CHAPTER – III

Study of Three-Phase Flow of Visco-Elastic Fluid with Application to Blood Flow

Introduction:

In biological sciences, the study of diffusivity of nutrients metabolic products, drugs and other solutes is of it most importance. Specially many life giving materials mixed in the blood, reach to the different parts of the body by the process of diffusion. Steady flow of non-Newtonian viscous fluids is important in view of blood flow in narrow vessels. Viscous non-Newtonian fluids where the shear stress is non-linearly proportional to the nth power of the velocity gradient may be considered to be appropriate model for discussion under consideration. It is further observed that Herschel – Bulkley model is valid over the range where the Casson’s plot can be used. Herschel – Bulkley constitutive equation includes one more parameter than Casson equation; therefore one can get more information in relevant discussion. Scott–Blair {14} suggested that the blood obeys the
Casson equation only in the limited range except at very high and very low shear rate and that there is no difference between the Casson and Herschel – Bulkley plots of experimental data over the range where the Casson plot is valid. It is also suggested that the assumptions used in the Casson equation are unsuitable for the Cow’s blood and that the Herschel – Bulkley equation represents fairly closed what is occurring in the blood.

Blood is a suspension of red cells in plasma, its flow in narrow tubes is different from that of the flow of Newtonian fluids. The experimental observation of Bugliarello and Sevilla {2} shows that blood flow in narrow tubes consists of two regions, the peripheral plasma layer [PPL] region and the core region. In view of these experimental findings, it is quite reasonable to consider a two layered model for blood flow through narrow tubes. Charm et al. {3}, Kiani and Hudetz {8} and Oka {12} have considered non-Newtonian visco-elastic fluid models of blood flow in narrow vessels. A two-layered model with core and PPL both obeying Newtonian constitutive law has been proposed by Haynes {6}, Mishra et al. {11} and Saran et al. {13}. Das and Sheshadri {5} considered a two layered model with
core of power law fluid and PPL of Newtonian fluid. Bagchi \{1\}, Leuprecht and Perktold \{9\}, Marella and Uday Kumar \{10\}, Secomb and Tsu \{15\}, Thurston \{16\}, Thurston \textit{et al.} \{17\}, Yilmaz \textit{et al.} \{18\}, Zhao \textit{et al.} \{19\} and Zwiefacht \{20\} have studied the model of red blood cell motion in capillaries. Iida \{7\} considered blood flow through narrow tubes with PPL of Newtonian fluid and core as Herschel – Bulkley fluid. Since this model assumes existence of yield stress, therefore central fluid will have a plug core behaving as a solid material around the axis of the tube. Thus in the present chapter, we have assumed a three layered model, that is –

The plug core solid fluid near the axis of the tube,

Core fluid surrounded by plasma and plug fluids,

and Plasma fluids (PPL) near the wall of the tube.

Core fluid obeys Herschel – Bulkley constitutive equation in the present analysis. We investigate in detail the effect of yield stress (the parameter which expresses the degree of shear thinning) on apparent viscosity and its variation with respect to other rheological parameters. Effect of plug core radius on whole blood concentration and core concentration has been found.
**Mathematical Analysis:**

In the subsequent study we have made the following assumptions to solve the problem.

1. Motion of the fluid is fully developed and has an axial symmetry.

2. Flow is steady and incompressible in the long rigid circular tube of uniform radius.

3. Flowing fluid in the tube obey the Herschel – Bulkley equation.

The shear stress $\tau$ and shear rate $\dot{\gamma}$ relationship is given by

$$
\dot{\gamma} = \frac{\partial u}{\partial r} = 0, \quad 0 < \tau < \tau_y
$$

$$
= \frac{1}{k} \left[ \tau - \tau_y \right], \quad \tau_y < \tau < \alpha \tau_w
$$

$$
= \frac{\tau}{\eta_p}, \quad \alpha \tau_w < \tau < \tau_e
$$

(3.1)

where $u$ is axial velocity in $z$-direction,

$\tau$ is shear stress,

$\tau_w$ is wall shear stress,

$\tau_y$ is yield stress

and $\eta_p$ is plasma viscosity.
The equation is reduced to that for Bingham fluid, when $n = 1$, to that for power law fluid when $\tau_y = 0$, to that for Newtonian fluid when $\tau_y = 0$, $n = 1$, and to that for Casson fluid when $n = 2$ ($\tau$ and $\tau_y$ are taken with under root sign).

We take cylindrical co-ordinate system $(r, \theta, z)$ whose origin lies on the axis of the tube. Momentum and continuity equations for fully developed laminar incompressible axi-symmetrical flow are

![Flow Diagram of Three Layered Model](image)

**Figure – 3.1 : Flow Diagram of Three Layered Model**
\[ 0 = -\frac{\partial p}{\partial r} \]  
(3.2)

\[ 0 = -\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \tau_{rz} \right) \]  
(3.3)

and \[ \frac{\partial u}{\partial z} = 0 \]  
(3.4)

where \( p \) is the pressure, \( \tau_{rz} \) is the shear stress normal to \( r \) along \( z \) direction, \( u \) is the velocity field along \( z \)-direction.

On integrating equation (3.3), we get

\[ r \tau_{rz} = \frac{1}{2} \frac{\partial p}{\partial z} r^2 + C. \]  
(3.5)

On assuming that the stress is finite at the axis of the tube, we get \( C = 0 \) and so equation (3.5) becomes.

\[ \tau_{rz} = \frac{r}{z} \frac{\partial p}{\partial z} \]  
(3.6)

If \( \tau_\omega \) is the shear stress at the wall, then,

\[ \tau_{\omega} = \frac{R}{z} \frac{\partial p}{\partial z} \]

or \[ \tau_{rz} = \frac{r}{2} \cdot \frac{\tau_\omega}{R} \]  
(3.7)

where \( R \) is the radius of the tube.
If \( \varepsilon \) denotes the thickness of PPL then the core radius \( R_1 \) is given by

\[
R_1 = R - \varepsilon = \alpha R
\]

where \( \alpha = 1 - \frac{\varepsilon}{R} \).

(3.8)

The Herschel–Bulkley viscosity \( \eta_H \) is defined by

\[
\eta_H = \frac{k}{\tau_\infty^{n-1}}
\]

(3.9)

We now assume that there exists a constitutive relation relating the stress with the strain rate in the form

\[
\left| \frac{du}{dr} \right| = \dot{\gamma}(\tau) = f(\tau)
\]

(3.10)

where \( f(\tau) \) is a function which describes the dependence of strain rate on the shear stress \( \tau \).

Integrating equation (3.10) we get

\[
\dot{u}(r) = \int_r^R f(\tau) \, d\tau
\]

(3.11)

or

\[
\dot{u}(r) = \frac{R}{\tau_\infty} \int_\tau^\infty f(\tau) \, d\tau
\]

(3.12)

**Velocity Field** – From equation (3.10) we get

\[
u = -\int f(\tau) \, d\tau + C
\]

(3.13)
For Herschel – Bulkley fluid, velocity field $u_H$ is given by

$$u_H = -\frac{1}{k} \int \left( \tau - \tau_y \right)^n dr + C \quad (3.14)$$

$$= -\frac{1}{k} \int \left( \frac{\tau_m r}{R} - \tau_y \right)^n dr + C$$

$$= -\frac{1}{k(n+1)} \left( \frac{\tau_m r}{R} - \tau_y \right)^{n+1} \cdot \frac{R}{\tau_{so}} + C,$$

constant of integration $C$ is calculated from the boundary condition,

$$u = u_p \quad \text{where} \quad r = R - \varepsilon = \alpha R$$

Thus,

$$u_p = -\frac{1}{k(n+1)} \cdot \left( \frac{\tau_m}{R} \cdot \alpha R - \tau_y \right)^{n+1} \frac{R}{\tau_{so}} + C$$

or

$$C = u_p \frac{R}{k(n+1)\tau_{so}} \left( \alpha \tau_{so} - \tau_y \right)^{n+1} \quad (3.15)$$

where

$$u_p = \frac{R \tau_m}{z} \left[ 1 - \left( \frac{r}{R} \right)^2 \right] \quad (3.16)$$

= plasma fluid velocity in PPL.

Iida {7} has found the apparent viscosity $\eta_a$ in the form

$$\eta_a = (n+1) \left[ \frac{4}{\eta_{Ht}} \left\{ \alpha^2 \left( \frac{R_p}{R} \right)^n - \frac{2\alpha \left( \alpha - \frac{R_p}{R} \right)}{n+2} \right\} \right]$$
where $R_p$ is plug core radius.

Bugliarello and Sevilla \cite{2} and Haynes \cite{6} have considered RBC concentration of core fluid as the concentration of while blood, but really it is not so. As PPL has been assumed to be depleted of RBC; hence the core will have more concentration than the whole blood. Also it may be noted that plug region will tend to have a very high concentration of RBC, difficult to measure experimentally. Taking into account of these factors core concentration was calculated by Chaturani \textit{et al.} \cite{4} from the expression.

\begin{equation}
\frac{2\left(\alpha - \frac{R_p}{R}\right)}{(n+2)(n+3)} + \frac{n+1}{\eta_p} \left(1 - \alpha^4\right)^{-1}
\end{equation}

(3.17)

It has been observed that plug core radius will have a maximum value for a given whole blood concentration. If $C_w$ denotes the whole blood concentration, $C_p$ the plug core concentration and $r_m$ the maximum non-dimensional plug core radius, then the table has been plotted.
Table – 3.1

<table>
<thead>
<tr>
<th>Cw</th>
<th>Cp</th>
<th>90%</th>
<th>98%</th>
<th>100%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.67</td>
<td>0.64</td>
<td>0.63</td>
</tr>
<tr>
<td>40%</td>
<td></td>
<td>0.71</td>
<td>0.68</td>
<td>0.67</td>
</tr>
<tr>
<td>45%</td>
<td></td>
<td>0.77</td>
<td>0.74</td>
<td>0.73</td>
</tr>
<tr>
<td>54%</td>
<td></td>
<td>0.77</td>
<td>0.74</td>
<td>0.73</td>
</tr>
</tbody>
</table>

From the table we observe that for 45% concentration of whole blood, the maximum plug core radius $r_m$ decreases as the concentration of plug core increases whereas for a fixed plug core concentration, $r_m$ increases as the concentration of whole blood increases.

**Single Phase Flow** –

For a single phase flow model, that is when there is no PPL and plug core regions and the whole flow region is a suspension of Herschel – Bulkley fluid, then the expression for velocity field is obtained as

$$u^{(l)}_{ij} = \frac{R \tau^n}{k(n+1)} \left[ \left( 1 - \frac{\tau_y}{\tau_0} \right)^{n+1} - \left( \frac{r}{R} - \frac{\tau_y}{\tau_0} \right)^{n+1} \right]$$

(3.19)

The volumetric flow rate $Q$ across a cross-section of radius $R$ is given by
\[ Q = \int_{0}^{R} 2\pi r u \, dr \]

or \[ Q = \pi \left( \frac{R}{\tau_{o}} \right)^{3} \int_{0}^{\tau_{o}} \tau^{2} f(\tau) \, d\tau \]  

(3.20)

Thus, for Herschel – Bulkley fluid flow rate \( Q_{H} \) is found as

\[ Q_{H} = \frac{\pi R^{3} \tau_{o}^{n}}{k} \left( 1 - \frac{\tau_{y}}{\tau_{o}} \right)^{n+1} \]

\[ \left[ \frac{1}{n+3} + \frac{2}{(n+2)(n+3)} \left( \frac{\tau_{y}}{\tau_{w}} \right) \right. \]

\[ \left. + \frac{2}{(n+1)(n+2)(n+3)} \left( \frac{\tau_{y}}{\tau_{o}} \right)^{2} \right] \]

(3.21)

From the equation (3.21) we find the apparent viscosity \( \eta_{a}^{(i)} \) as

\[ \frac{1}{\eta_{a}^{(i)}} = \frac{Q}{\pi R^{4} \tau_{o}} \]

or \[ \frac{1}{\eta_{a}^{(i)}} = \left\{ \frac{\tau_{o}^{n-1} \left( 1 - \frac{\tau_{y}}{\tau_{w}} \right)^{n+1}}{k (n+1) (n+2) (n+3)} \right\} \]

\[ \left[ (n+1)(n+2) + 2(n+1) \left( \frac{\tau_{y}}{\tau_{o}} \right) + 2 \left( \frac{\tau_{y}}{\tau_{o}} \right)^{2} \right] \]

(3.22)
Graph – 3.1: Variation of Apparent Viscosity with respect to Non-Newtonian Parameters $k$ and $n$. 
Table – 3.2

Variation of $\eta_a^{(i)}$ for different values of $k$ and $n$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$\eta_a^{(i)}$</th>
<th>$n = 1$</th>
<th>$n = 2$</th>
<th>$n = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.00345</td>
<td>0.01267</td>
<td>0.0259</td>
<td></td>
</tr>
<tr>
<td>0.03</td>
<td>0.00467</td>
<td>0.01901</td>
<td>0.0389</td>
<td></td>
</tr>
<tr>
<td>0.04</td>
<td>0.00632</td>
<td>0.02595</td>
<td>0.0518</td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>0.00778</td>
<td>0.03165</td>
<td>0.0648</td>
<td></td>
</tr>
</tbody>
</table>

From the Table – 3.2 and Graph – 3.1, we observe that apparent viscosity $\eta_a^{(i)}$ increases with respect to $k$ for all values of $n = 1, 2, 3$. At higher value of $n = 3$, the increment is faster in comparison to that obtained at lower value of $n = 1$. At fixed $k$, $\eta_a^{(i)}$ increases with $n$. 
References


