CHAPTER 4

U-TERNARY SEMIGROUPS
AND V-TERNARY SEMIGROUPS
CHAPTER-4
U-TERNARY SEMIGROUPS AND V-TERNARY SEMIGROUPS

INTRODUCTION:

GILMER [14] studied about U-rings and this notion was introduced by STYANARAYANA [62] in commutative semigroups. ANJANEYULU [5] introduced the notions of U-semigroups and V-semigroups in the class of arbitrary semigroups and he studied prime ideals and maximal ideals in a U-semigroup and characterize V-semigroups. In this thesis we study the notion of U-ternary semigroups and V-ternary semigroups in the class of arbitrary ternary semigroups. We study prime ideals and maximal ideals in U-ternary semigroup and characterize V-ternary semigroups.

This chapter is divided into 2 sections. In section 1, the term U-Ternary semigroup is introduced. It is proved that a ternary semigroup T is a U-ternary semigroup if either T has a left (lateral, right) identity or T is generated by an idempotent. It is proved that a ternary semigroup is U-ternary semigroup if and only if (1) every proper ideal of T is contained in a proper prime ideal of T, (2) every ideal A of T is semiprime ideal of T, (3) every ideal A of T is the intersection of all prime ideal of T contains A, (4) T\A is an n-system of T or empty where A is an ideal of T, (5) T\A is an m-system of T where A is an ideal of T. Further it is proved that if T be a U-ternary semigroup. Then T = T^3 and hence every maximal ideal is prime. Conversely if \{P_\alpha\} is a collection of all prime ideals in T and if P is a maximal element in this collection, then P is a maximal ideal of T. The term dimension n is introduced and it is proved that if A is a proper ideal of the finite dimensional U-ternary semigroup T. Then A is contained in maximal ideal.

In section 2, the term, V-ternary semigroup is introduced. It remarked that every U-ternary semigroup is V-ternary semigroup but V-ternary semigroup need not be U-ternary semigroup. It is proved that a ternary semigroup T is a V-ternary semigroup if and only if T has at least one proper prime ideal and if \{P_\alpha\} is the family of all proper prime ideals, then \( < x > = T \) for \( x \in T \cup P_\alpha \) or T is a simple ternary semigroup.

The contents of chapter 4 are published in “International Journal of Innovative Research & Development” under the title ‘U-Ternary Semigroups and V-Ternary Semigroups’ [42].
4.1. U-TERNARY SEMIGROUPS:

We introduce the notions of U-ternary semigroup.

DEFINITION 4.1.1: A ternary semigroup T is said to be \textit{U-ternary semigroup} provided for any ideal A in T, $\sqrt{A} = T$ implies $A = T$.

EXAMPLE 4.1.2: Let T be the ternary semigroup under the multiplication given in the following table.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>d</td>
<td>a</td>
<td>a</td>
<td>c</td>
<td>d</td>
</tr>
</tbody>
</table>

It can be easily verified that T is U-ternary semigroup.

THEOREM 4.1.3: A ternary semigroup T is a U-ternary semigroup if either T has an identity or T is generated by an idempotent.

\textit{Proof}: Suppose T has an identity e. Let A be any proper ideal such that $\sqrt{A} = T$. Since $\sqrt{A} = \{ x \in T : x^n \in A \text{ for some odd natural number } n \} = T$. So there is an odd natural number n such that $e^n \in A$ and hence $e \in A$. Thus $T = eeT = eTe = Tee \subseteq A$. This is a contradiction. Therefore T is a U-ternary semigroup.

NOTE 4.1.4: One can note that there are U-ternary semigroups neither containing left (lateral, right) identity nor generated by an idempotent.

EXAMPLE 4.1.5: In example 4.1.2, we remark that the U-ternary semigroup T neither containing left (lateral, right) identity nor generated by an idempotent.

THEOREM 4.1.6: A ternary semigroup T is U-ternary semigroup if and only if every proper ideal of T is contained in a proper prime ideal of T.

\textit{Proof}: Suppose T is a U-Ternary semigroup. Let A be any proper ideal of T. If A is not contained in any proper prime ideal of T, then $\sqrt{A} = T$. Since T is a U-ternary semigroup,
we have $A = T$, this is a contradiction. So every proper ideal is contained in a proper prime ideal of $T$.

Conversely suppose that every proper ideal is contained in a proper prime ideal of $T$, then clearly $T$ is a $U$-ternary semigroup.

**THEOREM 4.1.7**: A ternary semigroup $T$ is $U$-ternary semigroup if and only if every ideal $A$ of $T$ is semiprime ideal of $T$.

**Proof**: Suppose that $T$ is $U$-ternary semigroup. Let $A$ is an ideal of $T$ and $\sqrt{A} = T$. $\sqrt{A} = T$ and $T$ is $U$-ternary semigroup implies that $A = T$. Therefore $\sqrt{A} = A$. By theorem 1.4.49, $A$ is semiprime ideal of $T$.

Conversely suppose that $A$ is semiprime ideal of ternary semigroup $T$ and $\sqrt{A} = T$. By theorem 1.4.49, $\sqrt{A} = A$ and hence $\sqrt{A} = T$ implies that $A = T$. Therefore $T$ is $U$-ternary semigroup.

**COROLLARY 4.1.8**: A ternary semigroup $T$ is $U$-ternary semigroup if and only if every ideal $A$ of $T$ is the intersection of all prime ideals of $T$ contains $A$.

**Proof**: Suppose that $T$ is $U$-ternary semigroup. Let $A$ is an ideal of $T$. By theorem 4.1.9, $A$ is semiprime ideal of $T$. By corollary 1.4.50, $A$ is the intersection of all prime ideals of $T$ contains $A$.

Conversely suppose that every ideal $A$ of ternary semigroup $T$ is the intersection of all prime ideals of $T$ contains $A$. By corollary 1.4.50, $A$ is semiprime ideal of $T$. Therefore by theorem 4.1.7, $T$ is $U$-ternary semigroup.

**THEOREM 4.1.9**: If $T$ is a ternary semigroup and $A$ is an ideal of $T$, then $T$ is $U$-ternary semigroup if and only if $T \setminus A$ is an $n$-system of $T$ or empty.

**Proof**: Suppose that $T$ is $U$-ternary semigroup. By theorem 4.1.7, an ideal $A$ of ternary semigroup $T$ is semiprime ideal of $T$. By theorem 1.4.34, $T \setminus A$ is an $n$-system of $T$ or empty.

Conversely suppose that $T \setminus A$ is an $3^+$-system of $T$ or empty. By theorem 1.4.34, $A$ is semiprime ideal of $T$. By theorem 4.1.7, $T$ is $U$-ternary semigroup.

**COROLLARY 4.1.10**: If $T$ is $U$-ternary semigroup and $A$ is an ideal of $T$, then $T \setminus A$ is an $3$-system of $T$. 

124
**Proof :** Suppose that $T$ is $U$-ternary semigroup. By theorem 4.1.9, $T \setminus A$ is an $S'$-system of $T$. Therefore by theorem 1.4.33, $T \setminus A$ is an $S$-system of $T$.

**THEOREM 4.1.11 :** Let $T$ be a $U$-ternary semigroup. Then $T = T^3$ and hence every maximal ideal is prime. Conversely if $\{P_\alpha\}$ is a collection of all prime ideals in $T$ and if $P$ is a maximal element in this collection, then $P$ is a maximal ideal of $T$.

**Proof :** Clearly $\sqrt{T^3} = T$. Since $T$ is a $U$-ternary semigroup, we have $T^3 = T$ and hence every maximal ideal is prime. If $P$ is not maximal ideal of $T$, then there exists a proper ideal $A$ of $T$ containing $P$ properly. Since $P$ is a maximal element in the collection of all proper prime ideals in $T$, we have $A$ is not contained in any proper prime ideal. So $\sqrt{A} = T$. Since $T$ is a $U$-ternary semigroup, $A = T$, this is a contradiction. Therefore $P$ is a maximal ideal of $T$.

**DEFINITION 4.1.12 :** A ternary semigroup $T$ is said to have dimension $n$ or $n$-dimensional if there exist a strictly ascending chain $P_0 \subset P_1 \subset P_2 \subset \ldots \subset P_n$ of prime (proper) ideals in $T$, but no such a chain of $n + 2$ proper prime ideals exists in $T$.

**THEOREM 4.1.13 :** If $A$ is a proper ideal of the finite dimensional $U$-ternary semigroup $T$. Then $A$ is contained in maximal ideal.

**Proof :** By theorem 4.1.6, $A$ is contained in a proper prime ideal $P_0$. If $P_0$ is not a maximal ideal of $T$, then by theorem 4.1.7, there exists a proper prime ideal $P_1$ such that $P_0 \subset P_1$. If $P_1$ is maximal we are through. Otherwise $P_1$ is properly contained in proper prime ideal $P_2$ of $T$. The process of choosing $P_i$’s must cease in a finite number of steps because of the finite dimensionality of $T$. Hence $A$ is contained in a maximal ideal.

**NOTE 4.1.14 :** In a ternary semigroup, every finite dimensional $U$-ternary semigroup is not a union of maximal ideals.

**EXAMPLE 4.1.15 :** As the ternary semigroup $T$ in example 4.1.2, is a finite dimensional $U$-ternary semigroup with the unique maximal ideal $\{a, b, c\}$.

**4.2. V-TERNARY SEMIGROUP :**

In this section, the term, V-ternary semigroup is introduced. It is proved that a ternary semigroup $T$ is a V-ternary semigroup if and only if $T$ has at least one proper prime ideal and if $\{P_\alpha\}$ is the family of all proper prime ideals, then $< x > = T$ for $x \in T \setminus \cup P_\alpha$ or $T$ is a simple ternary semigroup.
We introduce the term V-Ternary semigroup.

**DEFINITION 4.2.1**: A Ternary semigroup T is said to be **V-Ternary semigroup** provided for any element \( a \in T \), \( \sqrt{< a >} = T \).

**NOTE 4.2.2**: Every U-ternary semigroup is a V-ternary semigroup. But V-ternary semigroup is not necessarily a U-ternary semigroup.

**EXAMPLE 4.2.3**: Let T be a ternary semigroup of all natural numbers greater than 1, under usual multiplication. The ideal \( A = \{3, 4, \ldots\} \) is not contained in any proper prime ideal and hence by theorem 4.1.8, T is not a U-ternary semigroup. Clearly every principal ideal is contained in a proper prime ideal. So T is V-ternary semigroup.

**THEOREM 4.2.4**: A ternary semigroup T is a V-ternary semigroup if and only if T has at least one proper prime ideal and if \( \{P_a\} \) is the family of all proper prime ideals, then \( < x > = T \) for \( x \in T \cup P_a \) or T is a simple ternary semigroup.

**Proof**: Let T be V-ternary semigroup which is not a simple ternary semigroup. If T has no proper prime ideals, then \( \sqrt{< a >} = T \) for every \( a \in T \). Thus implies \( < a > = T \) and hence T is a simple ternary semigroup. So assume T has no proper prime ideals. Then for any \( a \in T \cup P_a \), \( \sqrt{< a >} = T \), since \( a \) does not belong to any proper prime ideal. Thus \( < a > = T \). Conversely let \( a \) is any element of T such that \( < a > \neq T \). If \( a \in T \cup P_a \), then \( < a > = T \). So \( a \in \cup \mathcal{P} \) and hence \( \sqrt{< a >} \neq T \). Therefore T is a V-ternary semigroup.

* * * * *