CHAPTER III

REFLECTIVE SUBCATEGORIES OF THE CATEGORY OF TOPOLOGICAL SEMIGROUPS

This chapter consists of three sections. The first section deals with the category of topological semigroups. In the next section we introduce the notion of an ideal*semigroup and prove that epimorphisms in the category of ideal*semigroups are morphisms with dense range. Reflective Subcategories are discussed in section three and it is proved that the reflection of an object is unique up to topological isomorphism in $\mathcal{I}S$, the category of topological semigroups. In the last section we consider the Universal Problem defined by the adjoint functors and derive the sufficient condition for a pair of adjoint functors to induce a reflective subcategory.

3.1 Category of Topological Semigroups.

In category of topological semigroups, objects are topological semigroups and morphisms are continuous homomorphisms, except that in the monoid categories, morphisms are required to be identity preserving. The rule of composition in each category is ordinary composition of functions.

Let $\mathcal{KS}$- denote the category of discrete semigroups, $T\mathcal{KS}$- denotes the category of topological semigroups and $C\mathcal{KS}$- denotes the category of compact semigroups.[15]
If $S$ is a $\mathcal{C}K$-object, then the $\mathcal{C}K$ identity morphism for $S$ is the identity function from $S$ into itself (denoted as $I_S$). $\text{Hom}_{\mathcal{C}K}(S, T)$ is a set for each pair of $\mathcal{C}K$-objects $S$ and $T$ follows from the fact that $\text{Hom}_{\mathcal{C}K}(S, T) \subseteq T^S$.

Let $(S, m)$ be a semigroup. The dual of $(S, m)$ is the semigroup $(S, n)$, where $n(x, y) = m(y, x)$ for all $(x, y) \in S \times S$. Let $\bar{S}$ denote the dual semigroup of $S$. The dual functor from $\mathcal{C}K$ to $\mathcal{C}K$ is defined by $D(S) = \bar{S}$ and if $\phi : S \to T$ then $D(\phi) : \bar{S} \to \bar{T}$.

It is trivial to show that the dual functor $D : \mathcal{C}K : \to \mathcal{C}K$ defined above is a self inverse isomorphism. Many theorems have hypothesis and or conclusions dealing with left or right identities, zeros, ideals etc. [15]

If $S$ is a monoid, then $S$ may be identified with the category $\mathcal{I}$ whose single object is $S$ and whose morphisms are the elements of $S$. That is $\text{Obj}(\mathcal{I}) = \{S\}$ and $\text{Mor}(\mathcal{I}) = S$. For $x, y \in \text{Mor} \mathcal{I}$, $x \circ y = xy$. Conversely, if $\mathcal{I}$ is a category with exactly one object $A$, then $\text{Mor} \mathcal{I}$ is a monoid relative to composition.

**Definition 3.1.1.** The $\mathcal{C}K$-sections are precisely those $\mathcal{C}K$-morphisms $\phi : S \to T$ such that $\phi$ is a topological isomorphism onto $\phi(S)$ and $\phi(S)$ is a homomorphic retract of $T$. That is there exists a $\mathcal{C}K$-morphism $\gamma : T \to T$ such that $\gamma \circ \gamma = \gamma$ and $\gamma(T) = \phi(T)$.

**Definition 3.1.2.** The $\mathcal{C}K$-Retractions are precisely those $\mathcal{C}K$-morphisms $\phi : S \to T$ such that there exists a homomorphic retraction $\gamma : S \to T$ and a $\mathcal{C}K$-isomorphism $\delta : \gamma(S) \to T$ such
that $\phi = \delta \circ \gamma$. Hence, the $C\mathcal{H}$-retractions are up to $C\mathcal{H}$-morphisms, homomorphic retractions.

**Definition 3.1.3.** For each topological categories $C\mathcal{H}$-*isomorphisms* are precisely the topological isomorphisms (That is continuous homomorphisms which preserve multiplication).

In (1966) Hofmann and Mostert [35], established that the compact semigroup monomorphisms (that is the $C\mathcal{K}$-monomorphisms) are precisely the injective continuous homomorphisms of $C\mathcal{K}$ in any semigroup, (abelian group, semilattices, Lawson, inverse) then the $C\mathcal{K}$-morphisms are precisely the injective $C\mathcal{K}$-morphisms. The technique is to find a universal $C\mathcal{K}$-object generated by $N$, that is a $C\mathcal{K}$-object $F(N)$ and a homomorphism $\Phi : N \to F(N)$ (not necessarily a $C\mathcal{K}$-morphism, as $N$ need not be a $C\mathcal{K}$-object) such that for any $C\mathcal{K}$-object $S$ and any homomorphism $\gamma : N \to S$, there exists a unique $C\mathcal{K}$-morphism $\delta : F(N) \to S$ such that the diagram commutes.

![Diagram](image)

Once this is accomplished $\Psi : S \to T$ is a non-injective $C\mathcal{K}$-morphism and $\Psi (x) = \Psi (y)$ with $x \neq y$, then one obtains $\gamma_2$ from $N$ into $S$ by letting $\gamma_2(n) = x^n$ and $\gamma_2(n) = y^n$ for all $n \in N$, $\delta_1, \delta_2$ from the properties of $(F(N), \delta)$, $\Psi \circ \delta_1 = \Psi \circ \delta_2$ and $\delta_1 \neq \delta_2$. Hence $\Psi$ is not a $C\mathcal{K}$-monomorphism.

It is clear that each injective $C\mathcal{K}$-morphism is a $C\mathcal{K}$-monomorphism. Slight modifications of this technique yield that the $C\mathcal{K}$-monomorphisms are precisely the injective $C\mathcal{K}$-morphisms. This result also holds for category of discrete
(Topological) monoid, (Compact) monoid, locally compact monoid, Regular semigroups, uniquely divisible semigroups, -UD(TUD)(CUD) and the category of finite semigroups, but the techniques of proofs are different. This result fails to hold for the category of discrete (Topological) (Compact) Divisible semigroups.

Then natural map \( \Phi: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}/\{-1,1\} \) is a monomorphism in each of these categories as well as in the categories of connected (abelian) semigroups and compact connected (abelian) monoids.

3.2 Category of ideal* semigroups

**Definition 3.2.1.** In the non-topological and compact categories any surjective \( \mathcal{C} \mathcal{K} \)-morphism is a \( \mathcal{C} \mathcal{K} \)-epimorphism and in the topological categories any dense \( \mathcal{C} \mathcal{K} \)-morphism is a \( \mathcal{C} \mathcal{K} \)-epimorphism. The converse need not generally hold.

a. Let \( S \) be a semigroup of non negative integers under addition and let \( \phi: S \rightarrow \mathbb{Z} \) be the including homomorphism. Then \( \phi \) is not dense and easy to verify that \( \phi \) is a \( \mathcal{C} \mathcal{K} \)-epimorphism if \( \mathcal{C} \mathcal{K} \)-object belongs to the category of discrete semigroups, category of topological semigroups, category of discrete abelian semigroups, category of topological abelian semigroups, category of discrete monoids and category of topological monoids.

b. In category of discrete abelian group, epimorphisms are surjective. [Herrlich H. and Strecker G.E.(1973)[31]

c. \( \mathcal{C} \mathcal{K} \)-epimorphisms are surjective if \( \mathcal{C} \mathcal{K} \)-objects are belongs to the category of the category of discrete semilattices, category of

**Definition 3.2.2.** A topological semigroup $S$ is said to be an ideal* semigroup if all its non empty subsets are ideals of $S$.

**Example 3.2.3.** Let $S$ be a topological semigroup and $S= \{x_1, x_2, x_3, x_4\}$ for all $x_i \in S$.

\[
\begin{array}{cccc}
\cdot & X_1 & X_2 & X_3 & X_4 \\
X_1 & X_1 & X_1 & X_1 & X_1 \\
X_2 & X_2 & X_2 & X_2 & X_2 \\
X_3 & X_3 & X_3 & X_3 & X_3 \\
X_4 & X_4 & X_4 & X_4 & X_4 \\
\end{array}
\]

In this example we observe that each non empty subsets of $S$ are ideals in $S$. Then $S$ is a topological ideal*semigroup. Next we consider the categories of all topological ideal*semigroups with morphisms are continuous homomorphisms. Then we establish the following results in the category of all topological ideal*semigroups.

We consider the category of all topological ideal*semigroups and proved that epi-morphisms in the category of all topological ideal*semigroups are morphisms with dense range.

**Theorem 3.2.4.** The epimorphisms in the category of all ideal *semigroups (TIS) are morphisms with dense range.
**Proof:** Let $S$ and $T$ are ideal *semigroups and $f: S \to T$ be a continuous homomorphism such that $\overline{f(S)} = T$. Then $f$ is an epimorphism. For, if the equality $g \circ f = h \circ f$, we have to show that $g(x) = h(x)$ for all $x \in T$. If not assume that $g(x) \neq h(x)$ for at least one $x \in T/\mathfrak{f}(x)$.

Since $R$ is an ideal *semigroup, $R$ is a Hausdorff space. By the definition of Hausdorff axiom there exist disjoint open sets $U$ and $V$ containing $g(x)$ and $h(x)$ respectively. Choose a neighborhood $W$ of $x$ such that $g(W) \subseteq U$ and $h(W) \subseteq V$. This is possible since $g$, $h$ are continuous maps.

Since $x \in T = \overline{f(S)}$, $W$ intersects $f(x)$ in some point (say $y$) other than $x$. Then $g(y) \in U$, $h(y) \in V$. But $g(y) = h(y)$ (since $y \in f(x)$). That is $U \cap V \neq \emptyset$. This is not possible. This contradicts the fact that $U$ and $V$ are disjoint. Hence $g(x) = h(x)$ for all $x \in T$. That is $g = h$, therefore $f$ is an epimorphism.

Conversely, suppose $f: S \to T$ be an epimorphism, then $f(S)$ is dense in $T$. Given $f: S \to T$ be an epimorphism. Since $S$ is a ideal *semigroup, $f(S)$ is the continuous image of an ideal *semigroup which is an ideal in $T$.

Then we show that $\overline{f(S)} = T$. For this, assume that $\overline{f(S)} \neq T$. then we will show that $f$ cannot be an epimorphism. By constructing an ideal * semigroup $R$ and two continuous homomorphisms $p_1$ and $p_2$ from $T$ to $R$ which agree on $f(S)$.
Let $T_1 = T \times \{1\}$, $T_2 = T \times \{2\}$ where $T_1, T_2$ are ideal semigroups.

With product topology and multiplication defined by

$$(x_i, y_i) = (xy, i) \text{ for } i = 1, 2.$$  

Let $h_i : T \to T_i$ defined by $h_i(y) = (y, i)$ for $i = 1, 2$. 

Then clearly $h_1, h_2$ are topological isomorphisms.

Then $T_1 \cup T_2 = \{(x, i) : (x, i) \in Y_1 \text{ or } Y_2\}$ for each $i = 1, 2$.

Clearly $T_1 \cup T_2$ is a topological ideal semigroup with multiplication defined by $(x, i)(y, i) = (xy, \min\{i, j\})$. Multiplication is well defined, for, if $(x, i) = (x', i)$ then $x = x'$ and $i = j$.

If $(y, i) = (y', i)$ then $y = y'$ and $i = j$. Therefore $xy = (x'y', i=j)$

Now $(x, i)(y, j) = (xy, \min\{i, j\}) = (x'y', \min\{i, j\}) = (x', i)(y', i)$.

Also we can observe that multiplication is clearly associative and continuous. Again every non singleton subset of $T_1 \cup T_2$ is a topological ideal semigroup. Let $i_1 : T_1 \to T_1 \cup T_2$, $i_2 : T_2 \to T_1 \cup T_2$ be the inclusion maps. $(i_1 \circ h_1) (f(\mathcal{S})) \cup (i_2 \circ h_2) f(\mathcal{S})$ is the set of copies of $f(\mathcal{S})$ contained in $T_1 \cup T_2$. Let $R$ be the images of the quotient map $q$ obtained by identifying $(i_1 \circ h_1) y = i_1(y, 1), (i_2 \circ h_2)(y) = i_2(y, 2)$ if $y \in f(\mathcal{S})$. Define $q(x, i) \times q(y, i) = q(xy, \min\{i, j\})$. This multiplication is well defined.

For if $q(x, i) = q(x', j)$, then $x = x'$ and $i = j$ or $i \neq j$ and
If $q(y, j) = q(y', j)$ then either $y = y'$ or $i = j$ or $i \neq j$ and $y = y' \in f(S)$. Then there are four cases

Case 1: If $i = j$ and $x = x' \in f(S)$ then $(x, y) (x'y', i = j)$ then $q(xy, i) = q(x'y', j) = q(xy, i) = q(x'y', i)$, so $q(xy, \min \{i, j\})$.

That is $q(x, i) q(y, j) = q(x'y', j) q(y', i)$

Case 2: If $i \neq j$ and $y = y' \in f(S)$ then $x = x' \in f(S)$. $q(xy, i) = q(x'y', j) = q(xy, j) = q(x'y', i)$, so $q(x, i) q(y, j) = q(x'y', i) q(y', i)$

Case 3: If $i = j$ and $x = x' \in f(S)$ then $x'y' \in f(S)$. $q(xy, i) = q(x'y', i) = q(xy, j)$, so $q(x, i) q(y, j) = q(x'y', i) q(y', i)$

Case 4: If $i \neq j$ and $y = y' \in f(S)$ then $x = x' \in f(S)$ and $x'y' \in f(S)$. $q(xy, i) = q(x'y', i) = q(xy, j)$, so $q(x, i) q(y, j) = q(x'y', i) q(y', i)$

From all these we observe that multiplication is associative and $R$ is an ideal * semigroup. Again $q$ is a homomorphism, then $p_1 = q_1 \circ (i_1 \circ h_1)$ and $p_2 = q \circ (i_2 \circ h_2)$ are continuous homomorphism from $T$ to $R$.

Next show that $q \circ (i_1 \circ h_1) \neq q \circ (i_2 \circ h_2)$, that is $p_1 \neq p_2$. For if $x$ is a point in $S$, the maps $i_1 \circ h_1$ and $i_2 \circ h_2$ split the point $f(S)$ into two and is joined again by $q : T_1 \cup T_2 \rightarrow R$. Thus we see that $q \circ (i_1 \circ h_1) \circ f(x) = q \circ (i_2 \circ h_2) \circ f(x)$. Hence $q \circ (i_1 \circ h_1) \circ f = q \circ (i_2 \circ h_2) \circ f$.

However any point of $f$ lying outside $f(S)$ in $T$ is split by $i_1 \circ h_1$ and $i_2 \circ h_2$, but it is not joined by $q$.

Hence $q \circ (i_1 \circ h_1) \neq q \circ (i_2 \circ h_2)$ for a point $x \in T - f(S)$. This shows that $f$ cannot be an epimorphism. If we show that $R$ is an ideal * semigroup, for this it remains to show that $R$ is a Hausdorff
space. Let \( p \) and \( r \) be distinct points of \( R \). Then we have to find disjoint open sets containing \( p \) and \( r \).

**Case 1:** Let \( p, q \in T-\overline{f(S)} \). Since \( \overline{f(S)} \) is closed \( T-\overline{f(X)} \) is open. 

\[
(q \circ i_1 \circ h_1)^{-1}(p) = (q \circ (i_1 \circ h_1))^{-1}(r) \in T-\overline{f(S)}.
\]

There exist open sets \( U_p \) and \( U_r \) in \( T-\overline{f(S)} \). Again since \( T \) is Hausdorff we get disjoint neighbourhoods \( U_p \) and \( U_r \) of \( (q \circ i_1 \circ h_1)^{-1}(p) \) and \( (q \circ (i_1 \circ h_1))^{-1}(r) \) respectively. Thus the required neighbourhoods are 

\[
(q \circ i_1 \circ h_1)(U_p) \cap (q \circ (i_1 \circ h)(V_p)) \text{ and } (q \circ i_1 \circ h_1)(U_r) \cap (q \circ (i_1 \circ h_1)(V_r)).
\]

**Case 2:** If \( p, r \) belongs to \( (q \circ i_2 \circ h_2) (T-\overline{f(S)}) \). Here in different cases the elements belonging to \( \overline{f(S)} \) and \( T-\overline{f(S)} \). In all these cases disjoint neighborhoods have disjoint elements. Hence \( R \) is Hausdorff.

### 3.3. Reflective Subcategories.

**Definition 3.3.1.** Reflective subcategory means a subcategory which contains a "largest" model of any object of a given category. More precisely, a full subcategory \( \mathcal{C} \) of a category \( \mathcal{D} \) is called reflective if it contains a reflection (Reflection of an object of a category) for every object of \( \mathcal{D} \). Equivalently, \( \mathcal{C} \) is **reflective in** \( \mathcal{D} \) if and only if the inclusion functor \( F: \mathcal{C} \to \mathcal{D} \) has a left adjoint \( G: \mathcal{D} \to \mathcal{C} \). The functor \( G \) sends each object \( D \) of \( \mathcal{D} \) to its \( \mathcal{C} \)-reflection \( G(D) \); the morphisms \( \prod_D: D \to G(D) \) appearing in the definition of a reflection constitute a natural transformation from the identity functor on \( \mathcal{D} \) to the composite of \( G \) with the inclusion functor \( F \), which is the unit of the adjunction. The concept dual to that of a reflective subcategory is called a coreflective subcategory.
A reflective subcategory $\mathcal{C}$ inherits many properties from the ambient category $\mathcal{D}$. For example, a morphism $\mu$ of $\mathcal{C}$ is a monomorphism in $\mathcal{C}$ if and only if it is a monomorphism in $\mathcal{D}$. Therefore, every reflective subcategory of a well-powered category is well-powered. A reflective subcategory is closed under products, to the extent that they exist in the ambient category. The same holds for more general limits. A reflective subcategory need not be closed under colimits, but the functor $G$ transforms colimits in $\mathcal{D}$ into colimits in $\mathcal{C}$. Thus, a reflective subcategory of a complete (cocomplete) category is complete (cocomplete).

Suppose $\mathcal{D}$ is complete and has a bicategory (factorization) structure in which every object has only a set of admissible quotients. Then every full subcategory $\mathcal{C}$ of $\mathcal{D}$ which is closed under products and admissible subobjects is reflective. In this context, one may construct the $\mathcal{C}$-reflection of an object $D$ of $\mathcal{D}$ as follows: Choose a set of representatives $\gamma_i : D \to D_i$, $i \in I$, of those quotient objects of $D$ which lie in $\mathcal{C}$. The product $P = \prod_{i \in I} D_i$ belongs to $\mathcal{C}$, and the $\mathcal{C}$-reflection $G(D)$ is the image of the unique morphism $\gamma : D \to P$ such that $\prod_{i \in I} \gamma = \gamma_i$, $i \in I$.

A reflective subcategory is called epireflective if the canonical morphism $\prod_D : D \to G(D)$ is an epimorphism for every $D$. If every morphism in $\mathcal{D}$ factors as an epimorphism followed by a monomorphism, then a reflective subcategory of $\mathcal{D}$ will be epireflective provided it is closed under arbitrary subobjects in $\mathcal{D}$. The category of Abelian groups is epireflective in the category of all groups. The dual concept is that of a monocoreflective subcategory;
for example, the category of torsion Abelian groups is monocoreflective in the category of all Abelian groups.

Examples 3.3.2

- The category of fields is a reflective subcategory of the category of integral domains. The reflector is the functor which sends each integral domain to its field of fractions.

- The category of vector spaces over the field $K$ is a (non full) reflective subcategory of the category of sets. The reflector is the functor which sends each set $B$ in the free vector space generated by $B$ over $K$ that can be identified with the vector space of all $K$ valued functions on $B$ vanishing outside a finite set. In similar way, several free construction functors are reflectors of the category of sets onto the corresponding reflective subcategory.

- The category of completely regular spaces $CReg$ is a reflective subcategory of $Top$. The subcategory of Tychonoff spaces is also reflective.

- The full subcategory of compact Hausdorff topological spaces is reflective in the category of completely regular topological spaces. The Stone–Čech compactification provides the reflector.

- The category of all compact Hausdorff spaces is a reflective subcategory of the category of all Tychonoff spaces. The reflector is given by the Stone–Čech Compactification.

- The category of all complete metric spaces with uniformly continuous mappings is a reflective and full subcategory of the
category of metric spaces. The reflector is the completion of a metric space on objects, and the extension by density on arrows.

- The category of Hilbert spaces and bounded linear operators is a reflective subcategory of the category of sets; the reflector is the functor which sends a set $S$ into the free Hilbert space generated by $S$, which is the Lebesque space of all -valued functions on $S$ that are square-integrable with respect to the counting measure, that is, square-summable.

- The category of Banach spaces is a reflective and full subcategory of the category of normed spaces and bounded linear operators. The reflector is the norm completion functor.

**Definition 3.3.3.** A functor $r$ from a category $\mathcal{C}$ to a subcategory $\mathcal{R}$ of $\mathcal{C}$ is a **Reflective Functor** if there is a morphism $\eta_c: \mathcal{C} \rightarrow r\mathcal{C}$ and every morphism from $\mathcal{C}$ to an object $R$ of $\mathcal{R}$ factors through $r\mathcal{C}$ via $\eta_c$ so that the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{C} & \rightarrow & r\mathcal{C} \\
\downarrow r & & \downarrow h \\
\mathcal{R} & \downarrow & \\
\end{array}
\]

If $r: \mathcal{C} \rightarrow \mathcal{R}$ is a reflective functor, the subcategory $\mathcal{R}$ is called a reflective subcategory. The object $r\mathcal{C}$ is called the reflection of $C$ in $\mathcal{R}$.

**Proposition 3.3.4.** In the category of topological semigroups $\eta\mathcal{C}$, the reflection of an object is unique up to topological isomorphism.
Proof :— Let \( \mathcal{R} \) be a reflective subcategory of \( \eta \mathcal{C} \). Let \( rX \) be the reflection of an object \( X \). Let \( Y \) be any object of \( \mathcal{R} \) and let \( f : X \to Y \) be a continuous homomorphism such that any continuous homomorphism from \( X \) to an object of \( \mathcal{R} \) factors uniquely through \( Y \) (via \( f \)). Then we have the following commutative diagram in which the morphisms \( h \) and \( g \) exist and are unique by the reflective properties of \( Y \) and \( rX \) respectively. Because \( rX \) is an object of \( \mathcal{R} \), the continuous homomorphism \( g \) must factor uniquely through \( \eta_x \) one such factorization is \( I_r \) and the diagram shows that \( h \circ g \) is another factorization. Thus we must have \( h \circ g = I_{rx} \), since there is only one factorization.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & rX \\
| & \swarrow & \downarrow h \\
\eta_x & & \eta_x \\
| & \searrow & \\
rX & \xrightarrow{g} & Y \\
\end{array}
\]

\( g \circ \eta_x = f \) and \( h \circ f = \eta_x \)

\( h \circ (g \circ \eta_x) = \eta_x \)

\( (h \circ g) \circ \eta_x = \eta_x \).

Also \( I_{rx} \circ \eta_x = \eta_c \) then \( h \circ g = I_{rx} \).

\( h : Y \to rX \) is a continuous homomorphism.

\( g = h^{-1} : rX \to Y \) is a continuous homomorphism \( h \circ g = I_{rx} \) is an algebraic isomorphism and a homeomorphism. Hence the reflection of an object is unique up to topological isomorphism. Repeating the argument with \( rX \) and \( Y \) interchanged then \( g \circ h = I_y \) and \( g, \) and

65
topological isomorphisms between rX and Y. Hence rX and Y are toliqually isomorphic.

**Universal Problem 3.3.5.** Since \( \mathcal{C} \) is a subcategory of \( \mathcal{D} \), the universal problem corresponding to a reflective subcategory is easily represented. Let \( C \in \mathcal{C} \) and \( D \in \mathcal{D} \). There exists a morphism \( f: D \to R\mathcal{D} \) in \( \mathcal{D} \) induced by the natural transformation \( \text{Id}_\mathcal{D} \to \mathcal{E}: \mathcal{R} \). If \( g: D \to C \) is another morphism in \( \mathcal{D} \) then there exists exactly one morphism \( h \) in the subcategory \( \mathcal{C} \) which make the diagram commutative. That is \( h \circ f = g \)

![Diagram](image)

**Lemma 3.3.6.** Let \( \mathcal{C} \) be a full, reflective subcategory of the category of topological semigroup \( \mathcal{D} \) with reflector \( \mathcal{R} \). Then the restriction of \( \mathcal{R} \) to the subcategory \( \mathcal{C} \) is topologically isomorphic to \( \text{Id}_\mathcal{C} \).

**Proof:** Since \( \mathcal{C} \) is a full subcategory, we get for each \( C \in \mathcal{C} \) that the continuous homorphism \( I_C: C \to C \) is a universal solution for the universal problem defined by \( \mathcal{E}: \mathcal{C} \to \mathcal{D} \). By the uniqueness of the universal solution \( \mathcal{R} \circ C = C \) is natural in \( C \) for all \( C \in \mathcal{C} \).

In the case of reflective subcategory we have a simple presentation of the universal problem defined by the adjoint functors in semigroup theory, thus it is interesting to know when a pair of adjoint
functors induces a reflective subcategory. The following theorem gives a sufficient condition.

**Theorem 3.3.7.** Let the functor \( F : \mathcal{C} \to \mathcal{D} \) be left adjoint to the functor \( G : \mathcal{D} \to \mathcal{C} \) and let \( G \) be injective homomorphism on the objects. Then \( G(\mathcal{D}) \) is a reflective subcategory of \( \mathcal{C} \) with reflector \( G^\prime \).

**Proof:** The image of \( G \) is a subcategory of \( \mathcal{C} \). We define factorizations of the functors by the following commutative diagram of categories.

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow{G} & & \downarrow{G} \\
\mathcal{E} & \xleftarrow{F^\prime} & \mathcal{C}
\end{array}
\]

Where \( \mathcal{C}^\prime = G(\mathcal{D}) \). We know that \( G : \text{Mor}_{\mathcal{D}}(\cdot, \cdot) \to \text{Mor}_{\mathcal{C}^\prime}(\cdot, \cdot) \) is an injective homomorphism. Thus, \( G : \text{Mor}_{\mathcal{D}}(\cdot, \cdot) \to \text{Mor}_{\mathcal{C}^\prime}(\cdot, \cdot) \) is a natural topological isomorphism by the definition of \( \mathcal{C}^\prime \). We get \( \text{Mor}_{\mathcal{C}^\prime}(\cdot, \cdot) \equiv \text{Mor}_{\mathcal{D}}(\cdot, \cdot) \equiv \text{Mor}(\cdot, G(\cdot)) \equiv \text{Mor}(\cdot, \mathcal{E} \cdot) \).

Since each object in \( \mathcal{C}^\prime \) may uniquely be represented as \( G^\prime \mathcal{D} \), and since \( G^\prime \) is full, we get \( \text{Mor}(\cdot, G^\prime \mathcal{D} \cdot) \equiv \text{Mor}(\cdot, \mathcal{E} \cdot). \) \( F^\prime \) and \( G^\prime \) coincide up to the topological embedding of \( \mathcal{C}^\prime \) into \( \mathcal{C} \).

**Proposition 3.3.8.** Let \( \mathcal{C}^\prime \) be a reflective subcategory of \( \mathcal{C} \) with reflector \( \mathcal{R} \). For all \( A \in \mathcal{C}^\prime \) the morphism \( f : A \to \mathcal{R}A \) defined by the corresponding universal problem is a section in \( \mathcal{C} \).
Proof: - Let $\mathcal{E} : \mathcal{E} \rightarrow \mathcal{C}$ be the topological embedding. We have $(\mathcal{E} \xrightarrow{\phi \xi} \mathcal{E} \mathcal{E} \xrightarrow{\gamma \psi} \mathcal{E} ) = \text{id}_\mathcal{E}$, thus $(A \rightarrow \mathcal{R} A \xrightarrow{\xi \psi A} A) = \text{id}_A$ for all $A \in \mathcal{C}$. Observe that $f$ is a continuous homomorphism in $\mathcal{E}$, whereas $\xi \psi A$ is even in $\mathcal{C}$.

**Theorem 3.3.9.** Let $\mathcal{E} : \mathcal{C} \rightarrow \mathcal{D}$ be a full, reflective subcategory. If for each $C \in \mathcal{C}$ also each $D \in \mathcal{D}$ with $C = D$ in $\mathcal{D}$ is an object in $\mathcal{C}$, then $\mathcal{E}$ is a monadic functor.

**Proof: -** Let $\mathcal{H} = \mathcal{E} \mathcal{R}$ and $\mathcal{R}$ be the reflector to $\mathcal{E}$, then $\varepsilon(D) : D \rightarrow \mathcal{E} \mathcal{R} D$ is the universal solution of the universal problem defined by $\mathcal{E}$. Let $\delta : \mathcal{H} D \rightarrow D$ be a $\mathcal{D}$ morphism, such that the following figure:

\[
\begin{array}{ccc}
D & \xrightarrow{\varepsilon(D)} & \mathcal{H} D \\
\downarrow \varepsilon(D) & & \downarrow \delta \\
\mathcal{H} D & \xrightarrow{1_D} & D
\end{array}
\]

is commutative. Then $\varepsilon(D) \delta \varepsilon(D) = \varepsilon(D)$. Since $\mathcal{E}$ is full, we get $\varepsilon(D) \delta = \varepsilon(f)$ with $f : \mathcal{R} D \rightarrow \mathcal{R} D$. By the universal property of $\varepsilon(D)$ and the commutativity of the figure,
we get \( f = I_{\mathcal{R}D}, \) thus \( \varepsilon(D) \delta = I_{\mathcal{R}D}. \) This proves that \( \varepsilon \circ (D) \rightarrow \mathcal{H}D \) is an isomorphism and \( D \in \mathcal{C}. \) Furthermore, because
\[
(\Psi_{\mathcal{R}D}) (\mathcal{R} \varepsilon(D)) = I_{\mathcal{R}D} = (\mathcal{R} \delta)(\mathcal{R} \varepsilon(D)),
\]
we also have \( \Psi \mathcal{R}D = \mathcal{R} \delta \) thus \( \mu(D) = \mathcal{H} \delta. \) This implies that the following figure is commutative, and \((D, \delta)\) is an \( \mathcal{H} \) algebra. If \( D \in \mathcal{C}, \) then there exists exactly one \( \delta : \mathcal{H}D \rightarrow D \) with \( \delta \varepsilon(D) = I_D, \) because \( \varepsilon(D) \) is a universal solution. Let \( f : D \rightarrow D^1 \) be a morphism and \( D, D^1 \in \mathcal{C}. \) Let \((D, \delta)\) and \((D^1, \delta^1)\) be the corresponding \( \mathcal{H} \) algebras. Then the following figure is commutative, thus \( f \) is a morphism of \( \mathcal{H} \) algebras. Hence \( \mathcal{L} : \mathcal{C} \rightarrow \mathcal{D}_\mathcal{H} \) is an isomorphism of categories.