CHAPTER – I
INTRODUCTION

1.1 The old hazy notion of convergence of infinite series was placed on sound foundation with the appearance of Cauchy’s monumental work “Course d’Analyse algebrique” in 1821 and Abel’s researches on the Binomial series in 1826. However, it was observed that there were certain non convergent series which, particularly in Dynamical Astronomy, furnished nearly correct results. A theory of divergent series was formulated explicitly for the first time in 1890, when Cesàro published a paper on multiplication of series. Since, then, the theory of series, whose sequence of partial sums oscillates, has been the centre of attraction and fascination for the most of the pioneering mathematical analysts.

After persistent efforts, in which a number of celebrated mathematicians took part, like Frobenius, Cesàro, Holder, Housdorff, Riesz, Nörlund, etc., it was in the closing decade of the last century and in the early years of the century just passed that satisfactory methods were devised so as to associate with them by processes closely connected with Cauchy’s concept of convergence, certain values which may be called their “sums” in a reasonable way. Such processes of summation of series which were formerly tabooed being divergent have given rise to the modern rigorous theory of summability. These processes of associating generalized sums, known as methods of summability (Szasz, (1946) and Hardy, (1949)) provide a natural generalization
of the classical notion of Convergence (Hobson, (1909)) and are thus responsible for bringing within the applicability a wider class of erstwhile rejected series that used to be tabooed as divergent.

The idea of convergence having been thus generalized, it was quite natural to study the possibilities of extending the notion of absolute convergence. As a matter of fact, just as the notion of convergence had led to the development of its extension under the general title of summability, so also by analogy, the concept of absolute convergence led to the formulation of the various processes of absolute summability (Kogbatliantz, (1922)).

The theory of summability in the context of the Fourier series received its first impetus from the fundamental paper of Fejer in (1913), which followed the investigation of Lebesgue in 1903. These memories, kindled a wide interest among analysts in the theory of summability, thought it has already shown its utility in connection with the problem of analytic continuation around 1910 A.D. and the pioneering researches of great mathematical geniuses such as Holder, Cesaro, Housdroff, Borel and others were so spectacular that summability became an independently well established and fashionable branch of modern analysis.

Hardy and Littlewood (1913), for the first time introduced the notion of “strong summability” of Fourier series. Fekate (1916) defined that series $\sum a_n$ is strongly summable to the sum $S$ if

$$\sum_{v=1}^{n} |S_v - S| = o(n)$$
as \( n \to \infty \), where \( S_n \) is the partial sum of the series \( \Sigma a_n \). This type of summability is now known as strong Cesàro summability of order 1 or \([C,1]\) summability.

It is important to note that strong summability is weaker than absolute summability but stronger than ordinary summability.

Lorentz (1948), for the first time, defined almost convergence of bounded sequence \( <S_n> \) of an infinite series \( \Sigma a_n \). The idea of almost convergence led to the formulation of almost summability method.

1.2. Some of the most familiar methods of summability and with which we shall be concerned in the sequel, are those that are known as methods of Nörlund summability, absolute Cesaro summability, ordinary and absolute \( (\bar{N}, P_n) \) summability, strong Perron summability, Almost Euler summability \( (J, p_m, q_n) \) summability, Karamata summability, uniform triangular matrix summability and degree of approximation. It may, however, be mentioned that all these methods can be derived from two basic general processes, which are termed as

(i) \( \hat{\ } \)-process

(ii) \( \pi \)-process

\( \hat{\ } \)-methods are based on the formation of an auxiliary sequence \( \{t_m\} \) defined be the sequence-to-sequence transformation

\[
t_m = \sum_{n=0}^{m} C_{m,n} S_n \quad (m = 0,1,2, \ldots \ldots).
\]
where $S_n$ is the $n^{th}$ partial sum of a given infinite series $\Sigma a_n$. The matrix $^\wedge = (C_{m,n})$, in which $C_{m,n}$ is the element in the $m^{th}$ row and $n^{th}$ column is a Toeplitz matrix. If $t_m$ tends to a fixed and finite limit $S$, as $m \to \infty$ the series $\Sigma a_n$ or the sequence $\{S_n\}$ is said to be summable by the $^\wedge$-process to the sum $S$ and this is written as

$$^\wedge - \Sigma a_n = S$$

The $\phi$-methods are based upon the formation of a functional transform $t(x)$, defined by the sequence to function transformation

$$T(x) = \Sigma \phi_n(x) S_n$$

or more generally, by the integral transformation

$$t(x) = \int_0^\infty \phi(x,y) S(y) \, dy .$$

where $x$ is a continuous parameter and the function $\phi_n(x)$ or $\phi(x,y)$ is defined over a suitable interval of $x$ [ or $x$ and $y$].

An infinite series $\Sigma a_n$, with its $n^{th}$ partial sum $S_n$ is said to tend to a finite limit $S$, if

$$\lim_{n \to \infty} S_n = S.$$ 

By analogy, the series $\Sigma a_n$ is said to be summable by $^\wedge$-method [or $\square$-method to the sum $S$, if

$$\lim_{m \to \infty} t_m = S \quad \text{or} \quad \lim_{x \to a} t(x) = S$$

We know that a series $\Sigma a_n$ with the sequence of $\{S_n\}$ of its partial sums is absolutely convergent, if the sequence $\{S_n\}$ is bounded variation, i.e.
Defining similarly, the absolute summability of an infinite series, we say that the series $\sum a_n$ is said to be absolutely summable by $^\wedge$-method or simply summable $|^\wedge|$, if the corresponding auxiliary sequence $\{t_m\}$ is of bounded variation i.e.

$$\sum_{m=1}^{\infty} |t_m - t_{m-1}| < \infty$$

Absolute summability by $\phi$-method, or summability $|\phi|$, is defined in the same way with the obvious difference, that, in this case, the corresponding function $t(x)$ should be a function of bounded variation in an interval of the continuous parameter $x$.

A summability method $Q$ is said to include another summability method $P$, if the summability of a series (or sequence) by the method $P$ to a certain sum implies its summability to the same sum by the method $Q$ as well.

A method of summability $B$ is said to be absolutely inclusive of another method $A$, if the absolute summability by the method $A$ implies absolute summability by the method $B$ and this fact is symbolically denoted as $|A| \subseteq |B|$. If any two methods of summability are absolutely inclusive of each other, then they are said to be absolutely equivalent.

The sequence to sequence transformation is absolute conservative or absolute convergence preserving transformation, if the absolute convergence of the sequence $\{S_n\}$ implies that of the sequence $\{t_n\}$ in each case, and is said to
be absolutely regular if the transformation is absolute conservative and the limits of $S_n$ and $t_m$ are the same, i.e.

$$\lim_{n \to \infty} S_n = S = \lim_{m \to \infty} t_m$$

Now we give briefly salient details of some of the methods of summability to which we have been referred to and with which we shall be dealing in the present thesis.

1.3. Nörlund Summability

Although the method of summability, considered under this title, was first introduced by Woronoi (1902), it is customary to associate it with the name of Nörlund (1919), who, independently, introduced this method. (a) $(N, p_n)$ Summability

Let $\Sigma a_n$ be a given infinite series with the sequence $\{S_n\}$ of its partial sums. Let $\{p_n\}$ be a sequence of constants, real or complex and let

$$P_n = p_0 + p_1 + p_2 + \ldots + p_n \quad (P_n \neq 0)$$

Then the sequence to sequence transformation.

$$(1.3.1) \quad t_n = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} S_m = \frac{1}{P_n} \sum_{m=0}^{n} p_m S_{n-m}$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{S_n\}$ generated by the sequence of constants $\{p_n\}$. The series $\Sigma a_n$ or the sequence $\{S_n\}$ is said to be summable by Nörlund means or summable $(N, p_n)$ to the sum $S$, if

$$\lim_{n \to \infty} t_n = S$$

and is said to be absolutely summable by Nörlund mean, or, summable $[N, p_n]$, if
\[
\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty
\]

The conditions of regularity of the transformation (1.3.1) are

(i) \( p_n = 0 \) (P_n)

(ii) \( \sum_{m=0}^{n} p_m = O(|p_n|) \), as \( n \to \infty \) \hspace{1cm} (Hardy, 1949)

If \( \{p_n\} \) is real and non-negative, condition (ii) is automatically satisfied and, if in addition, \( \{p_n\} \) is non-increasing, condition (i) is also satisfied.

Mcfadden (1942) has established that absolute Nörlund summability implies the ordinary Nörlund summability but the converse implication is not necessarily true.

1.3 (b) \((N, p^\alpha)\) Summability

Let \( \Sigma u_n \) be an infinite series with the sequence \( \{S_n\} \) of its partial sums. Let \( \{p_n\} \) be a sequence of constants with \( p_0 > 0, p_n > 0, \) for \( n > 0 \). For \( \alpha > -1 \), we define

\[
P_n^{(\alpha)} = \sum_{r=0}^{n} p_r^\alpha
\]

where

\[
P_n = \sum_{r=0}^{n} p_r
\]

The sequence-to-sequence transformation given by

\[
t_n^{(\alpha)} = \frac{1}{P_n^{(\alpha)}} \sum_{r=0}^{n} p_r^\alpha S_r
\]

defines the \( n\)th Nörlund mean of order \( \alpha \), i.e. the \( n\)th \((N, p^\alpha)\) mean of the sequence \( \{S_n\} \) of partial sums of the infinite series \( \Sigma u_n \).
If

(1.3.5) \( t_n^{(\alpha)} \to S, \text{ as } n \to \infty \)

where \( S \) is a fixed and finite sum, then the series \( \Sigma u_n \), or the sequence \( \{S_n\} \) of its partial sums is said to be summable \( (N, p^\alpha_n) \) to the sum \( S \).

(c) \( (N, p, q) \) Summability

Let \( \Sigma a_n \) be an infinite series with \( \{S_n\} \) as the sequence of its partial sums. Let \( \{p_n\} \) and \( \{q_n\} \) be two non-negative sequences with

(1.3.6) \( R_n = \sum_{k=0}^{n} p_{n-k} q_k \)

The \( (N, p, q) \) transform \( t_n^{p,q} \) of the sequence \( \{S_n\} \) is defined by the sequence-to-sequence transformation

(1.3.7) \( t_n^{p,q} = \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k S_k \)

If \( t_n^{p,q} \to S \) as \( n \to \infty \), where \( S \) is fixed and finite quantity, then the series \( \Sigma a_n \) or the sequence \( \{S_n\} \) of its partial sums is said to be summable \( (N, p,q) \) to the sum \( S \). (Borwein, D. (1958)).

For \( p_n = \frac{1}{n+1} \) and \( q_n = 1 \ \forall \ n \), the \( (N, p,q) \) mean reduces to \( \left( N, \frac{1}{n+1} \right) \) mean called as the harmonic mean.
1.4 Cesàro Summability

(a) (C,1) Summability

An infinite series $\sum a_n$, with sequence $\{A_n\}$ of its partial sums, is said to be summable (C,1) to a fixed and finite sum $A$ if the sequence-to-sequence transformation

$$C_n^1 = \frac{1}{n+1} \sum_{\nu=0}^{n} A_\nu \rightarrow A,$$

as $n \rightarrow \infty$ \hspace{1cm} (Titchmarsh, (1939)).

(b) (C, $\alpha$) Summability

Given a sequence $\{S_n\}$ of partial sums of the series $\sum a_n$, let us write

$$\sigma_n^\alpha = \frac{S_n^\alpha}{A_n^\alpha}, \quad \alpha > -1$$

where $S_n^\alpha$ are defined by the formulae

$$\sum_{n=0}^{\infty} S_n^{\alpha}\cdot x^n = (1-x)^{-\alpha} \sum_{n=0}^{\infty} S_n\cdot x^n$$

$$= (1-x)^{-\alpha+1} \sum_{n=0}^{\infty} a_n\cdot x^n$$

and

$$\sum_{n=0}^{\infty} A_n^{\alpha}\cdot x^n = (1-x)^{-\alpha+1}$$

The expression $S_n^\alpha$ and $\sigma_n^\alpha$ are called respectively the Cesàro sums and Cesàro means of order $\alpha$ of the sequence $\{S_n\}$ (or the series $\sum a_n$). The numbers $A_n^\alpha$ are termed as the Cesàro numbers of order $\alpha$. If
where $S$ is a fixed and a finite number, then the series $\Sigma a_n$ is said to be summable $(C, \alpha)$ to the sum $S$. If $\{\sigma_n^\alpha\}$ is simply bounded, then the series $\Sigma a_n$ is said to be bounded $(C, \alpha)$.

If the sequence $\{\sigma_n^\alpha\}$ is of bounded variation, i.e., to say that if

$$\sum |\sigma_n^\alpha - \sigma_{n-1}^\alpha| < \infty,$$

we say that the sequence $\{S_n\}$, or the series $\Sigma a_n$ is absolutely summable $(C,\alpha)$ or summable $|C, \alpha|$, $\alpha > -1$.

From the definition it is evident that the absolute convergence and summability $|C,0|$ are equivalent properties.

To Kogbatliantz (1922) is due, the consistency theorem in its general form for the absolute Cesàro summability which asserts that if $\Sigma a_n$ is summable $|C,\alpha|$, $\alpha > -1$, then it is also summable $|C,\alpha+\delta|$ for every $\delta > 0$.

It is worth noticing that the Cesàro summability $(C, \alpha)$ is a special class of Nörlund summability $(N, p_n)$ for the case when

$$P_n = \binom{n+\alpha-1}{\alpha-1}, \quad \alpha > -1.$$

1.5 Harmonic Summability

An infinite series $\Sigma a_n$, with the sequence $\{S_n\}$ of its partial sums, is said to be summable by harmonic method or simply summable $\left(N, \frac{1}{n+1}\right)$ to the sum $S$, where $S$ is fixed and finite number if the sequence-to-sequence transformation
(1.5.1) \[ t_n = \frac{1}{\log n} \sum_{\nu=0}^{n} \frac{S_{\nu}}{\nu+1} \]

\[ \rightarrow S \text{ as } n \rightarrow \infty. \]

It is obvious that the transformation (1.5.1) is a special case of the transformation (1.3.1) for \( p_n = \frac{1}{n+1}, \forall n. \)

1.6 Euler Summability

Let \( \sum a_n \) be an infinite series with the sequence \( \{S_n\} \) of its partial sums and let \( q > 0 \). Let us write

(1.6.1) \[ t^n_q = \frac{1}{(q+1)^n} \sum_{k=0}^{n} q^{n-k} S_k \] (Hardy, (1949))

If \( t^n_q \rightarrow S \text{ as } n \rightarrow \infty \), then we say that the series \( \sum a_n \) is summable by Euler method of order \( q \) or simply summable \( (E,q) \) to the sum \( S \).

For \( q = 1 \), this method reduces to the summability \( (E, 1) \) method of summation which is given by

(1.6.2) \[ t_n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} S_k \]

It is well known that if \( \{S_n\} \) is summable by \( (E, 1) \) method, it is summable by Borel’s exponential method to the same value.

1.7 Strong Summability

(a) Strong \( (E, 1) \) Summability

Following the definition of strong \( (C,1) \)summability given by Hardy and Littlewood (1935), we define that the series \( \sum a_n \), the sequence \( \{S_n\} \) of its partial sums, is strongly summable \( (E,1) \) to the sum \( A \), if
\begin{equation}
\sum_{k=0}^{n} \left( \binom{n}{k} \right) |S_k - S| = 0(2^n)
\end{equation}

as \( n \to \infty \).

(b) Strong \((E,q)\) Summability

The series \( \Sigma a_n \), or the sequence \{\( S_n \)\} of its partial sums, is strongly summable \((E,q)\) to the sum \( A \), if

\begin{equation}
\sum_{k=0}^{n} \left( \frac{n}{k} \right) q^{n-k} |S_k - S| = 0[(q+1)^n]
\end{equation}

as \( n \to \infty \).

(c) Strong \((E, q)_{k}\) Summability

The series \( \Sigma a_n \), or the sequence \{\( S_n \)\} of its partial sums, is said to be strongly summable \((E, q)\) with index \( k > 0 \), or simply summable \((E, q)_{k}\) to the sum \( S \), if

\begin{equation}
\sum_{m=0}^{n} \binom{n}{m} q^{n-m} |S_n - S|^k = 0\left[ (q+1)^n \right]
\end{equation}

as \( n \to \infty \).

1.8 (a) \((N, p_n)\) \((C,1)\) Summability

Nörlund summability \((N, p_n)\) has been defined in section 1.3 by the transformation \((1.3.1)\) and \((C,1)\) summability has been defined in Section 1.4 (a) by \((1.4.1)\).

Now superimposing \((N, p_n)\) summability on the \((C, 1)\) mean \( t_n \) of the series \( \Sigma a_n \), we get \((N, p_n)\) \((C, 1)\) summability method of the series \( \Sigma a_n \) where the \( n \)th \((N, p_n)\) \((C,1)\) mean \( \sigma_n \) of the sequence \{\( A_n \)\} of partial sums of the series \( \Sigma a_n \) is defined by sequence-to-sequence transformation
(1.8.1) \[ \sigma_n = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} t_m. \]

If \( \sigma_n \) tends to \( A \) as \( n \to \infty \) then the series \( \Sigma a_n \) is said to be summable \((N, p_n)\) \((C,1)\) to the sum \( A \).

(b) \( \left( N, \frac{1}{n+1} \right) \) \((C,1)\) Summability

Harmonic summability \( \left( N, \frac{1}{n+1} \right) \) has been defined in section 1.5 by (1.5.1), which when superimposed on \((C,1)\) summability defined by (1.4.1) yields to \( \left( N, \frac{1}{n+1} \right) \) \((C,1)\) summability method, being defined by sequence to sequence transformation

(1.8.2) \[ \sigma_n = \frac{1}{\log n} \sum_{\nu=0}^{n} \frac{t_\nu}{n-\nu+1} \]

where \( \sigma_n \) denotes \( n^{th} \) \( \left( N, \frac{1}{n+1} \right) \) \((C,1)\) mean of the series \( \Sigma a_n \) whose \( \nu^{th} \) \((C,1)\) mean is denoted by \( t_\nu \). If \( \sigma_n \to S \) as \( n \to \infty \) then the series \( \Sigma a_n \) is said to be summable \( \left( N, \frac{1}{n+1} \right) \) \((C,1)\) to the sum \( S \).

(c) \((E,1)\) \((C,1)\) Summability

\((E,1)\) summability has been defined by (1.6.2) and \((C,1)\) summability has been defined in 1.4 (a).

Now superimposing \((E,1)\) summability on the \((C,1)\) mean \( C^1_n \) of the series \( \Sigma a_n \), we get \((E,1)\) \((C,1)\) summability method of the series \( \Sigma a_n \), where
the \( n^{\text{th}} \) \((E,1)\) \((C,1)\) mean \((E,C)^1_n\) of the sequence \(\{A_n\}\) of partial sums of the series \(\sum a_n\) is defined by the sequence to sequence transformation.

\[
(1.8.3) \quad (E,C)^1_n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} C_k^1
\]

If \((E,C)^1_n\) tends to \(A\) as \(n \to \infty\) then the series \(\sum a_n\) is said to be summable \((E,1)\) \((C,1)\) to the sum \(A\).

\textbf{(d) \((E,q)\) \((C,1)\) Summability}

\((E,q)\) summability has been defined in section 1.6 by \((1.6.1)\) and \((C,1)\) summability has been defined in 1.4 (a).

Now superimposing \((E, q)\) summability on the \((C, 1)\) mean of the series \(\sum a_n\), we get \((E, q)\) \((C, 1)\) summability method of the series \(\sum a_n\) where the \(n^{\text{th}}\) \((E,q)\) \((C,1)\) mean \((E_q,C)^1_n\) of the sequence \(\{A_n\}\) of partial sums of the series \(\sum a_n\) is defined by the sequence-to-sequence transformation

\[
(1.8.4) \quad (E_q,C)^1_n = \frac{1}{(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} C_k^1.
\]

If \((E_q,C)^1_n \to A\) as \(n \to \infty\), then the series \(\sum a_n\) is said to be summable \((E,q)\) \((C,1)\) to the sum \(A\).

\textbf{(e) \(T\) \((C, 1)\) Summability}

Let \(\{S_n\}\) denote the sequence of partial sums of a given infinite series \(\sum u_n\). Let \(\sigma_n\) denote the \(n^{\text{th}}\) \((C,1)\) mean of the series \(\sum u_n\) which is given by sequence-to-sequence transformation

\[
(1.8.5) \quad \sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} S_k \quad [\text{Titchmarsh (1939), p. 146}].
\]
Let $T = (A_{n,k})$ be an infinite triangular matrix with intrices $a_{n,k}$ over reals or complexes and $a_{n,k} = 0$, $k > n$.

The sequence-to-sequence transformation given by

$$(1.8.6) \quad T_n = \sum_{k=0}^{n} a_{n,k} \sigma_k$$

defines the $n$th matrix mean or the $n$th $T$ mean of the sequence $\{\sigma_n\}$ of (C,1) means of the sequence $\{S_n\}$ of partial sums of the series $\sum u_n$ or the $n$th $T$ (C, 1) mean of the series $\sum u_n$.

If $T_n \to S$ as $n \to \infty$, where $S$ is something fixed and finite, then the series $\sum u_n$ or the sequence $\{S_n\}$ of its partial sums is said to be summable $T(C,1)$ to the sum $S$.

The regularity conditions for $T$-summability method are given as follows (Hardy, (1949)).

$$(1.8.7) \quad \lim_{n \to 0} a_{n,k} = 0 \text{ for every fixed } k$$

$$(1.8.8) \quad \text{(ii)} \quad \sum_{k=0}^{n} |a_{n,k}| \leq M,$$

where $M$ is an absolute constant, independent of $n$ and

$$(1.8.9) \quad \text{(iii)} \quad \lim_{n \to \infty} \sum_{k=0}^{n} |a_{n,k}| = 1$$

These regularity conditions are also termed as Silverman Toeplitz’s conditions.

Following the definition of Strong (C,1) summability, given by Hardy and Littlewood (1935), we define that the series $\sum a_n$, or the sequence $\{A_n\}$ of its partial sums, is strongly summable $(E,q)$ (C,1) to the sum $A$, if
\[ \sum_{k=0}^{n} \binom{n}{k} q^{n-k} |C_k - A| = 0[(q+1)] \]

as \( q = 1 \), the series \( \sum a_n \), or the sequence \( \{A_n\} \) of its partial sums is strongly summable (E,1) (C,1) to the sum \( A \), if

\[ \sum_{k=0}^{n} \binom{n}{k} |C_k - A| = 0\left[2^n\right] \quad \text{as} \quad n \to \infty \]

1.9 (a) (C,1) (E,1) Summability

(C,1) summability has been defined in section 1.4(a) and (E,1) summability has been defined in (1.6.2).

The (C,1) transform \( T_n^i \) of the (E,1) transform \( t_n \) of the sequence \( \{S_n\} \) of partial sums of the infinite series \( \sum u_n \) i.e. the (C,1) (E,1) transform of \( \{S_n\} \) is given by

\[ T_n^i = \frac{1}{n+1} \sum_{k=0}^{n} t_k \]

\[ = \frac{1}{n+1} \sum_{k=0}^{n} \left\{ \frac{1}{2^k} \sum_{m=0}^{k} \binom{k}{m} S_m \right\} \]

If \( T_n^i \to S \) as \( n \to \infty \), then the series \( \sum u_n \) or the sequence \( \{S_n\} \) of its partial sums is said to be summable (C,1) (E,1) to the sum \( S \).

(b) (N, p, q) (E, 1) Summability

(N, p, q) summability has been defined in section 1.3 (b).

The generalized (N, p, q) transform \( t_n \) of the sequence \( \{S_n\} \) of partial sums of the infinite series \( \sum u_n \) i.e. the (N, p, q) (E, 1) transform of \( \{S_n\} \) is given by following Borwein (1958), as
\[ T_{n}^{p,q} = \frac{1}{R_{n}} \sum_{k=0}^{n} p_{n-k} q_{k} t_{k} \]

\[ = \frac{1}{R_{n}} \sum_{k=0}^{n} p_{n-k} q_{k} \left\{ \frac{1}{2^{k}} \sum_{m=0}^{k} \binom{k}{m} S_{m} \right\} \]

(1.9.2)

If \( T_{n}^{p,q} \to S \) as \( n \to \infty \), then the series \( \sum u_{n} \) or the sequence \( \{S_{n}\} \) of its partial sums is said to be summable \((N, p, q) (E, 1)\) to the sum \( S \).

(c) \( T(E, 1) \) Summability

The \((E, 1)\) summability method has been defined by (1.6.2) and \( T \) summability has been defined in (1.8.6). Superimposing \( T \) method on \((E, 1)\) method, we obtain \( T(E, 1) \) method given by the sequence-to-sequence transformation

\[ T_{n} = \sum_{k=0}^{\infty} a_{n,k} t_{k}, \]

(1.9.3)

where \( T_{n} \) denotes the \( n \)th \( T(E, 1) \) mean of the series \( \Sigma a_{n} \) and \( t_{k} \) is the \( k \)th \((E, 1)\) transform of the sequence \( \{S_{n}\} \) of partial sums of the series \( \Sigma a_{n} \). If \( T_{n} \to S \) as \( n \to \infty \),

where \( S \) is fixed and finite, then the series \( \sum a_{n} \), or the sequence \( \{S_{n}\} \) of its partial sums, is said to be summable \( T(E, 1) \) to the sum \( S \).

1.10 Borel Summability

Let \( \sum_{n=0}^{\infty} a_{n} \) be an infinite series with the sequence of partial sums \( \{S_{n}\} \). If

\[ \lim_{r \to 0^{+}} e^{-r} \sum_{n=0}^{\infty} \frac{r^{n}}{n!} S_{n} \]
exists and is equal to a finite limit $S$ then the series $\Sigma a_n$ or the sequence $\{S_n\}$ is said to be summable by Borel’s exponential method or simply summable (B) to the sum $S$.

1.11 Logarithmic Summability

Let $\Sigma a_n$ be an infinite series and $\{S_n\}$ the sequence of its partial sums. Then the sequence-to-function transformation

$$L(x) = \frac{1}{\log \left( \frac{1}{1-x} \right)} \sum_{n=1}^{\infty} \frac{S_n x^n}{n!}$$

Defines the logarithmic mean of the sequence $\{S_n\}$. If

$$L(x) \rightarrow S \quad as \quad x \rightarrow \infty$$

in the open interval $(0,1)$, where $S$ is fixed and finite limit, then the series $\Sigma a_n$ is said to be summable by logarithmic method, or simply summable (L) to the sum $S$ (Borwein, 1958)).

If $L(x)$ is of bounded variation in $(\delta, 1)$ for some $\delta$, satisfying $0<\delta<1$, then the sequence $\{S_n\}$ is said to be absolutely summable (L) or simply summable $|L|$ (Mohanty and Patnaik, 1968).

1.12 Perron Summability

Let $\Sigma a_n$ be an infinite series and $\{S_n\}$ the sequence of its partial sums. Let $\{p_n\}$ be a sequence of non-negative numbers and let us suppose that the series
(1.12.1) \[ p(x) = \sum_{n=1}^{\infty} p_n x^n \]

Converges for all \( x \) in \( 0 < x < 1 \) and

(1.12.2) \[ p(x) \uparrow \infty \text{ as } x \uparrow 1. \]

If

(1.12.3) \[ R(x) = \frac{1}{p(x)} \sum_{n=1}^{\infty} p_n S_n x^n \]

tends to a fixed and finite limit \( S \) as \( x \uparrow 1 \), then the series \( \Sigma a_n \), or the sequence \( \{ S_n \} \) of its partial sums, is said to be Perron summable or simply summable (P) to the sum \( S \) (Izumi & Izumi, (1970)).

For \( p_n = \frac{1}{n} \), Perron summability (P) reduces to the logarithmic summability (L) (Borwein, (1958)).

Again if \( R(x) \) is a bounded variation in \((\delta, 1)\) for some \( \delta \) satisfying \( 0 < \delta < 1 \), then the series \( \Sigma a_n \) is said to be absolutely Perron summable or simply summable \(|P|\) (Izumi & Izumi, (1970)).

1.13 (a) \( (J, p_n) \) Summability

Let \( p_n > 0 \) be such that the series \( \sum_{n=0}^{\infty} p_n \) diverges and radius of convergence of the power series.

(1.13.1) \[ p(x) = \sum_{n=0}^{\infty} p_n x^n \]
An infinite series $\sum a_n$, with the sequence of its partial sums $\{S_n\}$, is said to be summable $(J, p_n)$ to a fixed and finite sum $S$, if the series

$$(1.13.2) \quad p(x) = \sum_{n=0}^{\infty} p_n S_n x^n$$

converges in the right open interval $[0,1)$ and if

$$J(x) = \frac{p(x)}{\log(1-x)} = \frac{1}{p(x)} \sum_{n=0}^{\infty} p_n S_n x^n$$

$\to S$ as $x \to 1$-0 or $x \to 1$

For $p_n = \frac{1}{n+1}$, $(J, p_n)$ summability reduces to $(L)$ summability, called Logarithmic summability.

### 1.13 (b) $(J, p_m, q_n)$ Summability

We want to apply the above method of summability to a double series $\sum a_{m,n}$ whose $(m, n)^{th}$ partial sum is $S_{m,n}$. As $\{p_n\}$ taken above, so also, let us consider $\{q_n\}$. The series $\sum a_{m,n}$ or the sequence $\{S_{m,n}\}$ is said to be summable $(J, p_m, q_n)$ to a finite sum $S$, if

$$J_{m,n}(x,y) = \frac{1}{p(x)q(y)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_m q_n S_{m,n} x^m y^n$$

tends to $S$ as $x,y \to 1$-0.

### 1.14 (a) Triangular Matrix Summability

Let $\{S_n\}$ be the sequence of $n^{th}$ partial sums of an infinite series $\sum a_n$ and let $(\lambda_{n,k})$ $(n=0,1,..., \ k=0,1,..., \ n, \ \lambda_{n,0} = 1)$ be a triangular matrix of real or complex numbers. If
(1.14.1) \[ t_n = \sum_{v=0}^{n} \lambda_{n,v} a_v = \sum_{v=0}^{n} \Delta \lambda_{n,v} S_v, \]

where \[ \Delta \lambda_{n,m} = \lambda_{n,m} - \lambda_{n,m+1} \]

and \[ \Delta^2 \lambda_{n,m} = \Delta \lambda_{n,m} - \Delta \lambda_{n,m+1}; \]

tends to a finite limit \( S \) as \( n \to \infty \) then the series \( \Sigma a_n \), or the sequence \( \{S_n\} \) of its partial sums, is said to be summable by triangular matrix method, or simply summable \( (\wedge) \) to the sum \( S \) (Hardy, (1949), p. 42).

The necessary and sufficient conditions for the regularity of the summability \( (\wedge) \) are that

(i) There exist a constant \( M \) such that

\[ \sum_{v=0}^{\infty} |\Delta \lambda_{n,v}| < M, \] for every \( n \)

(ii) For every \( v \), \( \lim_{n \to \infty} \Delta \lambda_{n,v} = 0 \) and

(iii) \( \lim_{n \to \infty} \sum_{v=0}^{\infty} \Delta \lambda_{n,v} = 1 \)

1.14 (b) Uniform Triangular Matrix Summability

Let \( (\lambda_{n,m}) \) be a triangular matrix as defined in section (1.14(a)). An infinite series \( \sum_{n=0}^{\infty} u_n(x) \) with the sequence \( \{S_n(x)\} \) of partial sums is said to be summable \( (\wedge) \) to \( S(x) \) at a point \( x \), if the sequence-to-sequence transformation

(1.14.2) \[ \sigma_n(x) = \sum_{k=0}^{n} \lambda_{n,k} u_k(x) = \sum_{k=0}^{n} \Delta \lambda_{n,k} S_k(x) \]

where \( \Delta \lambda_{n,k} \) and \( \Delta^2 \lambda_{n,k} \) are as defined in section (1.14(a)), tends to a finite limit \( S(x) \) as \( n \to \infty \). If
as \( n \to \infty \), uniformly in a set \( E \) in which \( S(x) \) is bounded, then we say that the series \( \Sigma u_n(x) \) is summable \((\wedge)\) uniformly in the set \( E \) to the sum \( S(x) \).

(c) Following are some particular cases of triangular matrix summability

(i) If

\[
\Delta \lambda_{n,k} = \begin{cases} \frac{p_{n-k}}{p_n}, & k \leq n \\ 0, & k > n \end{cases}
\]

Then \( t_n \) in (1.14.1) \([or \sigma_n \text{ in (1.14.2)}]\) defines the \( n^{th} \) Nörlund mean or the \((N,p_n)\) mean of the sequence \( \{S_n\} \) \([or \{S_n(x)\}]\) generated by the sequence of coefficients \( \{p_n\} \).

(ii) If

\[
\Delta \lambda_{n,k} = \begin{cases} \left(\frac{n-k+\alpha-1}{\alpha-1}\right)^{\alpha}, & \alpha > 0 \text{ for } k \leq n \\ \left(\frac{n+\alpha}{\alpha}\right)^{\alpha}, & \text{for } k > n \end{cases}
\]

then the \( t_n\)-mean \([or \sigma_n\text{-mean}]\) implies the \((C, \alpha)\) mean, which is well-known Cesàro mean of order \( \alpha > 0 \), the sequence \( \{S_n\} \) \([or \{S_n(x)\}]\).

(iii) If

\[
\Delta \lambda_{n,k} = \begin{cases} \left[\left(\frac{n+k-1}{\log n}\right)^{-1}\right], & k \leq n \\ 0, & k > n \end{cases}
\]

then the \( t_n\)-mean \([or \sigma_n\text{-mean}]\) is the same as the harmonic mean \( \left(\frac{N}{n+1}\right) \) of the sequence \( \{S_n\} \) \([or \{S_n(x)\}]\).

1.15 Approximation to functions by Trigonometric Polynomials
Zygmund (1959) has defined the approximation to functions by trigonometric polynomials as follows.

Given a periodic and continuous function \( f(x) \), the deviation \( \delta (f,T) \) of a trigonometric polynomial \( T(x) \) from \( f \) is defined by the formula

\[
\delta (f,T) = \max |f(x) - T(x)|
\]

The lower bound of the numbers \( \delta (f,T) \), for all polynomials of given order \( n \) will be denoted by \( E_n(f) \) and called the best approximation of order \( n \).

By the theorem of Weierstrass \( E_n(f) \) tends to monotonically to zero as \( n \to \infty \). Weierstrass’s theorem read as follows.

If \( f \) is periodic and continuous, then for every \( \varepsilon > 0 \) there is a trigonometric polynomial \( T(x) \) such that

\[
|f(x) - T(x)| < \varepsilon \quad \text{for all } x
\]

1.16 **Riesz Logarithmic Summability**

Let \( \Sigma a_n \) be an infinite series with \( \{S_n\} \) as the sequence of its partial sums. If the sequence-to-sequence transformation defined by

\[
t_n = \frac{1}{\log n} \sum_{v=0}^{n} \frac{S_v}{v + 1}
\]

tends to a fixed and finite limit \( S \) as \( n \to \infty \), then the series \( \Sigma a_n \) or the sequence \( \{S_n\} \) is said to be summable by the method of first logarithmic mean or Riesz
logarithmic means or simply summable $(R, \log n, 1)$ to the sum $S$. $t_n$ is called the $n^{th}$ $(R, \log n, 1)$ mean of the series $\Sigma a_n$ or the sequence $\{S_n\}$.

The series $\Sigma a_n$ or the sequence $\{S_n\}$ is said to be absolutely summable by the method of Riesz Logarithmic mean or simply summable $|R, \log n, 1|$ if

$$ (1.16.2) \quad \sum_{n=1}^{m} \Delta \left( \frac{1}{\log(n+1)} \right) \left| \sum_{v=1}^{n} \log(v+1) a_v \right| = O(1) $$

as $n \to \infty$.

1.17 $(\overline{N}, p_n)$ Summability

Let $\{p_n\}$ be a sequence of real constants with $p_0 > 0$, $p_n \geq 0$ and

$$ (1.17.1) \quad R_n = \sum_{v=0}^{n} p_v \neq 0 $$

such that $R_n \to \infty$ as $n \to \infty$.

An infinite series $\Sigma a_n$, with the sequence $\{S_n\}$ of its partial sums, is said to be summable $(\overline{N}, p_n)$ to the sum $S$ if the sequence-to-sequence transformation

$$ (1.17.2) \quad t_n = \frac{1}{R_n} \sum_{v=0}^{n} p_v S_v $$

called as discontinuous Riesz mean (1924) tends to a fixed and finite limit $S$ as $n \to \infty$.

The transformation (1.17.2) is obtained after the use of the so called Hadamard type composition of the two sequences $\{S_n\}$ and $\{p_n\}$. The method $(\overline{N}, p_n)$ is regular.

The series $\Sigma a_n$ is said to be summable $|\overline{N}, p_n|$ if
(1.17.3) \[ \sum_{n=1}^{m} \Delta \left( \frac{1}{R_n} \right) \sum_{v=1}^{n} R_v a_v = 0(1) \]

as \( m \to \infty \).

For \( p_n = \frac{1}{n+1} \), the transformation (1.17.2) reduces to the transformation (1.16.1) where \( \log n \) is an asymptotic approximation of \( R_n \). In fact

\[ R_n - \log n \to C \text{ (Euler's constant)} \]

as \( n \to \infty \).

1.18. \( K^\lambda \) - Summability

Let us define, for \( n = 0, 1, 2, \ldots \), the numbers \( \left[ \begin{array}{c} n \\ m \end{array} \right] \) for \( 0 \leq m \leq n \) by

\[ \prod_{\nu=0}^{n-1} (x + \nu) = \sum_{m=0}^{n} \left[ \begin{array}{c} n \\ m \end{array} \right] x^m \]

where

\[ \prod_{\nu=0}^{n-1} (x + \nu) = x(x+1)(x+2)\ldots(x+n-1) \]

The numbers \( \left[ \begin{array}{c} n \\ m \end{array} \right] \) are known as the absolute values of the Sterlings numbers of the first kind.

Let \( \{S_n\} \) denote the sequence of partial sums of an infinite series \( \Sigma a_n \) and let

\[ S_n^\lambda = \frac{\Gamma \lambda}{\Gamma(\lambda + n)} \sum_{m=0}^{n} \left[ \begin{array}{c} n \\ m \end{array} \right] \lambda^m S_m \]

denote the \( n^{th} \) \( K^\lambda \) - mean of order \( \lambda > 0 \). If

\[ S_n^\lambda \to S \]
and \( n \to \infty \), where \( S \) is a fixed and finite quantity, then the sequence \( \{S_n\} \) or the series \( \sum a_n \) is said to be summable by Karamata method \( K^\lambda \) of order \( \lambda > 0 \) to the sum \( S \). The methods \( K^\lambda \) are regular for \( \lambda > 0 \).

The methods \( K^\lambda \) were first introduced by Karamata (1935). Lototsky (1953) reintroduced the special case \( \lambda = 1 \). Only after the paper of Agnew (1957), an intensive study of those and similar methods took place.

### 1.19 \( \bar{N}, p_n \) Summability

Let \( \{p_n\} \) be a sequence of positive real constants with \( P_n \) as its non-vanishing \( n^{th} \) partial sum. The sequence-to-sequence transformation given by

\[
\sigma_n = \frac{1}{P_n} \sum_{m=0}^{n} p_m \cdot S_m
\]

where \( \{S_n\} \) is the sequence of partial sums of an infinite series \( \sum a_n \) defines the \( n^{th} \) \( \bar{N}, p_n \) mean of the sequence \( \{S_n\} \) or the series \( \sum a_n \). If

\[
\sigma_n \to S \quad \text{as} \quad n \to \infty
\]

where \( S \) is fixed and finite quantity, then the series \( \sum a_n \) or the sequence of its partial sums is said to be summable \( [\bar{N}, p_n] \) to the sum \( S \) (Hardy, 1949). The method \( [\bar{N}, p_n] \) is regular (Hardy (1945)).

For \( p_n = 1 \) for every \( n \), the \( [\bar{N}, p_n] \) mean reduces to \( (C,1) \) mean.

Following Bor (1985), the series \( \sum a_n \) is said to be summable \( [\bar{N}, p_n]_k \), \( k \geq 1 \), if
(1.19.3) \[ \sum_{n=1}^{\infty} \left( \frac{P_n}{P_{n+1}} \right)^{k-1} |\sigma_n - \sigma_{n-1}|^k < \infty \]

If we denote the \( n \)th mean of the sequence \( \frac{P_n a_n}{p_n} \) by \( T_n \) then the condition (1.19.3) may be written as

(1.19.4) \[ \sum_{n=1}^{\infty} \frac{P_n |T_n|^k}{P_n} < \infty \]

For \( p_n = 1, \ \forall \ n \). So that

\[ P_n = p_0 + p_1 + p_2 + \ldots + p_n = n + 1 \]

\( |N, p_n| \) summability reduces to \(|C, 1|_k \) summability.

1.20 \( \ell_2 \) Summability

Let \( \Sigma a_n \) be a given infinite series with the sequence \( \{S_n\} \) of its partial sums.

Let us write

(1.20.1) \[ C_n = C_n (S_n) = \sum_{v=1}^{n} \frac{S_v}{v} \]

and

(1.20.2) \[ \tau_n = \frac{C_n (S_n)}{C_n (1)} \]

where

(1.20.3) \[ C_n (1) = \sum_{v=1}^{n} \frac{1}{v} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \sim \log n \]

We further write
(1.20.4) \[ C_n^2 = C_n^2 \left( S_n \right) = \sum_{v=1}^{n} \frac{C_v}{v} \]

and

(1.20.5) \[ \tau_n^2 = \frac{C_n^2 \left( S_n \right)}{C_n^2 \left( 1 \right)} \]

If

(1.20.6) \[ \lim_{n \to \infty} \tau_n = S \]

We say that the series is summable (\( \ell \)) to \( S \) and if

\[ \lim_{n \to \infty} \tau_n^2 = S \]

Then the series is summable (\( \ell_2 \)) to \( S \).

### 1.21 **K\(^3\)(E,1) Summability**

An infinite series \( \sum u_n \), with the sequence \( \{ S_n \} \) of its partial sums, is said to be summable (E, 1) to a fixed and finite sum \( S \), if the sequence to sequence transformation given by

(1.21.1) \[ t_n = 2^{-n} \sum_{k=1}^{n} \binom{n}{k} S_k \]

tends to \( S \) as \( n \to \infty \) [Hardy (1949)].

Let us define for \( n = 1,2,3, \ldots \) The numbers \( \left[ \binom{n}{m} \right] \), \( 0 \leq m \leq n \), by

(1.21.2) \[ \prod_{v=0}^{n-1} \left( x + v \right) = \sum_{m=0}^{n} \binom{n}{m} x^m \]

where

(1.21.3) \[ \prod_{v=0}^{n-1} \left( x + v \right) = x (x+1) (x+2) \ldots \ldots (x+n-1) \]
The numbers \[ \begin{bmatrix} n \\ m \end{bmatrix} \] are known as the absolute values of the Stirling numbers of the first kind. The sequence to sequence transformation given by

\[
S_n^\lambda = \frac{\Gamma(\lambda)}{\Gamma(\lambda + n)} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \lambda^m S_m
\]

(1.21.4)

Defines the \( n \)th mean of order \( \lambda > 0 \), of the sequence \( \{S_n\} \) of partial sums of the series \( \Sigma u_n \). If

\[
S_n^\lambda \to S, \quad \text{as} \quad n \to \infty
\]

then the series \( \Sigma u_n \), or the sequence \( \{S_n\} \) of its partial sums, is said to be summable \( K^\lambda (\lambda > 0) \) to the sum \( S \) [Karamata (1935)].

On superimposing \( K^\lambda \) summability over \((E,1)\) means of the sequence \( \{S_n\} \) of partial sums of the series \( \Sigma u_n \) we get \( K^\lambda (E, 1) \) summability method for the series \( \Sigma u_n \).

**1.22 Lipschitz’s Condition**

A \( 2\pi \) periodic function \( f \) is said to belong \( \text{Lip} (\alpha, p) \) for \( 0 < \alpha \leq 1 \) and \( p > 1 \), if

\[
\left( \int_0^{2\pi} \left| f(x + t) - f(x) \right|^p \, dx \right)^{1/p} = O(\left| t \right|^\alpha)
\]

(1.22.1)

We define the norm of a function \( f \) w.r.t. \( p \geq 1 \) by

\[
\|f\|_p = \left( \int_0^{2\pi} |f(x)|^p \, dx \right)^{1/p}
\]

(1.22.2)

The degree of approximation of \( f \) is defined by

\[
E_n(f) = \min \| f - T_n \|_p
\]

(1.22.3)
where $T_n(x)$ is a trigonometric polynomial of degree $n$ (Zygmund, 1959)).

1.23 Lip $(\xi(t), p)$ Condition

A $2\pi$-periodic and Lebesgue integrable function $f$ in the interval $(-\pi, \pi)$ is said to belong to the class Lip $(\xi(t), p)$, $1 < p < \infty$, if

$$
\left[ \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x+t) - f(x)|^p \, dx \right]^{1/p} = O[\xi(t)],
$$

where $\xi(t)$ is a positive monotonic increasing function of $t$ (McFadden 1942)).

If we put $\xi(t) = t^\alpha (0 < \alpha \leq 1)$, then $f \in$ Lip $(\alpha, p)$ (McFadden 1942)).

Further, $f$ is said to belong to the class Lip $\alpha$ ($0 < \alpha \geq 1$) if

$$
f(x+t) - f(x) = O(|t|^\alpha), \quad \text{as } t \to 0,
$$

uniformly with respect to $x$ (Titchmarsh 1939)).

Thus Lip $(\xi(t), p)$ is a general case of Lip $(\alpha, p)$ and Lip $(\alpha, p)$.

1.24 Fourier Series and its Derived Series

Let $f(t)$ be a $2\pi$-periodic and Lebesgue integrable function of $t$ in the interval $(-\pi, \pi)$ and then periodically extended beyond this interval to the left and to the right so as to satisfy the functional equation $f(x \pm 2\pi) = f(x)$. Then the Fourier series corresponding to the function $f(t)$ is defined by the correspondence.

$$
(1.24.1) \quad f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nt + b_n \sin nt \right)
$$

$$
= \sum_{n=0}^{\infty} A_n(t)
$$

where the coefficients $a_0$, $a_n$ and $b_n$ are known as Fourier coefficients given by
(1.24.2) \[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt \]

(1.24.3) \[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \ dt \]

(1.24.4) \[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \ dt \]

The notation in (1.24.1) is due to Hurtwitz (1903) and the formulae (1.24.3) and (1.24.4) are known as Euler Fourier formulae for the coefficients.

The series

(1.24.5) \[ \sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} nB_n(t) \]

which is obtained on differentiating the series (1.24.1) term by term is called the first differentiated series or the derived series of the Fourier series (1.24.1).

The series

(1.24.6) \[ \sum_{n=1}^{\infty}(b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t) \]

is called the conjugate series of Fourier series (1.24.1).

### 1.25 Legendre Series

The Legendre series associated with a Lebesgue integrable function \( f(x) \) in the range (-1,1) is given by

(1.25.1) \[ f(x) \sim \sum_{n=1}^{\infty} a_n P_n(x) \]

where

(1.25.2) \[ a_n = \left( n + \frac{1}{2} \right) \frac{1}{2} \int_{-1}^{1} f(x) P_n(x) dx \]
and the \( n \)th Legendre Polynomial \( P_n(x) \) which is orthogonal in the interval (-1,1) is defined by the generating function

\[
(1.25.3) \quad \frac{1}{\sqrt{1-2xz + z^2}} = \sum_{n=0}^{\infty} P_n(x) z^n
\]

We use the following notations.

\[
(1.25.4) \quad \psi(t) = \psi_0(t) = f[\cos (\theta-t)] - f(\cos \theta)
\]

and

\[
N_n(t) = \sum_{k=0}^{\infty} \Delta \lambda_{n,k} \frac{\sin(k+1)t}{\sin \frac{t}{2}}
\]

However, if the coefficients \( a_n \)’s are not necessarily restricted by the relation (1.25.2) then the series (1.25.1) is known as the series of the Legendre polynomials.

The general theory of convergence and summability of the Legendre series is important from the point of view of its application to problems in the potential theory. A detailed account of Legendre series is given by Sansone (1959) and Szegö (1959).

### 1.26 Jacobi Series

Let \( f(x) \) be defined in the closed interval \([-1, 1]\) such that the integral

\[
(1.26.1) \quad \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} f(x) dx
\]

with \( \alpha > -1, \beta > -1 \); exists in the sense of Lebesgue.

The Jacobi series corresponding to the function \( f(x) \) is given by
(1.26.2) \[ f(x) \sim \sum_{n=0}^{\infty} a_n J_{n}^{(\alpha,\beta)}(x) \]

where

(1.26.3) \[ a_n = \frac{1}{g_n} \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta f(x) J_{n}^{(\alpha,\beta)}(x) \, dx \]

with

(1.26.4) \[ g_n = \frac{2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1) \Gamma(n+1) \Gamma(n+\alpha+\beta+1)} \]

and \( J_{n}^{(\alpha,\beta)}(x) \) are the Jacobi polynomials defined by the generating function.

(1.26.5) \[ 2^{\alpha+\beta} \left(1-2xt+t^2\right)^{-\frac{1}{2}} \left[1-t^2(1-2xt+t^2)^{-\alpha}\right] \left[1+t^2(1-2xt+t^2)^{-\beta}\right] = \sum_{n=0}^{\infty} J_{n}^{(\alpha,\beta)}(x) \]

We write

(1.26.6) \[ F(\theta) = \{f(\cos \theta) - A\} \left(\frac{\sin \theta}{2}\right)^{2\alpha+1} \left(\frac{\cos \theta}{2}\right)^{2\beta+1} \]

A being the fixed constant

1.27 Fourier Laguerre Series

The Fourier Laguerre series associated with a Lebesgue integrable function \( f(x) \) in the interval \((0, \infty)\) is given by

(1.27.1) \[ f(x) \sim \sum_{n=0}^{\infty} a_n L_{n}^{\alpha}(x), \quad \alpha > -1 \]

Where the coefficient \( a_n \) are defined by the formula

(1.27.2) \[ \Gamma(\alpha+1) \left[\begin{array}{c} n+\alpha \\ n \end{array}\right] a_n = \int_{0}^{\infty} e^{-x} x^{\alpha} f(x) L_{n}^{\alpha}(x) \, dx \]
provided the integral on the right exists and $L_n^\alpha (x)$ is the $n$th Laguerre polynomial of order $\alpha > -1$ defined by the generating function

$$(1.27.3) \quad \sum_{n=0}^{\infty} L_n^\alpha (x) \; w^n = (1-w)^{-\alpha-1} \exp \left[ -\frac{wx}{1-w} \right].$$

We write

$$(1.27.4) \quad \phi(x) = \left[ \Gamma(\alpha+1) \right]^{-1} e^{-x} \; x^\alpha [f(x)-f(0)]$$

### 1.28 Fourier Hermite Series

The Fourier Hermite series of a Lebesgue integrable function $f(x)$ in $(-\infty, \infty)$ is given by

$$(1.28.1) \quad f(x) \sim \sum_{n=0}^{\infty} a_n H_n(x)$$

where

$$(1.28.2) \quad \left( \sqrt{n} \; 2^n \; \Gamma(n+1) \right) a_n = \int_{-\infty}^{\infty} e^{-y^2} \; H_n(y) \; f(y) \; dy$$

and $H_n(x)$ denotes the Hermite Polynomial given by

$$(1.28.3) \quad \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} w^n = \exp(2x \; w-w^2)$$

We write

$\Phi(y) = f(y) - f(x)$

$\psi(u) = f(x \pm u) - f(x)$

and

$$\chi(y) = \left( \pi^{\frac{1}{2}} 2^{n+1} \Gamma(n+1) \right) e^{-y^2} \; \phi(y) \; \frac{H_{n+1}(x)H_n(y) - H_n(x)H_{n+1}(y)}{x-y}$$