CHAPTER II

ADJUNCTION SEMIGROUP

COMPACTIFICATION

In this chapter study has been done on Adjunction Bohr Compactification of adjunction semigroups and investigated some results on Quotient Adjunction Bohr Compactification. Also studied Product Semigroups Compactification, and some properties of Projective System of Adjunction Semigroups.

2.1. Adjunction Semigroups

Definition 2.1.1. If \( f: X \rightarrow Y \) is a surjection, we can define topology on \( Y \) by declaring a subset \( W \) of \( Y \) to be open if and only if \( f^{-1}(W) \) is open in \( X \). This topology is called the quotient topology on \( Y \) induced by \( f \).

Lemma 2.1.2. Let \( S \) be a topological semigroup and \( R \) be a closed congruence on \( S \) such that \( \Pi \times \Pi : S \times S \rightarrow S/R \times S/R \) is a quotient map, then \( S/R \) is a topological semigroup.

Theorem 2.1.3. Let \( S \) be a compact semigroup and let \( R \) be a closed congruence on \( S \), then \( S/R \) is a compact semigroup.

Definition 2.1.4. If \( X \) and \( Y \) are two disjoint topological spaces, then we have \( X \cup Y \) the topology which is soherent with that of \( X \) and \( Y \), that is a subset \( U \) of \( X \cup Y \) is open if and only if \( U \cap X \) is open in \( X \) and \( U \cap Y \) is open in \( Y \).
If $X$ and $Y$ are both compact then $X \cup Y$ is compact.

**Definition 2.1.5.** If $S$ and $T$ are disjoint topological semigroups and $\Phi: S \to T$ is a continuous homomorphism, then we define multiplication on $S \cup T$ by

$$(x, y) \rightarrow \begin{cases} 
    m_S(x, y) & \text{if } x, y \in S \\
    m_T(x, y) & \text{if } x, y \in T \\
    m_T(\Phi(x), y) & \text{if } x \in S \text{ and } y \in T \\
    m_T(x, \Phi(y)) & \text{if } x \in T \text{ and } y \in S
\end{cases}$$

Where $m_S$ and $m_T$ are the multiplications on $S$ and $T$ respectively.

We denote $S \cup T$ with this multiplication by $S \cup T_\Phi$ is called the **Adjunction Semigroup of $S$ and $T$ relative to $\Phi$.**

**Theorem 2.1.6.** If $S$ and $T$ are disjoint topological semigroups and let $\Phi: S \to T$ be a continuous homomorphism, then $S \cup T_\Phi$ is a topological semigroup.

**Proof:** Since $S$ and $T$ are topological semigroups and let $\Phi: S \to T$ be a continuous homomorphism, then $S \cup T_\Phi$ is a topological semigroup with respect to the following multiplication

$$(x, y) \rightarrow \begin{cases} 
    m_S(x, y) & \text{if } x, y \in S \\
    m_T(x, y) & \text{if } x, y \in T \\
    m_T(\Phi(x), y) & \text{if } x \in S \text{ and } y \in T \\
    m_T(x, \Phi(y)) & \text{if } x \in T \text{ and } y \in S
\end{cases}$$

Where $m_S$ and $m_T$ are the multiplications on $S$ and $T$ respectively.
2.2. Adjunction Semigroup Compactification.

In the theory of topological semigroups, it is observed that associated with each topological semigroup S, there is a compact semigroup called the Bohr compactification of S which is universal over the compact semigroup containing dense continuous homomorphic images of S.

In the following theorem we consider two topological semigroups with Bohr compactification and studied adjunction Bohr compactification of adjunction semigroups.

**Theorem 2.2.1.** Let S be a topological semigroup with Bohr compactification \((\beta, B)\) and T be a topological semigroup with Bohr compactification \((\gamma, C)\) then there exist a Bohr compactification named adjunction Bohr compactification \(B \cup C\) of \(S \cup T\) where \(\psi : B \rightarrow C\) and \(\Phi : S \rightarrow T\) are continuous homomorphism respectively

**Proof** - Since \((\beta, B)\) be a Bohr compactification of a topological semigroup S, \((\gamma, C)\) be a Bohr compactification of a topological semigroup T, and \(\Phi : S \rightarrow T\) is a continuous homomorphism then there exist a continuous homomorphism \(\gamma : T \rightarrow C\) such that \(\gamma \circ \Phi : S \rightarrow C\) is a continuous homomorphism where C is a compact semigroup. Then by the definition of Bohr compactification of \((\beta, B)\) there exists a unique continuous homomorphism \(\psi : B \rightarrow C\) such that the diagram commutes. That is \(\psi \circ \beta = \gamma \circ \Phi\)
Since $B$ and $C$ are compact and $U$ a subset of $B \cup C$ is open if and only if $U \cap B$ is open in $B$ and $U \cap C$ is open in $C$. Also $B \cup C$ is compact, since $\psi : B \rightarrow C$ is a continuous homomorphism we define multiplication in $B \cup C$ by

$$m_B(x, y) \quad \text{if } x, y \in B$$

$$m_C(x, y) \quad \text{if } x, y \in C$$

$$(x, y) \rightarrow \begin{cases} 
  m_C(\psi(x), y) & \text{if } x \in B \text{ and } y \in C \\
  m_C(x, \psi(y)) & \text{if } x \in C \text{ and } y \in B
\end{cases}$$

Where $m_B$ and $m_C$ are multiplications on $B$ and $C$ respectively.

We denote it by $B \cup_C \psi$.

Since $S$ and $T$ are topological semigroups and let $\Phi : S \rightarrow T$ be a continuous homomorphism, then $S \cup_T \Phi$ is a topological semigroup by theorem [2.1.6]. Define a continuous homomorphism $h : S \cup_T \Phi \rightarrow B \cup C$ by

$$h(x, y) = \begin{cases} 
  \beta(xy) & \text{if } x, y \in S \\
  \gamma(xy) & \text{if } x, y \in T \\
  \gamma(\Phi(x)y) & \text{if } x \in S \text{ and } y \in T \\
  \gamma(x\Phi(y)) & \text{if } x \in T \text{ and } y \in S
\end{cases}$$

Where $\beta$ and $\gamma$ are multiplications on $S$ and $T$, respectively.
Clearly \( h \) is a dense continuous homomorphism. If \( g : S \cup T \rightarrow K \) is a continuous homomorphism from \( S \cup T \) into a compact semigroup \( K \), then by the definition of \( h \) and since \( K \) is a Hausdorff space there exists a unique continuous homomorphism \( f : B \cup C \rightarrow K \) such that \( f \circ h = g \). Hence the theorem.

Next theorem we consider adjunction semigroup of two disjoint topological semigroups with disjoint Bohr compactification and studied **adjunction semigroup compactification**.

**Theorem 2.2.2.** Let \( S \) and \( T \) be two disjoint topological semigroups with Bohr compactification \( B \) and \( C \) respectively. \( \Phi : S \rightarrow T \) be a continuous homomorphism, \( I \) a closed ideal of \( S \) and \( R \) the congruence on \( B \cup C \) generated by \( \{(x, \psi(x)) : x \in I \} \) where \( \psi : B \rightarrow C \) is a continuous homomorphism then \( B \cup C \mod R \) is a compact semigroup and the restriction of the natural map \( \Pi : B \cup C \rightarrow B \cup C \mod R \) is a topological embedding of \( C \) into \( B \cup C \mod R \). More over there exists an adjunction semigroup compactification \( (\eta, B \cup C \mod R) \) of \( S \cup T \).

**Proof** – Since \( S \) and \( T \) are topological semigroups with Bohr compactification \( B \) and \( C \) by theorem [2.2.1], there exists a Bohr compactification \( B \cup C \) of \( S \cup T \). Since \( I \) is a closed ideal of \( S \), we can observe that
Each set in this union is closed and so $R$ is closed. Also $B \cup C / R$ is a topological semigroup by Theorem [2.1.2].

Also we define $\eta : S \cup T \rightarrow B \cup C / R$ to $C$. Then $\eta = \Pi \circ h$.

Clearly $\eta$ is well defined and it is a dense continuous homomorphism hence $(\eta, B \cup C / R)$ determines an adjunction semigroup compactification of $S \cup T$. Also the restriction of $\Pi$ to $C$ is clearly a continuous isomorphism. Again $\Pi / C$ is open, let $U$ be an open subset of $C$ then $\Pi^{-1}(\Pi(U)) \cap C = U$ since $R \cap (C \times C) = \Delta(C)$. In view of the topology of $B \cup C$ and the fact that $\Pi$ is quotient, then $\Pi(U)$ is open in $\Pi(C)$.

Then $\Pi : B \cup C \rightarrow B \cup C / R$ to $C$ is a topological embedding of $C$ into $B \cup C / R$.

### 2.3 Product Semigroup Compactification.

One of the most useful tools in computing Bohr compactification is the product theorem. In proving the product
Theorem on semigroup compactification we need the following lemma.

**Lemma 2.3.1:** Let $S$ be a compact semigroup with identity $I$ and let $a$ and $b$ be distinct points of $S$. Then there exist open sets $M$, $N$ and $V$ in $S$ with $a \in M$, $b \in N$, $I \in V$, and $M \cap N = \emptyset$ such that for $p \in S$, there exists an open set $G_p$ containing $p$ so that either $G_p \cap M = \emptyset$ or $G_p \cap N = \emptyset$.

**Proof:** Let $M$ and $N$ be open sets in a compact semigroup $S$ such that $a \in M$, $b \in N$ and $M \cap N = \emptyset$. Then $\Delta \subset U \subset V$, where $V = (S \times S) / (M \times N)$, for some $\Delta$-ideal $U$. Let $V$ be an open set such that $1 \in V$ and $\overline{V} \times \overline{V} \subset U$. Now for $p \in S$, we have $(p, p)(\overline{V} \times \overline{V}) \subset U$, since $U$ is a $\Delta$-ideal. Hence there exists an open set $G_p$ containing $p$ with $(G_p \times G_p)(\overline{V} \times \overline{V}) \subset U \subset V$. It follows that either $G_p \cap M = \emptyset$ or $G_p \cap N = \emptyset$.

**Theorem 2.3.2.** Let $\{S_{\alpha}\}_{\alpha \in \Lambda}$ be a collection of topological semigroups with semigroup compactification $\{(f_{\alpha}, A_{\alpha})\}$ for each $\alpha \in \Lambda$. Define $F: \Pi\{S_{\alpha}\}_{\alpha \in \Lambda} \to \Pi\{A_{\alpha}\}_{\alpha \in \Lambda}$ by $F(x)_{\alpha} = f_{\alpha}p_{\alpha}(x)$ where $p_{\alpha}: \Pi(S_{\alpha})_{\alpha \in \Lambda} \to S_{\alpha}$ is a projection for each $\alpha \in \Lambda$. Then $(F, \Pi\{A_{\alpha}\}_{\alpha \in \Lambda})$ is a semigroup compactification of $\Pi\{S_{\alpha}\}_{\alpha \in \Lambda}$.

**Proof:** Let $S = \Pi\{S_{\alpha}\}_{\alpha \in \Lambda}$ and $A^* = \Pi\{A_{\alpha}\}_{\alpha \in \Lambda}$. Define $F: S \to A^*$ by $F(x)_{\alpha} = f_{\alpha}p_{\alpha}(x)$. It is known that $A^*$, being the Cartesian product of a family of compact semigroups with coordinate wise multiplication and Tychonoff topology, is a compact semigroup [6]. A straight
forward argument shows that $F$ is a dense continuous homomorphism. Again $F$ is dense ($F(S) = A^*$) for, let $y = \{y_\alpha\}$, $\alpha \in \Lambda$ be any element of $A^*$ where $y_\alpha \in A_\alpha$, $\alpha \in \Lambda$. To show that $A^*$ has a net in $F(S)$ converging to $y$. Since $\int_\alpha (S_\alpha) = A_\alpha$ and $f_\alpha(x_\alpha) \in A_\alpha$ for each $\alpha \in \Lambda$, $A_\alpha$ has a net in $f_\alpha(S_\alpha)$ converging to $y_\alpha$, $\alpha \in \Lambda$. That is for $i \in I_\alpha$, $\alpha \in \Lambda$, there exists $(x_\alpha) \in S_\alpha$ such that $y_\alpha(x_\alpha) \in I_\alpha \to y_\alpha$ for each $\alpha \in \Lambda$, where $(I_\alpha, < \alpha)$ is a directed set for each $\alpha \in \Lambda$. Then $(\Pi[I_\alpha : \alpha \in \Lambda], \leq)$ is a product directed set by defining $i \leq j \iff i_\alpha \leq j_\alpha$ that is $(i(\alpha) \leq j(\alpha))$ for each $\alpha \in \Lambda$.

Also we have $A \times \Pi[I_\alpha : \alpha \in \Lambda]$ is a directed set by defining $(\alpha, i) \leq (\alpha', j)$ and $\leftrightarrow \alpha \leq \alpha'$ and $i_\alpha \leq j_\alpha$ for every $\alpha \in \Lambda$, and $[y_\alpha(x_\alpha)] : \alpha \in A$ is a net in $F(S)$ such that $[y_\alpha(x_\alpha)] \to (y_\alpha)$, $\alpha \in \Lambda \Rightarrow y$. Therefore $F(S)$ contains a net converging to $y$. That is $F(S) = A^*$. Therefore $(F, A^*) = (F, \Pi \{A_\alpha\})$, $\alpha \in \Lambda$ is a semigroup compactification of $\Pi[S_\alpha]$, $\alpha \in \Lambda$.

**Theorem 2.3.3.** Let $[(\eta_\alpha, A_\alpha) : \phi^\beta_\alpha]$, $\alpha \leq \beta \in D$ be a projective system of semigroup compactification of a topological semigroup $S$, Where $\eta_\alpha = \phi^\beta_\alpha \eta_\beta$ for every pair $\alpha \leq \beta$ in a directed set $D$. Then $\lim (\eta_\alpha, A_\alpha)$, itself is a semigroup Compactification of $S$.

**Proof:** By the definition of semigroup compactification $\eta_\alpha : S \to A_\alpha$ is a dense continuous homomorphism for each $\alpha \in D$ and $A_\alpha$ is a compact semigroup. Moreover each bonding map $\phi^\beta_\alpha$ is surjective.

When $\alpha \leq \beta$, $A_\alpha = \eta_\alpha(S) = \phi^\beta_\alpha \eta_\beta(S)$
\[ \phi_\alpha^\beta \eta_\beta(S) = \phi_\alpha^\beta (A_\beta). \]

(since \( \phi_\alpha^\beta \) is continuous and closed, being a continuous map from compact semigroup is Hausdorff space). Hence the system 
\((A_\alpha, \phi_\alpha^\beta) \alpha \leq \beta \) is a strict projective system of compact semigroups \( \phi_\alpha^\beta \).

Then \( A^* = \lim_{\alpha \to \infty} [A_\alpha, \phi_\alpha^\beta] \) exists and is a compact semigroup. 

So it is enough to show that there exists a dense continuous homomorphism from \( S \) into \( A^* \). Define \( \eta: S \to \prod_{\alpha \in D} [A_\alpha, \phi_\alpha^\beta] \) by \( \eta(x)(\alpha) = \eta_\alpha(x) \) for each \( x \in S \) then \( \eta \) actually maps \( S \) into \( A^* \) for, if \( \alpha \in D \) and \( \alpha \leq \beta \in D \), 
\( \eta(x)(\alpha) = \eta_\alpha(x) = \phi_\alpha^\beta \eta_\beta(x) = \phi_\alpha^\beta [\eta(x)(\beta)] \). Then \( \eta(x) \in A^* \). Since each \( \eta_\alpha \) is a continuous homomorphism so is \( \eta \). Claim \( \eta(S) = A^* \).

For this, we show that each non-empty basic open set in \( A^* \) contains points of \( \eta(S) \). Since the system is strict projective, the restricted map \( \prod_{\alpha} \to \prod_{\alpha} \eta_\alpha: A \to A_\alpha \) is surjective for each \( \alpha \in D \). Given \( \eta_\alpha(S) = A_\alpha \) for every \( \alpha \in D \). Let \( U \) be an open set in \( A_\alpha \) containing points of \( \eta_\alpha(S) \). That is \( \prod_{\alpha}^{-1} \) contains points of \( \prod_{\alpha}^{-1} \eta_\alpha(S) \). That is \( \prod_{\alpha}^{-1} (U) \) contains points of \( \eta(S) \) for every \( \alpha \in D \). Therefore each non-empty basic open set in \( A^* \) contains points of \( \eta(S) \), since \( \prod_{\alpha}^{-1} \): all \( \alpha \), all \( U \subset A_\alpha \) forms a basis for \( A^* \). That is \( \eta(S) = A^* \). Therefore \( (\eta, A^*) \) is a semigroup compactification of \( S \).

### 2.4 Projective System and Adjunction Semigroups.

The concept of projective (or inverse) and injective (or direct) limits of topological semigroups are developed and some results on compact semigroups are obtained in this section.
**Definition 2.4.1.** A projective system of (topological) semigroups is a triple \( ((D, \leq), \{ S_\alpha \}_{\alpha \in D}, \{ \Phi_\alpha^\beta \}_{\alpha \leq \beta} ) \) where

a. \( (D, \leq) \) is a directed set.

b. \( \{ S_\alpha \}_{\alpha \in D} \) is a family of (topological) semigroups indexed by \( D \) and

c. \( \{ \Phi_\alpha^\beta \}_{\alpha \leq \beta} \) is a family of functions indexed by \( \leq \) such that

(i) \( \Phi_\alpha^\beta : S_\beta \to S_\alpha \) is a (continuous) homomorphism for each \( (\alpha, \beta) \in \leq \);

(ii) \( \Phi_\alpha^\alpha = I_{S_\alpha} \) identity map on \( S_\alpha \) for each \( \alpha \in D \) and

(iii) \( \Phi_\alpha^\beta \circ \Phi_\beta^\gamma = \Phi_\alpha^\gamma \) for all \( \alpha \leq \beta \leq \gamma \) in \( D \).

Each \( \Phi_\alpha^\beta \) is called a bonding map and \( \{ S_\alpha, \Phi_\alpha^\beta \}_{\alpha \in D} \) is said to be strict if each bonding map is surjective.

If \( S = \{ x \in \Pi \{ S_\alpha \}_{\alpha \in D} : \Phi_\alpha^\beta (x(\beta)) = x(\alpha) \text{ for all } \alpha \leq \beta \text{ in } D \} \) is non-empty, then \( S \) is called the projective limit of \( \{ S_\alpha, \Phi_\alpha^\beta \}_{\alpha \in D} \).

**Theorem 2.4.2.** Let \( \{ S_\alpha, \Phi_\alpha^\beta \}_{\alpha \in D} \) be a projective system of (topological) semigroups such that \( S = \text{lim } S \) exists. Then \( S \) is a closed subsemigroup of \( \Pi \{ S_\alpha \}_{\alpha \in D} \).

**Proof:** For fixed \( \beta \leq \gamma \) in \( D \), let \( T_\beta^\gamma = \{ x \in \Pi \{ S_\alpha \}_{\alpha \in D} : \Pi_\beta^\gamma (x(\gamma)) = x(\beta) \} \)

Then \( T_\beta^\gamma \) is a closed subsemigroup of \( \Pi \{ S_\alpha \}_{\alpha \in D} \) and, as \( S = \bigcap \{ T_\beta^\gamma : \beta \leq \gamma \} \),

\( S \) is a closed subsemigroup of \( \Pi \{ S_\alpha \}_{\alpha \in D} \).

Hence the theorem.

One of the most useful tools in computing Bohr compactification is the product theorem.
Theorem 2.4.3 (product theorem)

Let \( \{ S_\alpha : \alpha \in A \} \) be a collection of abelian topological monoids with \((\beta, B)\) the Bohr compactification of \( S_\alpha \) for each \( \alpha \in A \) and \( \beta: \Pi\{S_\alpha \} \rightarrow \Pi\{\beta_\alpha \} \) be the function defined by \( \beta(x)(\delta) = \beta_\delta \Pi_\delta(x) \) where \( \Pi_\delta: \Pi\{S_\alpha \} \rightarrow S_\alpha \) is projection for each \( \delta \in A \). Then \((\beta, \Pi(\beta_\alpha))\) is the Bohr compactification of \( \Pi\{S_\alpha \} \).

Proof: A straightforward argument works to demonstrate that \( \beta \) is a dense continuous homomorphism. Let \( S = \prod\{S_\alpha \}_{\alpha \in A} \), \( B = \prod\{B_\alpha \}_{\alpha \in A} \), and suppose that \( g: S \rightarrow T \) is a continuous homomorphism of \( S \) into a compact semigroup \( T \). With no loss of generality we can assume that \( g \) is dense. To complete the proof, we need to exhibit a continuous homomorphism \( f: B \rightarrow T \) such that the diagram:

\[
\begin{array}{ccc}
S & \xrightarrow{g} & T \\
\downarrow{\beta} & & \downarrow{f} \\
B & \xrightarrow{\beta} & T
\end{array}
\]

commutes. Let \( 1 \) denote the identity of \( S \) and let \( 1_\alpha \) denote the identity of \( S_\alpha \) and \( p_\alpha: B \rightarrow B_\alpha \) projection for each \( \alpha \in A \). Define \( \psi_\alpha: S_\alpha \rightarrow S \) so that

\[
\Pi_\delta \psi_\alpha(z) = \begin{cases} 
z & \text{if } \alpha = \delta \\ 
1_{\delta} & \text{if } \alpha \neq \delta
\end{cases}
\]
For each $\alpha, \beta \in \Lambda$ and $z \in S_{\alpha}$. Then $\psi_{\alpha}: S_{\alpha} \rightarrow S$ is a continuous homomorphism for each $\alpha \in \Lambda$, and there exists a unique continuous homomorphism $g_{\alpha}: B_{\alpha} \rightarrow T$ such that the diagram:

\[
\begin{array}{ccc}
B_{\alpha} & \xrightarrow{g_{\alpha}} & T \\
\uparrow{\beta_{\alpha}} & & \uparrow{g} \\
S_{\alpha} & \xrightarrow{\psi_{\alpha}} & S
\end{array}
\]

commutes, since $(\beta_{\alpha}, B_{\alpha})$ is the Bohr compactification of $S_{\alpha}$ for each $\alpha \in \Lambda$.

Now, in order to define $f: B \rightarrow T$, let $x \in B$ and let $U$ be the collection of all open neighbourhoods of $x$. Define $G(x) = \bigcap \{ g^{-1}(U) : U \in \mathcal{U} \}$. We will show that $G(x)$ is degenerate and define $f(x)$ to be that point. Since $\beta$ is dense, we have that $\{ g^{-1}(U) : U \in \mathcal{U} \}$ is a descending collection of non-empty compact subsets of $T$ and hence $G(x) \neq \emptyset$.

Suppose that $G(x)$ contains distinct points $a$ and $b$, and select $M$, $N$ and $V$ in $T$ according to lemma (2.3.1) with $a \in M$, $b \in N$, and $g(1) \in V$ (observe that $g(1)$ is the identity of $T$). Let $W$ be a basic open subset of $S$ containing 1 such that $g(W) \subset V$. Then there exists a finite subset $F$ of $\Lambda$ such that $\prod_{a} (W) = S_{\alpha}$ for $\alpha \in \Lambda \setminus F$. Let $p$ be the product of the elements of the finite set $\{ g_{\alpha} p_{\alpha}(x) : \alpha \in F \}$, and choose $G_{p}$ according to lemma (2.3.1). Without loss of generality we assume that $G_{p} V \cap M = \emptyset$. In view of the continuity of multiplication in $T$ and the continuity of $g_{\alpha}: B_{\alpha} \rightarrow T$ for each $\alpha \in \Lambda$, we see that for each $a \in F$, there exists an open subset $H_{a}$ of $B_{a}$ containing $p_{a}(x)$ such
that the product of the set \( \{ g_{\alpha}(H_{\alpha}) : \alpha \in F \} \) is contained in \( G_p \). Let \( H \) be the basic open subset of \( B \) containing \( x \) such that \( \beta(x) = H_\alpha \) for \( \alpha \in F \) and \( \rho_\alpha(H) = B_{\alpha} \) for \( \alpha \in A \setminus F \). Then there exists \( s \in S \) such that \( g(s) \in M \) and \( \beta(s) \in H \). Now let \( t \) and \( w \) be the elements of \( S \) such that for each \( \alpha \in A \),

\[
\Pi_{\alpha}(t) = \begin{cases} 
\Pi_{\alpha}(S) & \text{if } \alpha \in F; \\
1_{\alpha} & \text{if } \alpha \in A \setminus F
\end{cases}
\]

\[
\Pi_{\alpha}(w) = \begin{cases} 
\Pi_{\alpha}(S) & \text{if } \alpha \in A \setminus F \\
1_{\alpha} & \text{if } \alpha \in F
\end{cases}
\]

Then \( s = tw, \beta(t) \in H, \) and \( w \in W \). We obtain \( g(t) \in G_p \), and hence \( g(s) = g(tw) = g(t) \circ g(w) \in G_p \). On the other hand, we have that \( g(s) \in M \) contradicting the fact that \( G_p \cap M = \emptyset \), and we conclude that \( G(x) \) is degenerate. We define \( f(x) \) to be the point in \( G(x) \) for each \( x \in B \). Then \( g = f \circ \beta \).

To prove the continuity of \( f \), let \( x \in B \) and let \( Q \) be an open subset of \( T \) containing \( f(x) \). Since \( \{ g\beta^{-1}(U) : U \in \mathcal{U} \} \) is descending, there exists \( U \in \mathcal{U} \) such that \( g\beta^{-1}(U) \subset Q \), and thus \( f(U) \subset Q \). It follows that \( f \) is continuous. Since \( g = f \circ \beta, f|\beta(s) \) is a homomorphism and \( g \) is dense, we conclude that \( f \) is a homomorphism. Hence the theorem.

If we consider two topological semigroups \( S \) and \( T \) and their respective Bohr compactifications \( (\alpha, A) \) and \( (\beta, B) \), then each continuous homomorphism \( g : S \to T \) gives rise to a unique continuous homomorphism \( f : A \to B \) such that the following diagram commute. Then \( f \circ \alpha = \beta \circ g \).
Using above result we have the following proposition

**Proposition 2.4.4.** Let \( S_\alpha, S_\beta, S_\gamma \) be topological semigroups with Bohr compactification \( B_\alpha, B_\beta \) and \( B_\gamma \) respectively. \( \Phi_\rho^\alpha : S_\alpha \to S_\beta \), \( \beta \leq \alpha \). \( \Phi_\gamma^\beta : S_\beta \to S_\gamma, \gamma \leq \beta \) are continuous homomorphisms such that

\[
\Phi_\alpha^\alpha = \perp_{S_\alpha} \text{ and } \Phi_\gamma^\beta \circ \Phi_\rho^\alpha = \Phi_\gamma^\alpha \text{ for } \gamma \leq \beta \leq \alpha
\]

\[
\Psi_\beta^\beta : B_\alpha \to B_\beta \text{ and } \Phi_\gamma^\beta : B_\beta \to B_\gamma
\]

\[
\Phi_\rho^\beta : S_\alpha \to S_\beta \text{ gives a unique continuous homomorphism}
\]

\[
\Psi_\rho^\beta : B_\alpha \to B_\beta \text{ Such that } \Psi_\rho^\beta \circ \beta_\alpha = \beta_\beta \circ \Phi_\rho^\alpha. \text{ Also } \Phi_\gamma^\beta : S_\beta \to S_\gamma, \text{ is a continuous homomorphism gives rise to a unique continuous homomorphism } \Psi_\gamma^\beta : B_\beta \to B_\gamma \text{ such that } \Psi_\gamma^\beta \circ \beta_\beta = \beta_\gamma \circ \Phi_\gamma^\beta \text{ then}
\]

\[
\Psi_\gamma^\alpha \circ \beta_\alpha = \beta_\gamma \circ \Phi_\gamma^\alpha.
\]

**Proof:** \[
\Psi_\gamma^\beta \circ \beta_\alpha = (\Psi_\gamma^\beta \circ \Psi_\rho^\alpha) \circ \beta_\alpha = \Psi_\gamma^\alpha \circ (\Psi_\rho^\alpha \circ \beta_\alpha)
\]

\[
= \Psi_\gamma^\alpha \circ (\beta_\alpha \circ \Phi_\rho^\alpha)
\]

\[
= (\Psi_\gamma^\alpha \circ \beta_\alpha) \circ \Phi_\rho^\alpha
\]

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Theorem 2.4.5: Let $S$ and $T$ be disjoint topological semigroups with Bohr compactifications $(B_S, \beta_S)$ and $(B_T, \beta_T)$ respectively. Let $\Phi: S \to T$ be a continuous homomorphism then $S \cup T$ is a topological semigroup and $\Phi: S \to T$ determines a unique continuous homomorphism $\psi: B_S \to B_T$ such that $\psi \circ \beta_S = \beta_T \circ \Phi$ and $B_S \cup B_T$ is a topological semigroup called adjunction semigroup of $S \cup T$.

Proof: 

Let $S_\alpha$, $S_\beta$ be topological semigroups with Bohr compactifications $B_\alpha$ and $B_\beta$ respectively $\alpha \leq \beta$. $\Phi_\alpha^\beta: S_\beta \to S_\alpha$ be continuous homomorphism such that there exist a unique continuous homomorphism $\Psi_\alpha^\beta: B_\beta \to B_\alpha$ such that $\Psi_\alpha^\beta \circ \beta_\beta = \beta_\alpha \circ \Phi_\alpha^\beta$. That is the following diagram commutes.
Theorem 2.4.6. Let \( \{S_\alpha \cup S_\beta\}_{\alpha \leq \beta \in D} \) be a projective system of topological semigroups with Bohr compactification \( B_\alpha \), \( B_\beta \) respectively determines a projective system of Bohr compactification \( \{B_\alpha \cup B_\beta\}\).

**Proof:** Since \( B_\alpha \), \( B_\beta \) respectively the Bohr compactification of \( S_\alpha \) and \( S_\beta \), \( \Phi_\alpha^\beta : S_\beta \rightarrow S_\alpha \) is a continuous homomorphism and since \( \beta_\alpha : S_\alpha \rightarrow B_\alpha \) is a continuous homomorphism.

\[
\begin{array}{c}
\Psi_\alpha^\beta \\
B_\beta & \longrightarrow & B_\alpha \\
\beta_\beta & \uparrow & \beta_\alpha \\
S_\beta & \longrightarrow & S_\alpha
\end{array}
\]

\( \beta_\alpha \circ \Phi_\alpha^\beta \) is a continuous homomorphism. By the definition of Bohr compactification there exists a unique continuous homomorphism \( \Psi_\alpha^\beta : B_\beta \rightarrow B_\alpha \) such that \( \Psi_\alpha^\beta \circ \beta_\beta = \beta_\alpha \circ \Phi_\alpha^\beta \) \( \forall \alpha \leq \beta \in D \). and \( \Psi_\alpha^\beta \circ \beta_\alpha = \beta_\alpha \circ \Phi_\alpha^\beta \) for each \( \alpha \in D \).

\[
\begin{array}{c}
\Psi_\alpha^\beta \\
B_\alpha & \longrightarrow & B_\alpha \\
\beta_\alpha & \uparrow & \beta_\alpha \\
S_\alpha & \longrightarrow & S_\alpha
\end{array}
\]

Thus \( \{B_\alpha \cup B_\beta\} \) is a projective system of Bohr compactification denoted as \( (\mathcal{D}, \leq), \{B_\alpha \cup B_\beta\}_{\alpha \leq \beta}, \{\Psi_\alpha^\beta\} \) where
a. $(D, \leq)$ is a directed set.

b. $\{B_\alpha \cup B_\beta\}_{\alpha \leq \beta}$ is a family of union of compact semigroups

c. $\{\Psi^\beta_\alpha\}$ is a continuous function : $B_\beta \rightarrow B_\alpha, \alpha \leq \beta \in D$ such that

i. $\Psi^\beta_\alpha \circ \beta_\beta = \beta_\alpha \circ \Phi^\beta_\alpha, \forall \alpha \leq \beta \in D.$

ii. $\Psi^\alpha_\alpha \circ \beta_\alpha = \beta_\alpha \circ \Phi^\alpha_\alpha$ for each $\alpha \in D$ and

iii. $\Psi^\beta_\gamma \circ \Psi^\alpha_\beta \circ \beta_\alpha = \beta_\gamma \circ \Phi^\beta_\gamma \circ \Phi^\alpha_\beta, \alpha \leq \beta \leq \gamma \in D$

**Theorem 2.4.7.** Let $\{S_\alpha \cup S_\beta\}_{\alpha \leq \beta \in D}$ be a projective system of topological semigroups with Bohr compactification $B_\alpha, B_\beta$ respectively then $\{B_\alpha \cup B_\beta\}$ determine a projective system of Bohr compactification. Clearly

1. $\Psi^\beta_\alpha \circ \beta_\beta = \beta_\alpha \circ \Phi^\beta_\alpha, \forall \alpha \leq \beta \in D.$

2. $\Psi^\alpha_\alpha \circ \beta_\beta = \beta_\alpha \circ \Phi^\alpha_\alpha$ for each $\alpha \in D$.

3. $\Psi^\beta_\gamma \circ \Psi^\alpha_\beta \circ \beta_\alpha = \beta_\gamma \circ \Phi^\beta_\gamma \circ \Phi^\alpha_\beta \forall \alpha \leq \beta \leq \gamma \in D.$

**Projective System of Adjunction Topological Semigroup**

**Theorem 2.4.8:** $\{S_\alpha \cup S_\beta\}_{\alpha \leq \beta \in D}, \Phi^\beta_\alpha$ be a projective system of adjunction topological semigroups. $S = \{x \in \Pi \{S_\alpha \cup S_\beta\}_{\alpha \leq \beta \in D} : \Phi^\beta_\alpha(x(\beta)) = x(\alpha) \}$ for
all \( \alpha \leq \beta \in D \) is non-empty. Then \( S \) is a closed subsemigroup of \( \Pi \{ S_\alpha \cup S_\beta \} _{\alpha \leq \beta \in D} \).

**Proof:** For fixed \( \beta \leq \gamma \in D \), let \( T^\gamma_\beta = \{ x \in \Pi \{ S_\alpha \cup S_\beta \} _{\alpha \leq \beta \in D} \} \) such that

\[
T^\gamma_\beta (x(\gamma)) = x(\beta)
\]

then \( T^\gamma_\beta \) is a closed subsemigroup of \( \Pi \{ S_\alpha \cup S_\beta \} _{\alpha \leq \beta \in D} \), and \( S \cap \{ T^\gamma_\beta , \beta \leq \gamma \} \) is a closed subsemigroup of \( \Pi \{ S_\alpha \cup S_\beta \} _{\alpha \leq \beta \in D} \).

**Projective System of Adjunction Bohr Compactification**

**Theorem 2.4.9** :-Let \( \{ S_\alpha \cup S_\beta \} _{\alpha \leq \beta \in D} \) be a projective system of topological semigroups with projective system of adjunction Bohr compactification \( \{ B_\alpha \cup B_\beta \} _{\alpha \leq \beta \} \) if \( S = \{ x \in \Pi ( S_\alpha \cup S_\beta ) : \Phi^\alpha_\beta (x(\beta)) = x(\alpha), \forall \alpha \leq \beta \in D \} \) exists then \( \lim ( B_\alpha \cup B_\beta ) \) is a Bohr compactification of \( S \) is called the adjunction Bohr compactification of \( S \).

**Proof:** For fixed \( \gamma \in D \), let \( T^\gamma = \{ x \in \Pi \{ S_\alpha \cup S_\beta \} _{\alpha \leq \beta \in D} \} \) such that \( f(x) (\beta) = \begin{cases} x(\beta) & \text{if } \beta \not\leq \gamma \\ \Phi^\gamma_\beta (x(\gamma)) & \text{if } \beta \leq \gamma \end{cases} \) for all \( \beta \leq \gamma \) in \( D \). For each \( x \in \Pi \{ S_\alpha \cup S_\beta \} _{\alpha \leq \beta \in D} \) if \( f(x) \) is defined by

\[
f(x) (\beta) = \begin{cases} x(\beta) & \text{if } \beta \not\leq \gamma \\ \Phi^\gamma_\beta (x(\gamma)) & \text{if } \beta \leq \gamma 
\end{cases}
\]

for all \( \beta \in D \), then \( f(x) \in T^\gamma \). Hence \( T^\gamma \not= \Phi \) and \( f \) is a continuous function from \( \Pi \{ S_\alpha \cup S_\beta \} _{\alpha \leq \beta \in D} \) into \( T^\gamma \).

Since \( \Pi_\beta \circ f = \begin{cases} \Pi_\beta & \text{if } \beta \not\leq \gamma \\ \Phi^\gamma_\beta \circ \Pi_\gamma & \text{if } \beta \leq \gamma \end{cases} \)

It is obvious that \( f \) retracts onto \( T^\gamma \circ \Pi \{ S_\alpha \cup S_\beta \} _{\alpha \leq \beta \in D} \) is a closed subspace of \( S = \{ x \in \Pi \{ S_\alpha \cup S_\beta \} _{\alpha \leq \beta \in D} \} \not= \Phi \). Also
since \( \{B_\alpha \cup B_\beta\}_{\alpha \leq \beta \in \mathbb{D}} \) be a projective system of adjunction Bohr compactification.

Let \( \beta'_T = \{x \in \prod \{B_\alpha \cup B_\beta\} \mid \beta_\beta \circ \Phi'_\beta(x(\gamma)) = \Psi'_\beta \circ \beta_\gamma(x) \}

\[
\begin{array}{c}
\Psi'_\beta \\
B_\gamma \rightarrow B_\beta \\
\beta_\gamma \uparrow \\
S_\gamma \rightarrow S_\beta \\
\Phi'_\beta
\end{array}
\]

For each \( x \in \prod \{B_\alpha \cup B_\beta\} \), if \( g(x) \) is defined by

\[
g(x)(\beta) = \begin{cases} 
  x(\beta) & \text{if } \beta \leq \gamma \\
  \Psi'_\beta(x(\gamma)) & \text{if } \beta \leq \gamma, \forall \beta \in \mathbb{D}
\end{cases}
\]
then \( g(x) \in \beta'_T \)

Hence \( \beta'_T \neq \Phi \) and \( g \) is a continuous function from \( \prod \{B_\alpha \cup B_\beta\} \) into \( \beta'_T \) since

\[
\begin{array}{c}
\Pi_{\beta} \circ g = \\
\Pi_\beta \\
\prod_{\{B_\alpha \cup B_\beta\}} \\
\Psi'_\beta \circ \Pi_\gamma
\end{array}
\]

\( g \) retracts \( \prod \{B_\alpha \cup B_\beta\} \) onto \( \beta'_T \) and so \( \beta'_T \) is a compact subspace of \( \prod \{B_\alpha \cup B_\beta\} \) \( \alpha \leq \beta \in \mathbb{D} \) More over, if \( \gamma \leq \delta \) in \( \mathbb{D} \), then \( \beta'_T \leq \beta'_T \). So \( \{ \beta'_T \} \)

is a descending family of compact semigroups. Therefore \( \cap \{ \beta'_T \} \neq \Phi \) is a compact subspace of \( \prod \{B_\alpha \cup B_\beta\} \) \( \alpha \leq \beta \in \mathbb{D} \) Bohr compactification of \( S \).
Clearly \( \cap \{ \beta_r^\gamma \} \) is compact. Define \( g^1 \colon S \to \lim_t (B_a \cup B_\beta) \) is \( \Pi \beta_a \to \{ B_a \cup B_\beta \} \)

\[
\begin{align*}
\Pi \{S_a \cup S_\beta\} & \longrightarrow \Pi \{B_a \cup B_\beta\} \\
\Pi \{S_a \cup S_\beta\} & \longrightarrow \Pi \{B_a \cup B_\beta\} \\
\beta_a & \longrightarrow B_a
\end{align*}
\]

\[
(\Pi \beta_a \cdot (x(\alpha))) = \left( \begin{array}{c}
\Pi (B_a \cup B_\beta) \circ \Pi \beta_a \end{array} \right)(x)
\]

\[
= \beta_a \circ \left( \begin{array}{c}
\Pi (S_a \cup S_\beta) \cdot (x) \end{array} \right)
\]

3.1 Category of Topological Semigroups. In category of topological semigroups, objects are topological semigroups and morphisms are continuous homomorphisms, except that in the non-id small categories, morphisms are required to be identity preserving. The role of composition in each category is ordinary composition of functions. The definition of Bohr compactification \( g^1, \lim_t (B_a \cup B_\beta) \) is a Bohr compactification of \( S \) called adjunction

Bohr compactification of \( S \).