INTRODUCTION

The theory of Categories plays a significant role in the development of the structure theory of topological semigroups. Many algebraic and topological results can be developed by using category theory rather easily. In this thesis, we developed the theory of adjunction semigroup compactification with reflective subcategories and studied some problems on adjunction semigroup theory related with pushout and pullback in category of topological semigroups.

The theory of compactification in topological spaces is originated with the work of Tychonoff A.(1930)[54]. Later Čech E. (1937)[8] and Stone M.H.(1937) independently defined the maximal Hausdorff Compactification of a topological space named Stone-Čech compactification and established its fundamental properties related with quotients of compact space.

The lattice properties of Hausdorff Compactification of a topological space by considering Hausdorff quotients of Stone-Čech compactification was studied by many topologists. In the theory of topological Semigroups, there are some developments in the theory of semigroup compactification. The theory of topological semigroups has its origin with the work of Wallace A.D.(1950)[55]. It contains the characterization of semigroup compactification. As a result of the theory of compact semigroups, Bohr compactification, group compactification, one-point compactification, all have deep roots in the theory of topological semigroups[32]. Main developments and applications are available in the monograph, “The theory of topological

The concept of compactness and congruence plays an important role in the development of structure theory of topological semigroups [38]. Lawson J.D. and Madison B.L. (1971) [40] developed an important theorem on Rees Quotient Semigroup of S modulo ideal, which was of immense help in developing the subject.

Related with compactification theory in topological semigroups we observed the following:


b. Berglund J.F., Junghenun H.D. and Milnes P. (1978) [4] developed the theory of almost periodic and weakly almost periodic compactifications of semitopological semigroups. Possible techniques developed for semigroup compactification are:

(i) Bohr compactification of topological semigroups using Operator theory by Deleeuw K. and Glicksberg I. (1961) [16]


Semigroup compactification of a topological semigroup S is defined as an ordered pair (g, T) where T is a compact semigroup and
g : S→ T is a dense continuous homomorphism of S into T (g is a dense means g(S) is dense in T).

Associated with each topological semigroup S there is a compact semigroup called Bohr compactification of S which is universal over the compact semigroups containing dense continuous homomorphic images of S. The existence and uniqueness of Bohr compactification is proved in [6].

It is also proved that semigroup compactification of a topological semigroup S is precisely quotients of the Bohr compactification of S under the closed congruence on (β, B). Also proved that if (β, B) is a Bohr compactification of S and R is any closed congruence on B then the quotient space B/R is a semigroup compactification.

**Bohr Compactification** of a topological semigroup S is a pair (β, B) such that B is a compact semigroup, β : S→ B is a continuous homomorphism and if g : S→ T is a continuous homomorphism of S into a compact semigroup T, then there exists a unique continuous homomorphism f : B→ T such that f o β = g


In the theory of topological semigroups, disjoint topological semigroups have an important role in the formation of new topological semigroups. As a result, the theory of adjunction semigroup was

If \( S \) and \( T \) are disjoint topological semigroups and \( \Phi: S \to T \) is a continuous homomorphism, then we define multiplication on \( S \cup T \) by

\[
(x, y) \rightarrow \begin{cases} 
  m_S(x, y) & \text{if } x, y \in S \\
  m_T(x, y) & \text{if } x, y \in T \\
  m_T(\Phi(x), y) & \text{if } x \in S \text{ and } y \in T \\
  m_T(x, \Phi(y)) & \text{if } x \in T \text{ and } y \in S
\end{cases}
\]

Where \( m_S \) and \( m_T \) are the multiplications on \( S \) and \( T \) respectively. We denote \( S \cup T \) with this multiplication by \( S \cup_T \) and is called the **Adjunction Semigroup** of \( S \) and \( T \) relative to \( \Phi \).

Compactification theory associated with adjunction semigroup is not extensively studied. However we observed some results on the quotient of adjunction compactification semigroup which motivated the study of adjunction semigroup compactification, product of adjunction semigroup compactification and projective system of adjunction semigroup compactification.

In [6], the study of new semigroups from the old developed as **Cartesian products**, since internal structure theory of topological semigroups is preserved by forming cartesian products. Using Cartesian products, Bohr compactification of a topological semigroup \( S \) is developed.

It has been observed that one of the fundamental methods of obtaining the new topological semigroup from a given collection of topological semigroups is to form their **projective system** of topological semigroups. Important results based on these was studied
A projective system of (topological) semigroups is a triple 
\((\mathcal{D}, \leq), \{ S_\alpha \} _{\alpha \in \mathcal{D}}, \{ \Phi_\alpha^\beta \} _{\alpha \leq \beta} \) where

a. \((\mathcal{D}, \leq)\) is a directed set.

b. \(\{ S_\alpha \} _{\alpha \in \mathcal{D}}\) is a family of (topological) semigroups indexed by \(\mathcal{D}\) and

c. \(\{ \Phi_\alpha^\beta \} _{\alpha \leq \beta}\) is a family of functions indexed by \(\leq\) such that

(i) \(\Phi_\alpha^\beta : S_\beta \rightarrow S_\alpha\) is a (continuous) homomorphism for each 
\((\alpha, \beta) \in \leq\)

(ii) \(\Phi_\alpha^\alpha = I_{S_\alpha}\) identity map on \(S_\alpha\) for each \(\alpha \in \mathcal{D}\) and

(iii) \(\Phi_\alpha^\beta \cdot \Phi_\beta^\gamma = \Phi_\alpha^\gamma\) for all \(\alpha \leq \beta \leq \gamma\) in \(\mathcal{D}\).

This projective system is denoted by \(\{ S_\alpha, \Phi_\alpha^\beta \} _{\alpha \in \mathcal{D}}\).

Eilenberg and S.Mac Lane laid the foundation of the theory of Categories and Functors in 1945. Many different Mathematical fields may be interpreted as categories and that the techniques and theorems of this theory may be applied to these fields. The most important terms occurring in Mathematical branches in one way or another have been expressed in the language of categories. Categories consists of class of objects and morphisms. Results on semigroup can very often be deduced from the corresponding results on categories. (Note that a semigroup with unity is but a category with a single object.)

The thesis begins with a discussion on Category of Topological Semigroups. The notion of category of topological semigroups was introduced by J.W.Crawley Junior in the year 1973 [15]. In the
category of topological semigroups, objects are topological semigroups and morphisms are continuous homomorphisms. In the topological monoid categories, morphisms are required to be identity preserving continuous functions and the rule of composition of morphisms is the ordinary composition of functions.

The category of topological semigroups is denoted by $\mathcal{TS}$. If $S$ is a topological semigroup, then the identity morphism for $S$ is the identity function from $S$ into itself. (Usually denoted by $I_S$). For each pair of objects $S$ and $T$ in $\mathcal{TS}$, the morphism set denoted by $\text{hom}_{\mathcal{TS}}(S, T)$, contains precisely those morphisms $\Phi : S \to T$ such that $\Phi$ is a topological isomorphism onto $\Phi(S)$ and $\Phi(S)$ is a homomorphic retract of $T$; that is, there exists a $\mathcal{TS}$-morphism $\gamma : T \to T$ such that $\gamma \circ \gamma = \gamma$ and $\gamma(T) = \Phi(T)$. The $\mathcal{TS}$-retractions are precisely those $\mathcal{TS}$-morphism $\Phi : S \to T$ such that there exists a homomorphic retraction $\beta : S \to S$ (again $\beta \circ \beta = \beta$) and a $\mathcal{TS}$-isomorphism $\partial : \beta(S) \to T$ such that $\Phi = \partial \circ \beta$. Hence the $\mathcal{TS}$-retractions are, up to $\mathcal{TS}$-isomorphisms, homomorphic retractions. In each category of topological semigroup $\mathcal{TS}$, isomorphisms are precisely the topological isomorphisms (ie, homeomorphism which preserve multiplication).

In 1966, Hofmann K.H.and Mostert P.S.[35] established that the compact semigroup monomorphisms are precisely the injective continuous homomorphisms. Each injective $\mathcal{TS}$-morphism is a $\mathcal{TS}$-monomorphisms. That is, the $\mathcal{TS}$-monomorphisms are precisely the injective $\mathcal{TS}$-morphisms.

By a topological property $\mathcal{P}$ we mean a full subcategory of category of topological spaces which is closed under the formation of

In the category of topological semigroups $\mathcal{FS}$, the triple $(S, p_1, p_2)$ in the following diagram

![Diagram](image)

(the topological semigroup $S$ together with the continuous homomorphisms $p_1$ and $p_2$) is said to be the pushout of the diagram

![Diagram](image)

if every pair of continuous homomorphism $q_1 : Q \to R$ and $q_2 : T \to R$ such that $q_1 \circ h = q_2 \circ g$, there is a unique morphism $\Phi : S \to R$ such that $\Phi \circ p_1 = q_1$, $\Phi \circ p_2 = q_2$.

Similarly the diagram
is said to be **pullback** in a category $\mathcal{E}$ if for every pair of morphisms $j: Z \to X$ and $k: Z \to Y$ such that $f \circ j = g \circ k$ there exists a unique morphism $h: Z \to P$ such that $p_1 \circ h = j$ and $p_2 \circ h = k$. $P$ together with $p_1$ and $p_2$ is the pullback of $f$ and $g$.

The present work is on **Adjunction Semigroups in Reflective Subcategories of $\mathcal{JS}$, the Category of Topological Semigroups**. The first chapter consists of the preliminary concepts and terminology which are required in the sequel. In the four different sections, theory of Semigroups, Topological Semigroups and Categories have been discussed. **Adjunction Semigroups**, Adjunction Bohr Compactification, Product Semigroup Compactification and Projective System of Adjunction Semigroups are derived in chapter two. It consists of four
sections. In the first section adjunction semigroup is defined and proved that the adjunction semigroup of two disjoint topological semigroups is a topological semigroup. In the next section, we derive Adjunction Bohr Compactification of adjunction semigroups [cf theorem 2.2.1]. Adjunction semigroups of two disjoint Topological Semigroups with Bohr compactification is considered here and derived the quotient of adjunction semigroup compactification which is the adjunction of adjunction semigroups. [cf theorem 2.2.2], The product semigroup compactification [cf theorem 2.3.2] is discussed in section three. The last section deals with Adjunction Bohr compactification of projective system of Adjunction Bohr compactification [cf theorem 2.4.9].

The third chapter contains Reflective Subcategories of \( \mathcal{S} \), the Category of Topological Semigroups. It consists of three sections. The first section deals with the category of topological semigroups. In the next section we introduce the notion of an ideal*semigroup and prove that epimorphisms in the category of ideal*semigroups are morphisms with dense range[cf theorem 3.2.5]. Reflective Subcategories are discussed in section three and it is proved that the reflection of an object is unique up to topological isomorphism in \( \mathcal{S} \), the category of topological semigroups, [cf theorem 3.3.4]. In the last section we consider the Universal Problem defined by the adjoint functors and derived the sufficient condition for a pair of adjoint functors which induces a reflective subcategory [c.f. Theorem 3.3.7].

Reflection Theory in the Category of Topological Semigroups is considered in chapter four. It is divided into four
sections. The first section gives various types of reflectors. The second section gives some topological semigroup properties. Pullback Sripping Functors, $\Lambda$-product and $\Pi$-product in partially ordered category are discussed in the third section and prove that in Pullback category the two products are the same [cf theorem 4.3.8]. In section four various types of categories are studied.

Chapter five is the study of the Adjunction Semigroups in Reflective Subcategories of $\mathcal{TS}$, the Category of Topological Semigroups. This chapter consists of four sections. In section one Pushout in Category Theory and different examples are explained. Also the pushouts in Category of topological semigroups [cf theorem 5.1.1] and in Reflective Subcategories of the Category of Topological Semigroups [cf theorem 5.1.6] are characterized. The Pullbacks in category theory is discussed in section two. It has been proved that pullback exists in topological semigroups. [cf theorem 5.2.2]. The third section deals with difference kernel and difference cokernel. In fourth section it has been derived that category of topological semigroup with finite products has difference kernel if and only if it has pullback [cf theorem 5.4.3]. Finally we conclude that in Reflective Subcategories of the Category of Topological Semigroups the Pushout is the Adjunction Semigroup.

The Adjunction Semigroup has wide applications in Quantum Mechanics and can be extended with pushout and pullback techniques.