3.1 Introduction:

Recently it has been brought into consideration that the low-energy effective field theory includes black hole or, more generally, black p-brane, solutions which may possess properties which are qualitatively different from those that come in ordinary Einstein gravity, as presented by Gibbons and Maeda (1988), Garfinkle et al (1991), Horowitz and Strominger (1991), Shapere et al (1991). These solutions are generally characterised by one or more charges associated with the Yang-Mills fields or the antisymmetric tensor gauge field, and a nontrivial dilaton field. In the absence of any charge, the solution reduces to the ordinary Schwarzschild solution. Rotating charge-neutral black hole solutions may also be obtained in string theory and are identical to the Kerr solution as shown by Adler et al (1975), Thorne et al (1986), of ordinary Einstein gravity with the dilaton with a constant value. Horne and Horowitz have analysed, rotating charged black hole solutions in these theories in the limit of small angular momentum. They, in fact, presented a
more general class of theories than those which arise as the low-energy effective action in string theory, by allowing dilaton couplings to the Maxwell fields of the type which is not necessarily the one induced in string theory. However, they considered only the dilaton graviton system, and do not consider a more general action that also includes antisymmetric tensor gauge field.

In this chapter we have obtained an exact classical solution in the low-energy effective field theory describing heterotic string theory, that describes a black hole carrying a finite amount of charge and angular momentum. It is to be noted that this solution differs from others even in the limit of small angular momentum since it includes the antisymmetric field in a nontrivial way. We have used to method of twisting procedure given by Ferrara et al (1977), Cremmer et al (1977, 1979), Roo (1985), Bergshoef et al (1985), Castellani et al (1986), Cecotti et al (1988), Duff (1990), Veneziano (1991), Gasperini et al (1991), Sen (1991), Horne and Horowitz (1992), that generates inequivalent classical solutions starting from a given classical solution of string theory. Hasan and Sen (1991) have presented to generate charged black hole solutions starting from charge-neutral solution. Using the same transformations, we have obtained the rotating charged black hole solution by starting from a rotating black hole solution without charge, i.e. Kerr solution.
3.2 Mathematical Structure:

Let us consider the string theory effective action in four-dimensions

\[
A = -\int d^4x \sqrt{-\det g} e^{-\phi} \left( -R + \frac{1}{12} H_{abc} H^{abc} - g_{ab} \phi_{,a} \phi_{,b} + \frac{i}{2} F_{ab} F^{ab} \right)
\]

where \( g_{ab} \) be the metric, \( R \) as the scalar curvature

\[
F_{ab} = A_{b,a} - A_{a,b},
\]

be the field strength corresponding to the Maxwell field \( A_a \), and \( \phi \) as the dilaton field, and

\[
H_{abc} = B_{bc,a} + \text{cyclic permutations}
\]

\[
- \left[ \Omega_3(A) \right]_{abc},
\]

where \( R_{ab} \) be the antisymmetric tensor gauge field and

\[
\left[ \Omega_3(A) \right]_{abc} = \frac{1}{4} \left( A_a F_{bc} + \text{cyclic permutations} \right)
\]

as the gauge Chern-Simons term.
Hence, we have considered a theory where six of the ten-dimensions are compactified i.e. to a Calabi-Yau manifold. The massless fields coming from compactification have not been included in the effective action. We have taken only a U(1) component of the full set of non-Abelian gauge fields present in the theory. The metric $g_{ab}$ comes naturally in the $\sigma$–model and related to the Einstein metric $g^{(E)}_{ab}$ as

$$g^{(E)}_{ab} = e^{-\sigma} g_{ab}. \tag{3.4}$$

One may put i.e. by rescaling

$$\phi \rightarrow 2\phi, \quad \tag{3.5}$$

$$A_a \rightarrow \sqrt{2} A_a, \quad \tag{3.6}$$

one recovers the action given by Horne and Horowitz (1992), except the term $H_{abc} H^{abc}$.

### 3.3 The Solution:

Let us obtain solutions that are independent of the time coordinate $t$, and also let us use the matrix notion to represent the various fields. Hence, $g_{ab}$ and $B_{ab}$ may be treated as 4x4 matrices, and $A_a$ as a four-dimensional column vector, with the fourth row
and/or column corresponding to the time coordinate, t. Therefore, let us now represent $M, K, \eta$ as

$$K_{ab} = -B_{ab} - g_{ab} - \frac{1}{4} A_a A_b$$  \hspace{1cm} (3.7)

$$\eta_{ab} = \text{diag}(1,1,1,-1)$$  \hspace{1cm} (3.8)

$$M = \begin{bmatrix}
(K^T - \eta) g^{-1} (K - \eta) (K^T - \eta) g^{-1} (K + \eta) - (K^T - \eta) g^{-1} A \\
(K^T + \eta) g^{-1} (K - \eta) (K^T + \eta) g^{-1} (K + \eta) - (K^T + \eta) g^{-1} A \\
-A^T g^{-1} (K - \eta) - A^T g^{-1} (K + \eta) A^T g^{-1} A
\end{bmatrix},$$  \hspace{1cm} (3.9)

where $T$ stands for the transposition of a matrix. Equation (3.9) describes a 9x9 matrix $M$. Hasaan and Sen (1991) presented if

$$\{g_{ab}, B_{ab}, \varphi, A_a\}$$  \hspace{1cm} (3.10)

represents a time-independent solution of the classical equations of motion, derived from the action, then

$$\{g'_{ab}, B'_{ab}, \varphi', A'_a\}$$  \hspace{1cm} (3.11)

also describes a solution of the same equations of motion, if they are related through the relation

$$M' = \Omega M \Omega T,$$  \hspace{1cm} (3.12)

$$\varphi' - \ln \det g' = \varphi - \ln \det g,$$  \hspace{1cm} (3.13)
where

\begin{equation}
\Omega = \begin{bmatrix}
I_7 & \cosh \alpha & \sinh \alpha \\
\sinh \alpha & \cosh \alpha & 0 \\
0 & 0 & I_7
\end{bmatrix},
\end{equation}

where $I_7$ stands for $7 \times 7$ identity matrix, and $\alpha$ as an arbitrary number. Eq. (3.12) and (3.13) determine all the primed fields in terms of unprimed ones. Let us now apply these transformations to the charge-neutral rotating black hole solution i.e. standard Kerr solution as shown by Adler et al (1975), Thorne et al (1986).

\begin{equation}
\begin{aligned}
ds^2 &= \frac{\rho^2 + a^2 \cos^2 \theta - 2m\rho}{\rho^2 + a^2 \cos^2 \theta} dt^2 + \frac{\rho^2 + a^2 \cos^2 \theta d\rho^2}{\rho^2 + a^2 - 2m\rho} \\
&+ \left( \rho^2 + a^2 \cos^2 \theta \right) d\theta^2 \\
&+ \frac{\sin^2 \theta}{\rho^2 + a^2 \cos^2 \theta} \left[ \left( \rho^2 + a^2 \right) \left( \rho^2 + a^2 \cos^2 \theta \right) + \\
&+ 2m\rho a^2 \sin^2 \theta \right] d\varphi^2 \\
&- \frac{4m\rho a \sin^2 \theta}{\rho^2 + a^2 \cos^2 \theta} dt d\varphi,
\end{aligned}
\end{equation}

(3.16) $\varphi = 0,$

(3.17) $B_{ab} = 0,$
Therefore, one may obtain transformed solutions

\[ ds'^2 = -\frac{\left(\rho^2 + a^2 \cos^2 \theta - 2m\rho\right)\left(\rho^2 + a^2 \cos^2 \theta\right)}{\left[\rho^2 + a^2 \cos^2 \theta + 2m\rho \sinh^2 \left(\frac{\alpha}{2}\right)\right]} \, dt^2 \]

\[ + \frac{\rho^2 + a^2 \cos^2 \theta}{\rho^2 + a^2 - 2m\rho} \, d\rho^2 \]

\[ + \left(\rho^2 + a^2 \cos^2 \theta\right) d\theta^2 \]

\[ + \left\{\left(\rho^2 + a^2\right)\left(\rho^2 + a^2 + \cos^2 \theta\right) + 2m\rho a^2 \sin^2 \theta \right\} \]

\[ + 4m\rho \left(\rho^2 + a^2\right) \sinh^2 \left(\frac{\alpha}{2}\right) \]

\[ + 4m^2 \rho^2 \sinh^4 \left(\frac{\alpha}{2}\right) \}

\times \frac{\left(\rho^2 + a^2 \cos^2 \theta\right) \sin^2 \theta}{\left[\rho^2 + a^2 \cos^2 \theta + 2m\rho \sinh^2 \left(\frac{\alpha}{2}\right)\right]} \, d\varphi^2 \]

\[ - \frac{4m\rho a \cosh^2 \left(\frac{\alpha}{2}\right)\left(\rho^2 + a^2 \cos^2 \theta\right) \sin^2 \theta}{\left[\rho^2 + a^2 \cos^2 \theta + 2m\rho \sinh^2 \left(\frac{\alpha}{2}\right)\right]^2} \, dt \, d\varphi, \]

\[ \varphi' = -\ell n \frac{\rho^2 + a^2 \cos^2 \theta + 2m\rho \sinh^2 \left(\frac{\alpha}{2}\right)}{\rho^2 + a^2 \cos^2 \theta} \]
\[ A_{\varphi'} = -\frac{2m\rho \sinh \alpha \sin^2 \theta}{\rho^2 + a^2 \cos^2 \theta + 2m\rho \sinh^2 (\frac{\varphi}{2})'} \]

(3.21)

\[ A' = \frac{2m\rho \sinh \alpha}{\rho^2 + a^2 \cos^2 \theta + 2m\rho \sinh^2 (\frac{\varphi}{2})'} \]

(3.22)

\[ B'_{\varphi} = \frac{2m\rho \sinh \left(\frac{\varphi}{2}\right) \sin^2 \theta}{\rho^2 + a^2 \cos^2 \theta + 2m\rho \sinh^2 \left(\frac{\varphi}{2}\right)} . \]

(3.23)

The rest components of \( A'_{\varphi'} \) and \( B'_{ab} \) vanish. The Einstein metric reads

\[ ds_{E}^2 = e^{-\varphi'} ds' \]

(3.24)

or

\[ ds_{E}^2 = -\frac{\rho^2 + a^2 \cos^2 \theta - 2m\rho}{\rho^2 + a^2 \cos^2 \theta + 2m\rho \sinh^2 (\frac{\varphi}{2})} dt^2 

+ \frac{\rho^2 + a^2 \cos^2 \theta + 2m\rho \sinh^2 (\frac{\varphi}{2})}{\rho^2 + a^2 - 2m\rho} d\theta^2 

+ \left(\rho^2 + a^2 \cos^2 \theta + 2m\rho \sinh^2 (\frac{\varphi}{2})\right) d\theta^2 

- \frac{4m\rho \cosh^2 \left(\frac{\varphi}{2}\right) \sin^2 \theta}{\rho^2 + a^2 \cos^2 \theta + 2m\rho \sinh^2 \left(\frac{\varphi}{2}\right)} dtd\varphi \]
\[+ \left[ (\rho^2 + a^2)(\rho^2 + a^2 \cos^2 \theta) + 2m \rho a^2 \sinh^2 \theta \right.
\]
\[+ 4m \rho (\rho^2 + a^2) \sinh^2 \theta \left(\frac{\alpha}{2}\right) \]
\[+ 4m^2 \rho^2 \sinh^4 \left(\frac{\alpha}{2}\right) \]}
\[\times \frac{\sinh^2 \theta}{\rho^2 + a^2 \cos^2 \theta + 2m \rho \sinh^2 \left(\frac{\alpha}{2}\right)} d\varphi^2.\]

The above metric describes a black hole solution with mass 
M, charge q, angular momentum J and magnetic dipole moment u 
as

(3.26) \[M = \left(\frac{m}{2}\right)(1 + \cosh \alpha),\]

(3.27) \[q = \left(\frac{m}{\sqrt{2}}\right) \sinh \alpha,\]

(3.28) \[J = \left(\frac{ma}{2}\right)(1 + \text{Cosh} \alpha),\]

(3.29) \[\mu = \left(\frac{1}{\sqrt{2}}\right) ma \sinh \alpha,\]

and g factor reads

(3.30) \[g = \mu M / qJ = 2.\]
3.4 Analysis of the Solution:

Let us now analyse various features of this solution and with its extremal limit. In view of eqs. (3.26) - (3.29), one may obtain

\[ m = M - q^2 / 2M, \]
\[ \sinh \alpha = \sqrt[2]{2} qM / (2M^2 - q^2), \]
\[ a = J / M. \]

Let us evaluate the coordinate singularities or horizon which occurs on the surface

\[ \rho^2 - 2m\rho + a^2 = 0, \]
which yields

\[ \rho = m \pm \left( m^2 - a^2 \right)^{1/2} \]
\[ + M - \frac{q^2}{2M} \pm \left[ \left( M - \frac{q^2}{2M} - \frac{J^2}{M^2} \right)^{1/2} = \rho \right. \frac{+}{H}. \]

Now

\[ q = 1 \]
\[ \mu = \frac{1}{a} \]

or
(3.37) \[ a = \frac{\mu}{q}. \]

Hence,

(3.38) \[ J/M = \frac{\mu}{q}. \]

One may compute the area of the outer event horizon as

(3.39) \[ A = 8\pi M \left[ M - \frac{q^2}{2M} + \left( \frac{M - \frac{q^2}{2M}}{M^2/2} \right)^{1/2} \right]. \]

It is obvious from eq. (3.35) that horizon disappears unless

(3.40) \[ |J| \leq M^2 - q^2/2. \]

Now, one may obtain the extremal limit of the black hole as

(3.41) \[ |J| \to M - q^2/2M, \]

and in this limit

(3.42) \[ A \to 8\pi |J|. \]

Hence, the event horizon remains finite size in this limit, as expected from the general arguments of Horne and Horowitz. It is to be noted that the result that in extremal limit the area of the event
horizon depend only on the angular momentum $J$. This result is the same to the corresponding result for the rotating charged black hole in a different context as presented by Frolov, Zelnikov and Bleyer (1987).

Let us evaluate the angular velocity $\Omega$ at the horizon. It is obtained by demanding that the Killing vector

$$ (3.43) \quad \partial/\partial t + \Omega \partial/\partial \phi $$

be null at horizon, i.e.

$$ (3.44) \quad g_{tt} + 2g_{t\phi} \Omega + g_{\phi\phi} \Omega^2 = 0, $$

which gives

$$ (3.45) \quad \Omega = \frac{J}{2M^2} \frac{1}{M - q^2/2M + \left[ \left( M - q^2/2M \right)^2 - J^2/M^2 \right]^{1/2}}. $$

Again, for the extremal limit

$$ (3.46) \quad \Omega \rightarrow (1/2M) \text{sgn}(J), $$

as long as

$$ (3.47) \quad |J| \neq 0. $$

If $J = 0$, then
\[ \Omega = 0. \]

Hence, in the extremal limit, \( |\Omega| \) depends only on the mass of the black hole.

The surface gravity \( K \) reads at the pole as

\[ K = \lim_{\rho \to \infty} \rho_{\text{H}} \left( g^{\alpha\beta} \right)^{1/2} \partial_{\rho} \left( -g_{\nu\mu}^{\frac{1}{2}} \right)_{\theta=0}, \]

\[ = \frac{\left[ (2M^2 - q^2)^2 - 4J^2 \right]^{1/2}}{2M \left[ 2M^2 - q^2 + \left( (2M^2 - q^2)^2 - 4J^2 \right)^{1/2} \right]}. \]

Hence, in the extremal limit

\[ K \to 0 \]

If \( J \neq 0 \). If \( J = 0 \), then

\[ K = 1/4M, \]

which is in agreement with the results of Shapere et al (1991).

One may now evaluate the Hawking temperature as

\[ T_H = K/2\pi. \]

If \( J = 0 \), then
(3.53) \[ T_H = \frac{1}{8\pi M}; \]

and if \( J \neq 0 \)

(3.54) \[ T_H \to 0, \]

at the pole.

### 3.5 Concluding Remarks:

We have presented a solution of the classical equations of motion arising in the low-energy effective field theory for heterotic string theory. The solution describes a black hole in four dimensions carrying mass \( M \), charge \( q \) and the angular momentum \( J \). We have also presented the extremal limit of the solution so obtained. Hence, we have constructed a rotating charged black hole solution in four dimensional heterotic string theory and investigated its various features. The extremal limit of the solution for \( J \neq 0 \) is obtained to have properties that are qualitatively similar to the extremal rotating black hole rather than extremal charged black hole.