CHAPTER-5
TERNARY FILTERS

PETRICH. M. [49] made a study on filters in general semigroups. LEE. S. K and LEE. S. S. [30], introduced the notion of a left (right) filter in a po-semigroup and gave a characterization of the left(right)-filter of S in term of the right(left prime ideals. N. KEHAYOPULU [47], gave the characterization of the filter of S in terms of the prime ideals in ordered semigroups. Y. B. JUN [25] introduced the notion of a Γ-filter in po-Γ-semigroups and gave a characterization of po-Γ-filter of a po-Γ-semigroup in terms of a prime po-Γ-ideals. LEE. S. K and KWON. Y. I. [31] introduced the notions of a left (right) Γ-filter in po-Γ-semigroup and gave a characterization of a left (right) Γ-filter of a po-Γ-semigroup in terms of right (left) prime Γ-ideals. SUBRAHMANYESWARA RAO SEETAMRAJU. VB, ANJANEYULU. A and MADHUSUDANA RAO. D [69], [70] made a study on po-Γ-filters and extended many characterizations of semigroups, po-semigroups and Γ-semigroups to po-Γ-semigroups. We introduce the notion of a po-Γ-filter in po-Γ-semigroups and gave a characterization of po-Γ-filters of po-Γ-semigroups in terms of completely prime po-Γ-ideals, c-system, m-system, d-system and n-system.

In this chapter, the terms ternary filter, ternary filter of a ternary semigroup T generated by A, principal ternary filter are introduced. It is proved that (1) The nonempty intersection of two ternary filters of a ternary semigroup T is also a ternary filter and (2) The nonempty intersection of a family of ternary filters of a ternary semigroup T is also a ternary filter. Further it is proved that a nonempty subset F of a ternary semigroup T is a ternary filter if and only if T\F is a completely prime ideal of T or empty.

Further it is also proved that if F is a ternary filter of a ternary semigroup T, then T\F is (i) a prime ideal, (ii) a completely semiprime ideal, (iii) a semiprime -ideal of T or empty and finally (iii) a nonempty subset F of a commutative ternary semigroup T is a ternary filter if and only if T\F is a prime ideal of T or empty. It is proved that a ternary semigroup T does not contain proper ternary filters if and only if T does not contain proper completely prime ideals. Further it is proved that every ternary filter F of a ternary semigroup T is (1) a 3-system of T (2) a 3-system of T and (3) a 3*-system of T. It is proved that the ternary filter of a ternary semigroup T generated by a nonempty subset A of T is the intersection of all ternary filters of T containing A. It is proved that if T is a ternary semigroup and a ∈ T, then N(a) is the least filter of T containing {a}. Further it is
proved that if \( N(b) \subseteq N(a) \), then \( N(a) \setminus N(b) \), if it is nonempty, is a completely prime ideal of \( N(a) \). Finally it is proved that (1) if \( a, b \in T \) and \( b \in N(a) \) then \( N(b) \subseteq N(a) \), (2) if \( a, b \in T \) then \( N(b) \subseteq N(a) \).

**TERNARY FILTERS** :

**Definition 5.1.1**: A Ternary Subsemigroup \( F \) is said to be **ternary filter** of \( T \) if \( a, b, c \in T \) and \( abc \in F \) implies \( a, b, c \in F \).

**Example 5.1.2**: Let \( T = \{ a, b, c, d \} \) with the ternary multiplication defined by

\[
xyz = \begin{cases} 
  b & \text{if } x = y = z = b \\
  c & \text{if } x = y = z = c \\
  d & \text{if } x = y = z = d \\
  a & \text{otherwise}
\end{cases}
\]

Then \( T \) is a ternary semigroup and \( \{ a, b, c, d \} \), \( \{ b \} \), \( \{ c \} \), \( \{ d \} \) are all filters of \( T \).

**Theorem 5.1.3**: The nonempty intersection of two ternary filters of a ternary semigroup \( T \) is also a ternary semigroup of \( T \).

**Proof**: Let \( A, B \) be two ternary filters of \( T \).

Let \( a, b, c \in T \) and \( abc \in A \cap B \).

\( abc \in A \cap B \Rightarrow abc \in A \) and \( abc \in B \).

\( a, b, c \in T \) , \( abc \in A \) and \( A \) is a ternary filter of \( T \) \( \Rightarrow a, b, c \in A \).

\( a, b, c \in T \) , \( abc \in B \) and \( B \) is a ternary filter of \( T \) \( \Rightarrow a, b, c \in B \).

\( a, b, c \in A \) and \( a, b, c \in B \Rightarrow a, b, c \in A \cap B \).

Therefore \( A \cap B \) is a ternary filter of \( T \).

**Theorem 5.1.4**: The nonempty intersection of a family of ternary filters of a ternary semigroup \( T \) is also a ternary filter of \( T \).

**Proof**: Let \( \{ F_a \}_{a \in \Delta} \) be a family of ternary filters of \( T \) and let \( F = \bigcap_{a \in \Delta} F_a \).

Let \( a, b, c \in T, abc \in F \). Now \( abc \in F \Rightarrow abc \in \bigcap_{a \in \Delta} F_a \Rightarrow abc \in F_a \) for each \( a \in \Delta \).

\( abc \in F_a, F_a \) is a ternary filter of \( T \) \( \Rightarrow a, b, c \in F_a \).
⇒ a, b, c ∈ \bigcap_{a \in \Lambda} F_a \Rightarrow a, b, c \in F. Therefore F is a ternary filter of T.

NOTE 5.1.5: In general, the union of two ternary filters is not a ternary filter.

EXAMPLE 5.1.6: As in the example 5.1.2, T is a ternary semigroup and \{b\}, \{c\}, \{d\}, are ternary filters, but \{b\} ∪ \{c\} ∪ \{d\} is not a ternary filter of T, because bcd = a is not in \{b\} ∪ \{c\}∪ \{d\}.

We now prove a necessary and sufficient condition for a nonempty subset to be a ternary filter in a ternary semigroup.

THEOREM 5.1.7: A nonempty subset F of a ternary semigroup T is a ternary filter if and only if T \ F is a completely prime ideal of T or empty.

Proof: Assume that T \ F ≠ ∅. Let x, y, z ∈ T \ F.

Suppose that xyz ∉ T \ F, then xyz ∈ F. Since F is ternary filter and hence x, y, z ∈ F.

It is a contradiction. Thus xyz ∈ T \ F, and so T(T \ F)T ⊆ T \ F.

Therefore T \ F is an ideal of T.

Now we shall prove that T \ F is completely prime.

Let x, y, z ∈ T \ F, for x, y, z ∈ T

Suppose that x ∉ T \ F, y ∉ T \ F and z ∉ T \ F. then x ∈ F, y ∈ F and z ∈ F.

Since F is a ternary subsemigroup of T , xyz ∈ F.

It is a contradiction. Thus x ∈ T \ F or y ∈ T \ F or z ∈ T \ F.

Hence T \ F is completely prime and hence T \ F is a completely prime ideal of T.

Conversely suppose that T \ F is completely prime ideal of T or empty.

If T \ F is empty, then F = T. Thus F is ternary filter of T.

Assume that T \ F is completely prime ideal of T.

Suppose that for x, y, z ∈ F, xyz ∉ F. Then xyz ∈ T \ F, for x, y, z ∈ F.

Since T \ F is completely prime, x ∈ T \ F or y ∈ T \ F or z ∈ T \ F.

It is contradiction. Thus xyz ∈ F and hence F is ternary subsemigroup of T.

Let x, y, z ∈ T and hence xyz ∈ F. If x, y, z ∈ F. Then x, y, z ∈ T \ F.

Since T \ F is completely prime ideal of T, then xyz ∈ T \ F. It is a contradiction.

Thus x, y, z ∈ F. Therefore F is a ternary filter of T.

COROLLARY 5.1.8: Let T be a ternary semigroup. If F is a ternary filter, then T \ F is a prime ideal of T or empty.
Proof: Since $F$ is a filter of $T$. By theorem 5.1.7, $T \setminus F$ is a completely prime ideal of $T$ or empty. By theorem 1.4.9, $T \setminus F$ is a prime ideal of $T$ or empty.

COROLLARY 5.1.9: A nonempty subset $F$ of a commutative ternary semigroup $T$ is a ternary filter if and only if $T \setminus F$ is a prime ideal of $T$ or empty.

Proof: Suppose that $T \setminus F$ is a filter of commutative ternary semigroup $T$. By corollary 5.1.8, $T \setminus F$ is prime ideal of $T$ or empty.
Conversely suppose that $T \setminus F$ is a prime ideal of $T$ or empty. If $T \setminus F = \emptyset$, then $F = T$. Thus $F$ is a filter of $T$. Assume that $T \setminus F$ is a prime ideal of $T$. By theorem 1.4.10, $T \setminus F$ is a completely prime ideal of $T$ or empty. By theorem 5.1.7, $F$ is a ternary filter of $T$.

THEOREM 5.1.10: A ternary semigroup $T$ does not contain proper ternary filters if and only if $T$ does not contain proper completely prime ideals.

Proof: Suppose that a ternary semigroup $T$ does not contain proper ternary filters.
Let $A$ be a completely prime ideals of $T$ and $A \subset T$. Then $\emptyset \neq T \setminus A \subseteq T$. and $T \setminus (T \setminus A) = A$ is a completely prime ideal of $T$. Since $T \setminus A$ compliment of $A$ to $T$. By theorem 5.1.7 $T \setminus A$ is ternary filter of $T$. Then $T \setminus A = T$ and hence $A = \emptyset$, It is a contradiction.
Therefore $T$ does not contain proper completely prime ideals.
Conversely suppose that $T$ does not contain proper completely prime ideals.
Let $F$ be a ternary filter of $T$ and $F \subset T$. Since $T \setminus F \neq \emptyset$, by theorem 5.1.7, $T \setminus F$ is completely prime ideal of $T$. Then $T \setminus A = T$, and hence $A = \emptyset$. It is a contradiction.
Therefore $T$ does not contain proper completely prime ideals.

THEOREM 5.1.11: Every ternary filter $F$ of a ternary semigroup $T$ is a 3-system of $T$.

Proof: Suppose that $F$ is a ternary filter of a ternary semigroup $T$. By corollary 5.1.8, $T \setminus F$ is a prime ideal of $T$. By theorem 1.4.12, $T \setminus (T \setminus F) = F$ is a 3-system of $T$ or empty.

THEOREM 5.1.12: Let $T$ be a ternary semigroup. If $F$ is a ternary filter, then $T \setminus F$ is a completely semi prime ideal of $T$.

Proof: Since $F$ is a ternary filter of a ternary semigroup $T$, by theorem 5.1.7, $T \setminus F$ is a completely prime ideal of $T$. By theorem 1.4.22, $T \setminus F$ is a completely semiprime ideal of $T$.  

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THEOREM 5.1.13: Every ternary filter $F$ of a ternary semigroup $T$ is a $d$-system of $T$.

*Proof*: Since $F$ is a ternary filter of a ternary semigroup $T$, by theorem 5.1.12, $T \backslash F$ is a completely semiprime ideal of $T$. By theorem 1.4.24, $T \backslash (T \backslash F) = F$ is a $d$-system of $T$ or empty.

THEOREM 5.1.14: Let $T$ be a ternary semigroup. If $F$ is a filter of $T$, then $T \backslash F$ is a semiprime ideal of $T$.

*Proof*: Since $F$ is a ternary filter of a ternary semigroup $T$. By theorem 5.1.7, $T \backslash F$ is a completely prime ideal of $T$. By theorem 1.4.22, $T \backslash F$ is a completely semiprime ideal of $T$. By theorem 1.4.29, $T \backslash F$ is a semiprime ideal of $T$.

THEOREM 5.1.15: Every ternary filter $F$ of a ternary semigroup $T$ is a $3^*$-system of $T$.

*Proof*: Since $F$ is a ternary filter of a ternary semigroup $T$. By theorem 5.1.14, $T \backslash F$ is a semiprime ideal of $T$. By theorem 1.4.34, $T \backslash (T \backslash F) = F$ is a $3^*$-system of $T$.

DEFINITION 5.1.16: Let $T$ be a ternary semigroup and $A$ be a nonempty subset of $T$. The smallest ternary filter of $T$ containing $A$ is called *ternary filter of $T$ containing $A*$ and it is denoted by $N(A)$.

THEOREM 5.1.17: The ternary filter of a ternary semigroup $T$ generated by a nonempty subset $A$ of $S$ is the intersection of all ternary filters of $T$ containing $A$.

*Proof*: Let $\Delta$ be the set of all ternary filters of $T$ containing $A$. Since $T$ itself is a ternary filter of $T$ containing $A$, $T \in \Delta$. So $\Delta \neq \emptyset$.

Let $F^* = \bigcap_{F \in \Delta} F$. Since $A \subseteq F$ for all $F \in \Delta$, $A \subseteq F^*$. So $F^* \neq \emptyset$.

By theorem 5.1.4, $F^*$ is a ternary filter of $T$.

Let $K$ be a ternary filter of $T$ containing $A$.

Clearly $A \subseteq K$ and $K$ is a ternary filter of $T$.

Therefore $K \in \Delta \Rightarrow F^* \subseteq K$. Therefore $F^*$ is the smallest ternary filter of $T$ containing $A$ and hence $F^*$ is the ternary filter of $T$ generated by $A$.
DEFINITION 5.1.18: A ternary filter $F$ of a ternary semigroup $T$ is said to be a principal ternary filter provided $F$ is a ternary filter generated by $\{a\}$ for some $a \in T$. It is denoted by $N(a)$.

EXAMPLE 5.1.19: As in the example 5.1.2, $T$ is a ternary semigroup and $N(a) = \{a, b, c, d\}$, $N(b) = \{b\}$, $N(c) = \{c\}$ and $N(d) = \{d\}$ are all the principal ternary filters of the ternary semigroup $T$.

COROLLARY 5.1.20: Let $T$ is a ternary semigroup and $a \in S$. Then $N(a)$ is the least ternary filter of $T$ containing $\{a\}$.

NOTE 5.1.21: For every $a \in T$, the intersection of all ternary filters containing $\{a\}$ is again a ternary filter and thus the least ternary filter containing $\{a\}$.

THEOREM 5.1.22: If $N(b) \subseteq N(a)$, then $N(a) \setminus N(b)$, if it is nonempty, is a completely prime ideal of $N(a)$.

Proof: By theorem 5.1.9, $N(a) \setminus N(b)$ is a completely prime ideal of $N(a)$.

LEMMA 5.1.23: Let $a, b \in T$ and $b \in N(a)$, then $N(b) \subseteq N(a)$.

Proof: From the definition of the principal ternary filter, it is clear.