CHAPER V

On the degree of approximation to a function belonging to the class Lip α $(0<\alpha\leq 1)$, by $K^{\lambda}(E,1)$ means of its Fourier series.

5.1 An infinite series $\sum u_n$ with the sequence $\{s_n\}$ if its partial sums is said to be summable (E,1) to a fixed and finite sum s, if the sequence-to-sequence transformation given by

(5.1.1) $$t_n = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} S_k$$

tends to s as $n \to \infty$. [Hardy (1949), p. 19]

Let us define, for n = 0,1,2, .., the numbers $\left[{n \atop m}\right], 0 \leq m \leq n,$ by

(5.1.2) $$\prod_{\nu=0}^{n-1}(x+\nu) = \sum_{m=0}^{n}\left[{n \atop m}\right] x^m,$$

where

(5.1.3) $$\prod_{\nu=0}^{n-1}(x+\nu) = x(x+1)\,(x+2)...(x+n-1)$$

The numbers $\left[{n \atop m}\right]$ are known as the absolute values of Stirling numbers of the first kind.

The sequence-to-sequence transformation given by

(5.1.4) $$S_n^{\lambda} = \frac{\lceil\lambda}{\lceil\lambda+n} \sum_{m=0}^{n}\left[{n \atop m}\right] \lambda^m\, s_m$$

defines the n^{th} k^{λ}-mean of order $\lambda>0$ of the sequence $\{S_n\}$ of partial sums of the infinite series Σu_n.

If

(5.1.5) $S_n^{\lambda} \to S$ as $n \to \infty$,

then the series Σu_n, or the sequence $\{S_n\}$ of its partial sums is said to be summable K^{λ} ($\lambda>0$) to the sum S. [Karamata (1935)].

On superimposing K^{λ}- summability on (E,1), means of the sequence $\{S_n\}$ of partial sums of the series Σu_n, we get K^{λ}(E,1) summability method. Thus, following (5.1.4), we define K^{λ}(E,1) transform σ_n^k by

(5.1.6) $$\sigma_n^{\lambda} = \frac{\lceil\lambda}{\lceil\lambda+n} \sum_{m=0}^{n} \left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right] \lambda^m \sigma_m$$

where σ_m is given by following (5.1.1) as

(5.1.7) $$\sigma_m = 2^{-m} \sum_{k=0}^{m} \left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right] S_k$$

If, now,

(5.1.8) $\sigma_n^{\lambda} \to S,$ as $n \to \infty$

then the series $\sum u_n$, or the sequence $\{S_n\}$ of its partial sums is said to be summable K^{λ}(E,1) to the sum S.

The Fourier series of a 2π-periodic and Lebesgue integrable function f(t) in the interval $(-\pi,\pi)$ is given by

(5.1.9) $$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nt + b_n \sin nt\right) = \sum_{n=0}^{\infty} A_n(t)$$

We write, at a point x,

$$\phi(t) = f(x+t)+f(x-t)-2f(x)$$

$$k_n(t) = \frac{\sqrt{\lambda}}{2\pi\sqrt{\lambda+n}} \sum_{m=0}^{n} \frac{\binom{n}{m} \lambda^m \cos^m t/2 \, \sin(m+1) t/2}{\sin t/2}$$

and

$\tau = \left[\frac{1}{t}\right]$, the integral part of $\frac{1}{t}$; that is the greatest integer, not greater than $\frac{1}{t}$.

Zygmund [1959), p.81] has defined the approximation to functions by trigonometrical polynomials as follows:

Given a periodic and continuous function f(x), the deviation $\delta(f,t)$ of a trigonometrical polynomial T(x) from f is defined by the formula

(5.1.10) $$\delta(f,T) = \max |f(x)-T(x)|$$

The lower bound of the numbers $\delta(f,T)$ for all polynomials

$$T(x) \sim \frac{1}{2} a_0 + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx)$$

of given order n will be denoted by $E_n(f)$ and called the best approximation of order n.

By the theorem of Weirstrass [Zygmund (1959) p.81] $E_n(f)$ tends monotonically to zero as $n \to \infty$.

Weirstrass's theorem reads as follows.

If f is periodic and continuous, then, for every $\in > 0$, there is a trigonometrical polynomial T(x) such that

(5.1.11) $$|f(x)-T(x)| < \in,$$

for all x.

A 2π- periodic and Lebesgue integrable function f(t) in the interval $(-\pi,\pi)$ is said to belong to the class Lip $\alpha(0<\alpha\leq 1)$, if

(5.1.12) $$f(x+t)-f(x)=O(|t|^{\alpha})$$

as $t\to 0$, uniformly with respect to x. [Titchmarsh (1939), p.70].

The degree of approximation of function $f:[-\pi,\pi]\to R$ bya trigonometrical polynomial σ_n of order n is defined by

(5.1.13) $$\|\sigma_n - f\| = \sup_{-\pi\leq x\leq\pi} |\sigma_n(x)-f(x)|$$

5.2 For the function $f\in$ Lip α $(0<\alpha\leq 1)$, the degree of approximation by Cesàro means and by Nörlund means of the Fourier series of the function f have been studied by Alexit (1961), Sahney and Goel (1973), Chandra (1975), Quereshi (1982) and Quereshi and Neha (1990).

Lal and Yadav (2000) have discussed the degree of approximation of conjugates of Lipschitz's function by (C,1) (E,1) mean by proving the following

Theorem A

If $f:R\to R$ is 2π–periodic and Lip α, then the degree of approximation of its conjugate function $\bar{f}$ by (C,1) (E,1) means of conjugate series of f satisfies, for n = 0, 1, 2, 3, …

$$\left\|(CE)_n' - \bar{f}\right\| = \begin{cases} 0\left[\dfrac{1}{(n+1)^{\alpha}}\right], \text{ if } 0<\alpha\langle 1\rangle \\ \hline 0\left[\dfrac{\log_e(n+1)\pi}{n+1}\right], \text{ if } \alpha=1, \end{cases}$$

where $(CE)_n'$ denotes the (C,1) (E,1) means of conjugate series of Fourier series (5.1.9) of the function f $\in$ Lip α, which is given by

$$\sum_{n=1}^{\infty}\left(b_n \cos nx - a_n \sin nx\right) = \sum_{n=1}^{\infty} B_n(x),$$

and a_n, b_n are the Fourier coefficients of the function f.

The methods K^λ were first introduced by Karamata (1935). Lototsky (1953) reintroduced the special case λ=1. Only after the paper of Agnew (1957), an intensive study of those and similar methods took place.

Here, in the present chapter, we have studied the degree of approximation to a function belonging to Lip α $(0<\alpha \leq 1)$, by $k^\lambda(E,1)$ mean of its Fourier series and established the following.

Theorem Let $f \in$ Lip α $(0<\alpha \leq 1)$ be a 2π- periodic and Lebesgue integrable function of t in the interval $(-\pi,\pi)$. Let $\left(K^\lambda E\right)_n^1(x)$ denote the n^{th} $K^\lambda(E,1)$ mean of the Fourier series (5.1.9) of the function f(t) at any point x in the interval $(-\pi,\pi)$. Then the degree of approximation to the function f $\in$ Lip α of its Fourier series (5.1.9) by $K^\lambda(E,1)$ means is given by

$$\left\|\left(K^\lambda E\right)_n'(x) - f(x)\right\| = \begin{cases} O\left[(n+1)^{-\alpha}\right], & \text{if } 0<\alpha<1 \\ O\left[\dfrac{\log_e (n+1)\pi}{n+1}\right], & \text{if } \alpha=1 \end{cases}$$

5.3. Lemma

Let us write

$$k_n(t) = \frac{\Gamma\lambda}{2\pi\,\Gamma(\lambda+n)} \sum_{m=0}^{n} \frac{\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right] \lambda^m \cos^m t/2 \,\sin(m+1)\, t/2}{\sin t/2}$$

Then

$$|k_n(t)| = \begin{cases} O\,(n+1), & \text{for} \quad 0 \le t \le \dfrac{1}{n+1} \\ O\left[\dfrac{1}{(n+1)\,t^2}\right], & \text{for} \quad \dfrac{1}{n+1} \le t \le \pi \end{cases}$$

Proof of the Lemma

For $0 \le t \le \frac{1}{n}$,

$$|k_n(t)| = O\left(\frac{\Gamma\lambda}{\Gamma(\lambda+n)}\right) \sum_{m=0}^{n} \frac{\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]\lambda^m \left|\cos^m t/2\right| \left|\sin(m+1)\, t/2\right|}{\left|\sin t/2\right|}$$

$$= O\left(\frac{\Gamma\lambda}{\Gamma(\lambda+n)}\right) \sum_{m=0}^{n} \frac{\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]\lambda^m (m+1) \left|\sin t/2\right|}{\left|\sin t/2\right|}$$

$$= O\left(\frac{\Gamma\lambda}{\Gamma(\lambda+n)}\right) \sum_{m=0}^{n} \left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]\lambda^m (m+1)$$

$$= O\left[(n+1)\, \frac{\Gamma\lambda}{\Gamma(\lambda+n)} \sum_{m=0}^{n} \left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right] \lambda^m\right]$$

$$= O\,(n+1)$$

and, for $\frac{1}{n+1} \le + \le \pi$

$$|k_n(t)| = O\left[\frac{1}{(n+1)\,t^2}\right]$$

5.4 Proof of the theorem

The n^{th} partial sum $S_n(x)$ of the Fourier series (5.1.9) of the function f(t) at a point x in $(-\pi,\pi)$ is given by

(5.4.1) $$S_n(x)-f(x)=\frac{1}{2\pi}\int_{\infty}^{\pi}\phi(t)\,\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dt$$

The (E,1) transform $E_n^{'}$ of $\{S_n\}$ is given by

$$E_n^1(x)-f(x)=\frac{1}{\pi.2^{n+1}}\int_0^{\pi}\frac{\phi(t)}{\sin t/2}\sum_{k=0}^{n}\binom{n}{k}\sin\left(k+\frac{1}{2}\right)t\,dt$$

$$=\frac{1}{\pi.2^{n+1}}\int_0^{\pi}\frac{\phi(t)}{\sin t/2}\left\{I_m\sum_{k=0}^{n}\binom{n}{k}e^{i\left(k+\frac{1}{2}\right)t}\right\}dt$$

$$=\frac{1}{\pi.2^{n+1}}\int_0^{\pi}\frac{\phi(t)}{\sin t/2}\left\{I_m\,e^{it/2}\sum_{k=0}^{n}\binom{n}{k}e^{ikt}\right\}dt$$

$$=\frac{1}{\pi.2^{n+1}}\int_0^{\pi}\frac{\phi(t)}{\sin t/2}\,I_m\left\{e^{it/2}\left(1+e^{it}\right)^n\right\}dt$$

$$=\frac{1}{\pi.2^{n+1}}\int_0^{\pi}\frac{\phi(t)}{\sin t/2}\,I_m\left\{e^{it/2}\left(2^n\cos^n t/2\right)\left(\cos\frac{nt}{2}+\sin\frac{nt}{2}\right)\right\}dt$$

$$=\frac{1}{2\pi}\int_0^{\pi}\frac{\phi(t)}{\sin t/2}\,I_m\left\{\left(\cos t/2+i\sin t/2\right)\cos^n t/2\right.$$

$$\left.\times\left(\cos\frac{nt}{2}+i\sin\frac{nt}{2}\right)\right\}dt$$

(5.4.2) $$=\frac{1}{2\pi}\int_{\infty}^{\pi}\phi(t)\,\frac{\cos^n t/2\,\sin(n+1)t/2}{\sin t/2}dt$$

Further, K^{λ}-mean of the (E,1) transform E'_n of $\{S_n\}$ is given by

$$\left(K^{\lambda}E\right)_n^1(x)-f(x)=\frac{1}{2\pi}\frac{\Gamma\lambda}{\Gamma\lambda+n}\sum_{m=0}^{n}\left[{n \atop m}\right]\lambda^m\left\{E'_m(x)-f(x)\right\}$$

$$=\frac{1}{2\pi}\frac{\Gamma\lambda}{\Gamma\lambda+n}\int_0^{\pi}\frac{\phi(t)}{\sin t/2}\left[\sum_{m=0}^{n}\left[{n \atop m}\right]\lambda^m\cos^m t/2\sin(m+1)t/2\right]dt$$

(5.4.3) $$=\int_0^{\pi}\phi(t)\,k_n(t),\ \text{say},$$

where

$$k_n(t)=\frac{1}{2\pi}\frac{\Gamma\lambda}{\Gamma\lambda+n}\sum_{m=0}^{n}\frac{\left[{n \atop m}\right]\lambda^m\cos^m t/2\sin(m+1)t/2}{\sin t/2}$$

Let us write

$$I=\int_0^{\pi}\phi(t)\,k_n(t)\,dt$$

$$=\left(\int_0^{1/(n+1)}+\int_{1/(n+1)}^{\pi}\phi(t)\,k_n(t)\,dt\right)$$

(5.4.4) $$=I_1+I_2,\ \text{say}$$

Then, we have

$$\|I\| = \|I_1+I_2\|$$

(5.4.5) $$\le \|I_1\|+\|I_2\|,$$

using Minkowski's inequality.

Firstly, we consider I_1. Now,

$$\|I_1\| = \left|\int_0^{1/(n+1)}\phi(t)\quad k_n(t)dt\right|$$

$$\le \int_0^{1/(n+1)}|\phi(t)|\quad|k_n(t)|dt$$

$$= O(n+1) \int_0^{1/(n+1)} |\phi(t)| dt$$

$$= O(n+1)\, O\left[\int_0^{1/(n+1)} t^{\alpha} dt\right], \quad \text{using (5.1.12)}$$

$$= O(n+1)\, O\left[\frac{t^{\alpha+1}}{\alpha+1}\right]_0^{1/(n+1)}$$

$$= O\left[(n+1).(n+1)^{-(\alpha+1)}\right]$$

(5.4.6) $$= O\left[(n+1)^{-\alpha}\right], \quad \text{as} \quad n \to \infty$$

Again, considering I_2, we have

$$\| I_2 \| = \left| \int_{1/(n+1)}^{\pi} \phi(t) \quad k_n(t) dt \right|$$

$$\leq \int_{1/(n+1)}^{\pi} |\phi(t)| \quad |k_n(t)| dt$$

$$= \int_{1/(n+1)}^{\pi} |\phi(t^{\alpha})| \ O\left[\frac{1}{(n+1)t^2}\right] dt$$

$$= O\left(\frac{1}{n+1}\right) \int_{1/(n+1)}^{\pi} t^{\alpha-2} dt$$

$$= O\left(\frac{1}{n+1}\right) \int_{1/(n+1)}^{\pi} t^{-1} dt, \quad \text{for} \quad \alpha=1$$

$$= O\left(\frac{1}{n+1}\right) [\log t]_{1/(n+1)}^{\pi}$$

$$= O\left(\frac{1}{n+1}\right) \left[\log \pi - \log \frac{1}{n+1}\right]$$

$$= O\left(\frac{1}{n+1}\right) [\log \pi + \log (n+1)]$$

$$= O\left[\frac{\log_e\ (n+1)\ \pi}{n+1}\right], \text{ as } n \to \infty. \tag{5.4.7}$$

Combining (5.4.5), (5.4.6) and (5.4.7) we get

$$\|I\| = \left\|\left(K^{\lambda}E_n^{'}\right)(x) - f(x)\right\| = \begin{cases} O\left[(n+1)^{-\alpha}, \quad \text{if } 0<\alpha<1\right] \\ O\left[\frac{\log_e(n+1)\pi}{n+1}\right], \quad \text{if } \alpha=1 \end{cases}.$$

This completes the proof of our theorem.