\Omega)| = m \), \( u = y_0, v = y_m \) and \( \Omega : y_0 y_1 \rightarrow y_m \). Consider these vertices of the path \( \Omega \):

\[
\begin{align*}
h &= \max_{0 \leq i \leq m-1} \{ i \mid y_i \in V(P_1) \}, \\
k &= \min_{h \leq j \leq m} \{ j \mid y_j \in V(P_2) \}
\end{align*}
\]

Observe that the \( u, v \)-paths \( P \) and \( \Omega \) can be described as follows:

\[
P : u \xrightarrow{P_1} y_h \xrightarrow{P_2} x \xrightarrow{P_2} k \xrightarrow{P_2} v, \quad \Omega : u \xrightarrow{\Omega_1} y_h \xrightarrow{\Omega_2} y_k \xrightarrow{\Omega_3} v
\]

Notice also that \( Q_1 : u \xrightarrow{P_1} y_h \xrightarrow{\Omega_2} y_k \xrightarrow{P_2} x \) is a \( u, x \)-path and \( Q_2 : x \xrightarrow{P_2} y_h \xrightarrow{\Omega_2} y_k \xrightarrow{P_2} v \) is a \( x, v \)-path. As both \( P_1 \) and \( P_2 \) are longest paths, then:

\[
\begin{align*}
|E(P_{11})| + |E(\Omega_2)| + |E(P_{21})| &\leq D(u, x) = |E(P_1)| = |E(P_{11})| + |E(P_{12})| \\
|E(P_{21})| + |E(\Omega_2)| + |E(P_{22})| &\leq D(x, v) = |E(P_2)| = |E(P_{21})| + |E(P_{22})|
\end{align*}
\]

Hence, \(|E(P_{12})| = |E(P_{21})| = |E(\Omega_2)| = 0\). In other words, \( x = y_h = y_k \in V(\Omega) \).

Therefore, \( u \) and \( v \) lies in adjacent blocks separated by the cut vertex \( x \).

Conversely suppose \( x \) is a cut vertex between two blocks \( B_1 \) and \( B_2 \) with \( u \in V(B_1) \) and \( v \in V(B_2) \). Any longest \( u, v \)-path should pass through \( x \) since \( V(B_1) \cap V(B_2) = \{ x \} \). Therefore \( x \in L(u, v) \) and the sections \( u \rightarrow x \) and \( x \rightarrow v \) are longest \( u, x \) and \( x, v \)-paths and we have \( L(u, x) \cap L(x, v) = \{ x \} \).

Now we have the following proposition

**Proposition 5.2:** Let \( u \) and \( v \) be vertices of \( G \) lying in two different blocks separated by a cut vertex \( x \), then \( L(u, v) = L(u, x) \cup L(x, v) \).
Lemma 5.3: $L$ satisfies (b1) if and only if $G$ is a tree.

Proof. If $G$ is a tree then (b1) is satisfied. If $G$ contains a cycle $C = u_0 u_1 u_2 \ldots u_n u_0$, then $L(u_0, u_n)$ will contain $u_1$ and $u_n \in L(u_1, u_0)$, a contradiction. □

Since (b3) implies (b1) and (b3') for any transit function by Theorem 2.5, and in a tree any path transit function coincides with the geodesic transit function which is a geometric transit function, as a corollary to Lemma 5.3, we have the following theorem.

Theorem 5.4: $L$ satisfies (b3) and (b3') if and only if $G$ is a tree.

Lemma 5.5: Let $G = (V, E)$ be a Hamiltonian graph. Then the longest path transit function $L$ of $G$ satisfies (b2) if and only if $L(u, v) = V$ for any two distinct $u, v \in V$.

Proof. If for any two distinct $u, v \in V$, $L(u, v) = V$, then $L$ satisfies (b2). Conversely, let $L$ satisfy (b2). Let $|V| = n$. Since $G$ is Hamiltonian, there is a Hamiltonian cycle $C = u_1 u_2 \ldots u_n u_1$. So any longest path in $G$ has length $n - 1$. Assume to the contrary that not every $L$-interval equals $V$, so that there exist distinct vertices $u$ and $v$ such that $L(u, v) \neq V$. Note that $u$ and $v$ cannot be adjacent in $C$, and that $n \geq 4$. Without loss of generality we may take $u = u_1$ and $v = u_j$ with $2 < j \leq \frac{1}{2} n + 1$. Take any neighbour of $u$ on $C$, say $u_2$. Then, using the Hamiltonian cycle, we see that $L(u_1, u_2) = V$. Now, if we would have $u_2 \in L(u, v)$, then, by (b2), we would have $V \subseteq L(u, v)$, which contradicts the choice of $u, v$. So $u_2$ as well as $u_n$ are not in $L(u, v)$. Similarly, we deduce that the neighbours $u_{j-1}$ and $u_{j+1}$ of $v$ of $C$ are not in $L(u, v)$. If $3 < j - 1$, then we proceed as follows. Note that now we have $|L(u, v)| \leq n - 4$. On the other hand, using the Hamiltonian cycle, we have $|L(u, v)| \geq n - 1$. So, as above by (b2), it follows that $u_3 \notin L(u, v)$. Similarly, $u_{n-1}, u_{n-2}, u_{j+2}$ are not in $L(u, v)$. If $4 < j - 2$, then we continue this process and
prove that \( u_4 \) and \( u_{j-3} \) are not in \( L(u,v) \), and so on. Thus we can show that all of \( u_2, u_3, \ldots, u_{j-1} \) are not in \( L(u,v) \) as well as \( u_n \) and \( u_{j+1} \). So \( |L(u,v)| \leq n - j \). On the other hand, along the Hamiltonian cycle \( v = u_j u_{j+1} \ldots u_n u_1 = u \) is a path with \( n - j + 2 \) vertices, so we should have \(|L(u,v)| \geq n - j + 2\). This provides us with the required contradiction. Hence \( L(u,v) = V \) for any two distinct vertices \( u \) and \( v \).

**Theorem 5.6**: Let \( G = (V, E) \) be any connected graph such that every block of \( G \) is Hamiltonian. Then the following are equivalent for the longest path transit function \( L \) of \( G \).

1. Every block of \( B \) of \( G \) satisfies \( L(u,v) = V \), for any two distinct \( u, v \in V(B) \).
2. \( L \) satisfies \((b2)\).
3. \( L \) satisfies \((m)\).
4. \( L \) coincides with the all-paths transit function \( A \) of \( G \).

**Proof.** Using Lemma 5.5, the theorem follows easily. \(\square\)

Using Theorem 5.4 and Theorem 5.6, we have a straightforward Corollary.

**Corollary 5.7**: The longest path transit function \( L \) of a connected graph \( G \) is geometric and monotone if and only if \( G \) is a tree. \(\square\)

**Remark 5.8**: There exist many 2-connected non-Hamiltonian graphs that satisfy \( L(u,v) = V(G) \), for every \( u, v \in V(G) \). For example, the graphs in figures (Figure 5.1, 5.2 and 5.3) satisfy this condition. One of the graphs of Figure 5.1 is the \( K_{2,n} \) and the other is \( K_{2,n} + e \).
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In general, there are non-Hamiltonian 2-connected graphs that satisfy $L(u, v) = V(G)$, for every $u, v \in V(G)$. For example, replace each of the $n$ vertices of $K_{2,n}$ in Figure 5.1 by a copy of $K_n$ and join the vertices $u$ and $v$ to all the vertices of $K_n$ by edges. The graph in Figure 5.2 is an instance of this construction with $n = 3$.

We construct another family of non-Hamiltonian graphs with the betweenness ($b2$) property: Replace each of the $n$ vertices of $K_{2,n}$ in Figure 5.2 by a copy of $C_{2n}$ with vertices labeled as $\{1, \ldots, 2n\}$ and join the vertex $u$ to the odd vertices $1, 3, \ldots, 2n - 1$ and $v$ to the even vertices $2, 4, \ldots, 2n$ by edges. The graph in Figure 5.3 is an instance of this construction with $C_4$. Similarly, we construct with the odd cycle $C_{2n+1}$ with similar vertex labeling and join by edges the vertex $u$ to the odd vertices $1, 3, \ldots, 2n$.
vertices $1, 3, \ldots, 2n - 1$ and $v$ to the even vertices $2, 4, \ldots, 2n$ and the vertex $2n + 1$ to both vertices $u$ and $v$.

![Graph](image)

**Fig. 5.3:** Graph for which $L(u, v) = V(G)$, for every $u, v \in V(G)$ - (3)

If a graph $G$ has the property that every block $B$ of $G$ satisfies $L(u, v) = V(B)$, for any $u, v \in V(B)$ — (α), we have that $L(x, z) \subseteq L(x, y)$, for any $z \in L(x, y)$, for any $x, y \in V(G)$, which is equivalent to the fact that if $G$ satisfies condition α, then $L$ satisfies the (b2) axiom, if and only $L$ satisfies the monotone axiom. From the discussions above, we can see that $G$ satisfies condition α is equivalent to the fact that $L$ coincides with the all paths transit function $A$.

### 5.3 Extreme sets of the detour distance

There are various concepts of boundary sets in a graph like boundary, periphery, eccentric and contour sets. All these boundary sets in a graph $G$ can be studied using any metric $\delta(u, v)$ between the vertices $u, v$ in $G$. These extreme sets and their
relationships in a graph $G$, defined using the geodesic distance, have been studied by various authors. The most important contributions on these concepts are by Caceres et.al in [14], [15]. In [14], they establish a realization theorem, namely, for each triple $a, b, c$ of integers with $2 = a = b = c$, there is a connected graph $G$ such that the Periphery of $G$ has order $a$, Eccentric set has order $b$, and boundary has order $c$, by correcting a corresponding theorem in [24].

An attempt to study the boundary sets using the detour metric in $G$, like the detour eccentricity and periphery, is made by Chartrand et.al. [25], [26], [32], where they also studied the center, diameter, and radius. These notions using another metric is studied in [27].

In this section, we make an attempt to study the extreme sets like boundary, periphery, eccentric and contour sets defined using the detour metric in search of an analogue of geodetic boundary sets as in Cáceres et.al in [14]. We start with the necessary definitions. Since we are dealing the detour distance, the metric $\delta(u, v)$ is the detour metric $D(u, v)$. The detour diameter of $G$ is denoted as $D(G)$.

**Definition 5.9: (Boundary of $G$)** A vertex $v$ is said to be a boundary vertex of $u$ if no other neighbour of $v$ is farther away from $u$ than $v$. The boundary of $G$ is the set of all boundary vertices.

That is, $\partial(G) = \{ v \in V \mid \text{such that for all} \ w \in N(v); D(u, w) \leq D(u, v) \}$, for some $u \in V(G)$.

**Definition 5.10: (Eccentric vertex)** Eccentricity of a vertex $u \in V$ is defined as $\text{ecc}_G(u) = \max \{ D(u, v) \mid \forall v \in V \}$. A vertex $v$ is called an eccentric vertex of $u$ if no vertex in $V$ is farther away from $u$ than $v$. That is $D(u, v) = \text{ecc}_G(u)$. Therefore a vertex $v$ is called a eccentric vertex of $G$ if it is the eccentric vertex of some vertex $u \in V$.

**Definition 5.11: (Eccentricity of $G$)** The eccentricity $\text{Ecc}(G)$ of $G$ is the set of all its eccentric vertices.
\[ Ecc(G) = \{ v \in V \mid \text{there exists } u \in V \text{ such that } ecc(u) = D(u, v) \}. \]

**Definition 5.12: (Periphery of G)** A vertex \( v \in V \) is called a **peripheral vertex** of \( G \) if no vertex in \( V \) has eccentricity greater than \( ecc(v) \). The periphery \( Per(G) \) of \( G \) is the set of all its peripheral vertices. That is \( Per(G) = \{ v \in V \mid ecc(v) = D(G) \} \).

**Definition 5.13: (Contour of G)** A vertex \( v \in V \) is called a **contour vertex** of \( G \) if no neighbour vertex of \( v \) has an eccentricity greater than \( ecc(v) \). The contour set \( Ct(G) \) is the set \( \{ v \in V \mid ecc(u) \leq ecc(v) \text{ for all } u \in N(v) \} \).

**Definition 5.14: (Simplicial set of G)** \( Simp(G) \) consists of the set of all simplicial vertices of \( G \), where a vertex is a simplicial vertex if its neighbourhood is complete. That is, \( Simp(G) = \{ v \in V \mid N(v) \text{ is a clique} \} \).

The following proposition with the shortest path metric is proved in [14]. We prove that the proposition is true for the detour metric also.

**Proposition 5.15:**

Let \( G = (V, E) \) be a graph, then we have the following results

1. \( Per(G) \subseteq \partial(G) \)
2. \( Per(G) \subseteq Ct(G) \cap Ecc(G) \)
3. \( Ecc(G) \cap Ct(G) \subseteq \partial(G) \)
4. \( Simp(G) \subseteq Ct(G) \subseteq \partial(G) \)

**Proof.**

1. Follows from the definition of \( Per(G) \) and \( \partial(G) \)
2. Let \( x \in Per(G) \). Therefore we have \( ecc(x) = D(G) \). So for all \( x' \in N(x) \) we have \( ecc(x') \leq ecc(x) \Rightarrow x \in Ct(G) \). Since the diameter \( D(G) \) of \( G \) is the maximum detour distance, also we have \( x \in Ecc(G) \). Therefore \( Per(G) \subseteq Ct(G) \cap Ecc(G) \).
3. \( x \in Ecc(G) \cap Ct(G) \). \( x \in Ecc(G) \Rightarrow ecc(x) = \max\{D(x, v) \mid v \in V(G)\} \) and \( x \in Ct(G) \Rightarrow ecc(x) \geq ecc(x') \), for all \( x' \in N(x) \) so that \( x \in \partial(G) \).
(4) Let \( x \in \text{Simp}(G) \). Therefore \( H = x \cap N(x) \) is a complete graph. Any longest path starting from any vertex in \( H \) will contain all the vertices of \( H \). Therefore we must have \( \text{ecc}(x) \geq \text{ecc}(x') \), for all \( x' \in N(x) \). Therefore \( x \in \text{Ct}(G) \). Now let \( x \in \text{Ct}(G) \), therefore we have \( \text{ecc}(x) \geq \text{ecc}(x') \) for all neighbours \( x' \) of \( x \). That is \( D(x, u) \geq D(x', u) \) for all \( u \in V(G) \). Hence we have \( x \in \partial(G) \) \( \square \)

**Remark 5.16:** The proof of (1),(2) and (3) in Proposition 5.15, uses only the metric property and therefore holds for any metric \( \delta(u, v) \) defined on \( V(G) \). Again, it can be easily verified that (4) holds for the other metric like the truncated metric or a weighted path metric (on a graph with positively weighted edges) or a tree metric (for different notions of metrics defined on graphs, see[35]). But (4) need not hold for arbitrary metric in general, in a graph. For example, consider the tree with vertex set \( V = \{a, b, c\} \) and edges \( ab, be \). Define a metric \( \delta \) on \( V \) as follows: \( \delta(a, b) = 2, \delta(a, c) = 1, \delta(b, c) = 3 \) and \( \delta(a, a) = \delta(b, b) = \delta(c, c) = 0 \). Here \( a \in \text{Simp}(G) \), but \( \delta(b, c) > \delta(a, c) \). Therefore \( a \notin \text{Ct}(G) \).

We can see that most of the results which are true in the geodesic may not hold for the detour. For example, we know that \( | \text{ecc}_d(u) - \text{ecc}_d(v) | = 1 \) if \( \{u, v\} \in E(G) \). But the corresponding result for the case of detour is not true in general. See the following graph.

**Fig. 5.4:** \( uv \in E(G) \) and \( \text{ecc}(u) = 10, \text{ecc}(v) = 8 \).
Lemma 5.17: If $\text{Per}(G) = C_{t}(G)$ and $x \notin \text{Per}(G) = C_{t}(G)$ then $\deg(x)$ will be at least 2.

Proof. Let $x \notin \text{Per}(G) = C_{t}(G)$. Therefore there exists some neighbour $x'$ of $x$ so that $\text{ecc}(x') \geq \text{ecc}(x)$. If $\deg(x) = 1$ then, we can see that $\text{ecc}(x) \geq \text{ecc}(x') \Rightarrow x \in C_{t}(G)$; a contradiction. Therefore $\deg(x) \geq 2$. \qedsymbol

We have the following lemmas

Lemma 5.18: $G$ is a connected graph with $D(G) = \lambda$. If $\text{Per}(G) = \{a, b\}$, then $\deg(a) = \deg(b) = 1$.

Proof. Let $\rho$ be a longest $a, b$-path. Note that $|V(\rho)| = \lambda + 1$. Suppose $\deg(a) \geq 2$

Case 1: $ab \in E(G)$

In this case there exists $c \in V(\rho)$ so that $cb \in E$. See Figure 5.5. Then the path $c \overset{L}{\rightarrow} a \rightarrow b$ is of length at least $\lambda$. Hence $\text{ecc}(c) \geq \lambda$ and therefore $c \in \text{Per}(G)$; a contradiction

\begin{center}
\begin{figure}
\centering
\includegraphics[width=0.4\textwidth]{fig55.png}
\caption{Case 1 from Lemma 5.18}
\end{figure}
\end{center}

Case 2: $ad \in E(G); d \notin V(\rho)$

In this case the path $d \rightarrow a \overset{L}{\rightarrow} b$ is of length at least $\lambda + 1$ and therefore we have $\text{ecc}(d) > \lambda$; a contradiction to our assumption. See Figure 5.6
Case 3: $ad \in E(G); d \in V(\rho) \setminus b$

In this case the path $b \rightarrow d \rightarrow a \xrightarrow{\rho} e$ is of length at least $\lambda + 1$ and therefore we have $ecc(e) > \lambda$; a contradiction to our assumption. See Figure 5.7

Hence in all the cases, we have obtained a contradiction. Therefore $\deg(a) = \deg(b) = 1$

\[\Box\]

**Lemma 5.19:** Let $G$ be a graph having diameter $\lambda > 2$ with $Per(G) = Ct(G) = \{a, b\}$ and $\Lambda_1 = \{ x \in V(G) : ecc(x) = \lambda - 1 \}$. Then $|\Lambda_1| = 2$ and every vertex in $\Lambda_1$ has degree exactly 2.

**Proof.** Let $\rho$ be a longest $a, b$-path and $a'$ and $b'$ be the neighbours of $a$ and $b$ respectively lying in $\rho$. Then clearly $a', b' \in \Lambda_1$. We prove that $|\Lambda_1|$ cannot be greater than 3.

Suppose $|\Lambda_1| \geq 3$. By Lemma 5.18 we can see that $\deg(a) = \deg(b) = 1$. Let $c \in \Lambda_1$. Therefore we have $ecc(c) = \lambda - 1$. If $cc' \in E(G)$ with $\deg(c) = 1$ then we can see that $c$ will be a contour vertex since eccentricity of $c'$ will be at least $\lambda - 1$. 

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**Fig. 5.6:** Case 2 from Lemma 5.18

**Fig. 5.7:** Case 3 from Lemma 5.18
This will not hold since $Per(G) = C_t(G)$. Therefore $deg(c) \geq 2$. Then following the similar steps as in the proof of Lemma 5.18, we can arrive at a contradiction that either $c$ will be a peripheral vertex or a contour vertex, which is a contradiction. Hence $|\Lambda_1| = 2$.

Thus we have $\Lambda_1 = \{a', b'\}$. Also $deg(a'), deg(b') \geq 2$. Suppose $deg(a') \geq 3$.

**Case 1:** $a'c \in E(G), c \notin V(\rho)$

Then the path $c \rightarrow a' \xrightarrow{\rho} b$ will be of length at least $\lambda$; so that $ecc(c) \geq \lambda \Rightarrow c \in Per(G)$; a contradiction. see Figure 5.8

![Fig. 5.8: Case 1 from Lemma 5.19](image)

**Case 2:** $a'c \in E(G)$ and $c \in V(\rho) \setminus \{a, a''\}$

Let $a''$ be the neighbour of $a'$ as we trace from $a$ to $b$ along $\rho$ and $d$ be the neighbour of $c$ as we trace from $b$ to $c$. See Figure 5.9. Here $ecc(d) \geq \lambda - 1$, since the path $d \xrightarrow{\rho} a'c \xrightarrow{\rho} b$ is of length $\lambda - 1$. This is a contradiction since $|\Lambda_1| = 2$ and $a', b' \in \Lambda_1$, $Per(G) = \{a, b\}$.

![Fig. 5.9: Case 2 from Lemma 5.19](image)

Similarly we can prove $deg(b') = 2$. □

Next two lemmas are similar to Lemma 5.19 and the proof is also similar. So we
omit the proof.

**Lemma 5.20:** Let $G$ be a graph having diameter $\lambda > 2$ with $Per(G) = Ct(G) = \{a, b\}$ and $\Lambda_2 = \{x \in V(G) : ecc(x) = \lambda - 2\}$. Then $|\Lambda_2| = 2$ and every vertex in $\Lambda_2$ has degree exactly 2.

**Lemma 5.21:** Let $G$ be a connected graph having diameter $\lambda \geq h + 1$, $h \geq 2$ with $Per(G) = Ct(G) = \{a, b\}$ and $\Lambda_i = \{x : ecc(x) = \lambda - i\}$. If $\forall i \in \{1, 2, \ldots, h - 1\}$, $|\Lambda_i| = 2$ and every vertex in $\Lambda_i$ has degree exactly 2, then $|\Lambda_h| = 2$ and every vertex in $\Lambda_h$ has degree exactly 2.

**Theorem 5.22:** For a graph $G$, $|Per(G)| = |Ct(G)| = 2$ if and only if $G$ is a path.

**Proof.** If $G$ is a path then it is clear that $|Per(G)| = |Ct(G)| = 2$. Conversely, suppose $|Per(G)| = |Ct(G)| = 2$ and $D(G) = h + 1$. Let $a, b \in Per(G)$. Therefore there exists a $a, b$ longest path $\rho$ of length $h + 1$ and by lemma 5.18 both $a$ and $b$ have degree 1. By lemma 5.20, 5.21 every internal vertex in $\rho$ have degree exactly 2. From this it follows that $G$ is a path. 

As a consequence of the above theorem we have the following proposition.

**Theorem 5.23:** The following are equivalent for a graph $G$.

(i) $G$ is a path

(ii) $|Per(G)| = |Ct(G)| = 2$

(iii) $|\partial(G)| = 2$.

**Proof.** (i) implies (ii) and (i) implies (iii) follows trivially.

(ii) implies (i) By Theorem 5.22.

(ii) implies (iii) Follows from Theorem 5.22, since a path has boundary 2.

(iii) implies (ii) By Proposition 5.15 (1), (2) and (4) $Per(G) \subseteq Ct(G) \subseteq \partial(G)$. Since $Per(G) > 1$ by hypothesis $|\partial(G)| = 2$, we have $|Per(G)| = |Ct(G)| = 2$. 

$\square$
5.4 Concluding remarks

From the foregoing discussions and examples of non-Hamiltonian graphs, in the last part of Section 5.2, we believe that the following conjecture holds:

**Conjecture 5.24:** For a connected graph $G$ with the longest path transit function $L$, the following statements are equivalent.

1. $L$ satisfies the betweenness axiom $(b2)$
2. $L$ satisfies monotone axiom $(m)$
3. $L$ coincides with $A$, the all paths transit function of $G$.

We conclude this chapter with a similar analysis of the implications of the betweenness axioms for the longest path transit function, as we did it for the functions $I_1$ and $I^A$ in the previous chapters. This is explained in the following table.

<table>
<thead>
<tr>
<th>(b1)</th>
<th>(b2)</th>
<th>(b3) = (b3')</th>
<th>(m)</th>
<th>Graph</th>
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<td>×</td>
<td>Tree</td>
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<td>No graph</td>
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<td>Figure 5.1</td>
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</tr>
</tbody>
</table>

Tab. 5.1: Longest path transit function with possible betweenness relations. × denotes that the corresponding axiom is satisfied.