CHAPTER 3

TERNARY SEMIGROUPS
IN WHICH PRIME IDEALS ARE MAXIMAL
AND PRIMARY IDEALS ARE PRIME AND
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TERNARY SEMIGROUPS IN WHICH PRIME IDEALS ARE MAXIMAL AND PRIMARY IDEALS ARE PRIME AND MAXIMAL

INTRODUCTION:

SCWARTZ [63] initiated a study of semigroups in which every proper prime ideals is maximal ideal and obtained some interesting results regarding the classical radical in ring theoretic sense. SATYANARAYANA [60] studied the structure of commutative cancellative primary semigroups and characterized the commutative semigroups in which proper prime ideals are maximal and idempotents form a chain under natural ordering. Further he [62] characterized commutative semigroups possessing cancellable elements in which every proper prime ideals is a maximal ideal. ANJANEYULU [3] studied the structure of quasi commutative cancellative primary semigroups and show that in a cancellative quasi commutative semigroups S, the statements; (1) S is a primary semigroup (2) proper prime ideals in S are maximal (3) semiprimary ideals in S are primary, are equivalent. Further, he characterize (1) semipseudo symmetric semigroups with identity in which the proper prime ideals are maximal and (2) semipseudo symmetric semigroups without identity in which the proper prime ideals are maximal and globally idempotent principal ideals form a chain. In thesis we extent the same concept and results to ternary semigroups.

It is well known that the class of all commutative rings with identity in which every proper primary ideal is prime as well as maximal is precisely the class of all commutative regular rings with identity. Such a statement is not true in the case of commutative semigroups and to this effect an example was given by SATYANARAYANA [59]. STYANARAYANA [59] showed that a commutative semigroup with identity, in which every proper primary idealas is prime as well as maximal, has only one proper ideal and is a union of two groups. He also showed that in a commutative semigroup with identity and with the unique maximal ideal M, in which every nonzero primary ideal is prime, either $M^2 = 0$ or $M^2 = M$. In thesis we study the arbitrary ternary semigroups in which proper primary ideal is prime as well as maximal. We show that the class of all ternary semigroups without zero in which every proper
primary ideal is prime as well as maximal is a subclass of class of all semi simple ternary semigroups.

This chapter is divided into 3 sections. In section 1, the terms left primary ternary ideal, lateral ternary primary ideal, right primary ternary ideal, ternary ideal are introduced. It is proved that A be an ideal in a ternary semigroup T. Then (1) X, Y, Z are three ideals of T such that \( XYZ \subseteq A \) and \( Y \nsubseteq A, Z \nsubseteq A \Rightarrow X \subseteq \sqrt{A} \) if and only if \( x, y, z \in T \), \(< \ x \ < \ y \ < \ z \ > \subseteq A \) and \( y, z \notin A \Rightarrow x \in \sqrt{A} \), (2) X, Y, Z are three ideals of T such that \( XYZ \subseteq A \) and \( X \nsubseteq A, Z \nsubseteq A \Rightarrow Y \subseteq \sqrt{A} \) if and only if \( < \ x \ > < \ y \ > < \ z \ > \subseteq A \) and \( x, z \notin A \Rightarrow y \in \sqrt{A} \), (3) X, Y, Z are three ideals of T such that \( XYZ \subseteq A \) and \( X \nsubseteq A, Y \nsubseteq A \Rightarrow Z \subseteq \sqrt{A} \) if and only if \( < \ x \ > < \ y \ > < \ z \ > \subseteq A \) and \( x, y \notin A \Rightarrow z \in \sqrt{A} \). Further it is proved that T be a commutative ternary semigroup with identity and let M be the unique maximal ideal in T. If \( \sqrt{A} = M \) for some ideal A in T, then A is a primary ideal. It is proved that the ideal A is a semiprime ideal of a ternary semigroup T, then (1) A is a prime ideal, (2) A is a primary ideal, (3) A is a left primary ideal, (4) A is a lateral primary ideal, (5) A is a right primary ideal and (6) A is a semiprimary ideal are equivalent. Further it is proved that A ternary semigroup T is semiprimary iff prime ideals of T form a chain under set inclusion.

In section 2, the terms left cancellative, lateral cancellative, right cancellative, cancellative, invertible elements and ternary group are introduce and we study the structure of cancellative quasi commutative ternary semigroups. In fact we prove that if T is a cancellative quasi commutative ternary semigroup, then (1) T is a primary ternary semigroup, (2) proper prime ideals in T are maximal, (3) semiprimary ideals in T are primary, are equivalent. We obtain a characterization for semipseudo symmetric ternary semigroups with identity in which proper prime ideals are maximal and also we characterize semipseudo symmetric semigroups without identity in which proper prime ideals are maximal and globally idempotent principal ideals from a chain. Further we characterize quasi commutative ternary semigroups containing cancellable elements in which proper prime ideals are maximal.
In section 3, it is proved that T be a ternary semigroup containing 0 and identity with the maximal ideal M. Then every nonzero primary ideal is prime as well as maximal if and only if T/M is a 0-simple ternary semigroup with either (1) \( M = (T \setminus M) a (T \setminus M) b (T \setminus M) \cup \{0\} \), \( a, b \in M \) and \( <a>^3 = 0, <b>^3 = 0 \) or (2) \( M \) is a 0-simple ternary semigroup. Further it is also proved that T be a ternary semigroup containing identity and not containing 0. Then every primary ideal is prime as well as maximal if and only if T is either a simple ternary semigroup or a 0-simple extension of a simple ternary semigroup. It is proved that T be a ternary semigroup containing 0 and identity with the maximal ideal M and suppose every nonzero primary ideal is prime. Then T/M is a 0-simple ternary semigroup such that either (1) \( M = (T \setminus M) a (T \setminus M) a (T \setminus M) \cup \{0\} \), \( a \in M \) and \( <a>^3 = 0 \) or (2) \( M^n = M \) for every odd natural number \( n \). Further it is also proved that T be a ternary semigroup containing identity and not containing 0 in which primary ideals are prime. Then T is a 0-simple ternary semigroup extension of a globally idempotent ternary semigroup.

The contents of chapter 3 are published in “Elixir Advances in pure Mathematics” under the title ‘Ternary semigroups in which prime ideals are maximal and primary ideals are prime and maximal’ [38].

### 3.1. PRIMARY IDEALS:

In this section, the terms left primary ternary ideal, lateral ternary primary ideal, right primary ternary ideal, ternary ideal are introduced. It is proved that A be an ideal in a ternary semigroup T. Then (1) \( X, Y, Z \) are three ideals of T such that \( XYZ \subseteq A \) and \( Y \not\subseteq A, Z \not\subseteq A \Rightarrow X \subseteq \sqrt{A} \) if and only if \( x, y, z \in T, \; <x> <y> <z> \subseteq A \) and \( y, z \not\in A \Rightarrow x \in \sqrt{A} \), (2) \( X, Y, Z \) are three ideals of T such that \( XYZ \subseteq A \) and \( X \not\subseteq A, Z \not\subseteq A \Rightarrow Y \subseteq \sqrt{A} \) if and only if \( <x> <y> <z> \subseteq A \) and \( x, z \not\in A \Rightarrow y \in \sqrt{A} \), (3) \( X, Y, Z \) are three ideals of T such that \( XYZ \subseteq A \) and \( X \not\subseteq A, Y \not\subseteq A \Rightarrow Z \subseteq \sqrt{A} \) if and only if \( <x> <y> <z> \subseteq A \) and \( x, y \not\in A \Rightarrow z \in \sqrt{A} \). Further it is proved that T be a commutative ternary semigroup and A be a ternary ideal of T. Then the conditions (1) A is primary ideal, (2) X, Y are two ideals of S, \( XYZ \subseteq A \) and \( Y \not\subseteq A, Z \not\subseteq A \) then \( X \subseteq \sqrt{A} \), (3) \( x, y, z \in T, xyz \in A, y, z \not\in A \) then \( x \in \sqrt{A} \) are equivalent. It is proved that T be a ternary semigroup with identity and let M be the unique maximal ideal in T. If \( \sqrt{A} = M \) for some ideal A in T, then A is a primary ideal. It is proved that the ideal A is a semiprime ideal of a ternary semigroup T, then (1) A is a prime ideal, (2) A is a primary ideal, (3) A is a left primary ideal, (4) A is a lateral primary ideal, (5) A is a right primary ideal and (6) A is a semiprimary ideal are
equivalemt. Further it is proved that A ternary semigroup T is semiprimary iff prime ideals of T form a chain under set inclusion.

We now introduce the notion of left primary ideal of a ternary semigroup.

**DEFINITION 3.1.1:** An ideal A of a ternary semigroup T is said to be a left primary ideal provided
(i) If X, Y, Z are three ideals of T such that XYZ \(\subseteq A\) and Y \(\not\subseteq A\), Z \(\not\subseteq A\) then X \(\subseteq \sqrt{A}\).
(ii) \(\sqrt{A}\) is a prime ideal.

**EXAMPLE 3.1.2:** Let T = \(\{ a, b, c, d \}\) be a ternary semigroup under the operation given by

\[
\begin{array}{ccc}
. & a & b & c & d \\
a & a & a & a & a \\
b & a & a & a & b \\
c & a & a & b & a \\
d & a & b & b & a
\end{array}
\]

Define the ternary operation \([ \) as \([xyz] = x(yz) = (xy)z\). Then \((T, [ \) ) is a ternary semigroup. Let A = \(\{ a, c \}\), B = \(\{ a, b \}\), C = \(\{ a, b, c \}\) and D = \(\{ a, b, d \}\). Then A, B, C, D are all ideals of T. Now BAC \(\subseteq D\) and A \(\not\subseteq D\), C \(\not\subseteq D\) then B \(\subseteq \sqrt{D}\) and \(\sqrt{D}\) is a prime ideal of T. Therefore D is a left primary ideal of T.

We now introduce the notion of lateral primary ideal of a ternary semigroup.

**DEFINITION 3.1.3:** An ideal A of a ternary semigroup T is said to be a lateral primary ideal provided
(i) If X, Y, Z are three ideals of T such that XYZ \(\subseteq A\) and X \(\not\subseteq A\), Z \(\not\subseteq A\) then Y \(\subseteq \sqrt{A}\).
(ii) \(\sqrt{A}\) is a prime ideal.

**EXAMPLE 3.1.4:** In the example 3.1.2, ABC \(\subseteq D\) and A \(\not\subseteq D\), C \(\not\subseteq D\) then B \(\subseteq \sqrt{D}\) and \(\sqrt{D}\) is a prime ideal of T. Therefore D is a lateral primary ideal of ternary semigroup T.

We now introduce the notion of right primary ideal of a ternary semigroup.
DEFINITION 3.1.5: An ideal A of a semigroup T is said to be a right primary ideal provided

(i) If X, Y, Z are three ideals of T such that XYZ ⊆ A and X ⊈ A, Y ⊈ A then Z ⊆ √A.
(ii) √A is a prime ideal.

EXAMPLE 3.1.6: In the example 3.1.2, ACB ⊆ D and A ⊈ D, C ⊈ D then B ⊆ √D and √D is a prime ideal of T. Therefore D is a right primary ideal of ternary semigroup T.

We now introduce the notion of primary ideal of a ternary semigroup.

DEFINITION 3.1.7: An ideal A of a ternary semigroup T is said to be a primary ideal provided A is a left primary ideal, lateral primary ideal and right primary ideal.

EXAMPLE 3.1.8: In example 3.1.2., the subset D is a primary ideal of ternary semigroup T.

THEOREM 3.1.9: An ideal A in a ternary semigroup T satisfies condition (i) of definition 3.1.1 iff x, y, z ∈ T, <x> <y> <z> ⊆ A and y, z ⊈ A, x ∈ √A.

Proof: Suppose that an ideal A of a ternary semigroup T satisfies the condition (i) of definition 3.1.1. Let x, y, z ∈ T ⇒ <x> <y> <z> ⊆ XYZ and y, z ⊈ A.

Since y, z ⊈ A, <y> ⊈ A, <z> ⊈ A.

Then by assumption, <x> <y> <z> ⊆ A and <y> ⊈ A, <z> ⊈ A ⇒ <x> ⊆ √A. Therefore x ∈ √A.

Conversely suppose that x, y, z ∈ T, <x> <y> <z> ⊆ A and y, z ⊈ A then x ∈ √A.

Let X, Y, Z be three ideals of T such that XYZ ⊆ A and Y ⊈ A, Z ⊈ A.

Suppose if possible X ∉ √A. Then there exists x ∈ X such that x ∉ √A.

Since Y ⊈ A, Z ⊈ A, let y ∈ Y, z ∈ Z so that y, z ∉ A.

Now <x> <y> <z> ⊆ XYZ ⊆ A and y, z ⊈ A ⇒ x ∈ √A. It is a contradiction. Therefore X ⊆ √A. Therefore A satisfies the condition (i) of definition 3.1.1.

THEOREM 3.1.10: An ideal A in a ternary semigroup T satisfies condition (i) of definition 3.1.3, iff <x> <y> <z> ⊆ A and x, z ∉ A ⇒ y ∈ √A.

Proof: The proof is similar to the proof of theorem 3.1.9.
THEOREM 3.1.11 : An ideal A in a ternary semigroup T satisfies condition (i) of definition 3.1.5, if and only if \( <x> <y> <z> \subseteq A \) and \( x, y \not\in A \Rightarrow z \in \sqrt{A} \).

Proof : The proof is similar to the proof of theorem 3.1.9.

THEOREM 3.1.12 : Let T be a commutative ternary semigroup and A be an ideal of T. Then the following conditions are equivalent.

1. A is primary ideal.
2. X, Y are two ideals of S, \( \text{XYZ} \subseteq A \) and \( Y \not\subseteq A \), \( Z \not\subseteq A \) then \( X \subseteq \sqrt{A} \).
3. \( x, y, z \in T, xyz \in A, y, z \not\in A \) then \( x \in \sqrt{A} \).

Proof : (1) \( \Rightarrow \) (2) : Suppose that A is a primary ideal. Then A is a left primary ideal. So by definition 3.1.1, we get \( X, Y, Z \) are three ideals of T, \( \text{XYZ} \subseteq A \), \( Y \not\subseteq A \), \( Z \not\subseteq A \) \( \Rightarrow \) \( X \subseteq \sqrt{A} \).

(2) \( \Rightarrow \) (3) : Suppose that \( X, Y, Z \) are three ideals of T, \( \text{XYZ} \subseteq A \), \( Y \not\subseteq A \), \( Z \not\subseteq A \) \( \Rightarrow \) \( X \subseteq \sqrt{A} \).

Let \( x, y, z \in T, xyz \in A \) and \( y, z \not\in A \). \( xyz \in A \Rightarrow <xyz> \subseteq A \Rightarrow <x<y<z> \subseteq A \).

Also \( y, z \not\in A \Rightarrow <y> \not\subseteq A \) and \( <z> \not\subseteq A \).

Now \( <x> <y> <z> \subseteq A \) and \( <y> \not\subseteq A, <z> \not\subseteq A \).

Therefore by assumption \( <x> \subseteq \sqrt{A} \Rightarrow x \in \sqrt{A} \).

(3) \( \Rightarrow \) (1) : Suppose that \( x, y, z \in T, xyz \in A \) and \( y, z \not\in A \Rightarrow x \in \sqrt{A} \).

Let \( X, Y, Z \) be three ideals of T such that \( \text{XYZ} \subseteq A \) and \( Y \not\subseteq A, Z \not\subseteq A \).

\( Y \not\subseteq A, Z \not\subseteq A \Rightarrow \) there exists \( y \in Y \) and \( z \in Z \) such that \( y, z \not\in A \).

Suppose if possible \( X \not\subseteq \sqrt{A} \) \( \Rightarrow \) there exists \( x \in X \) such that \( x \not\in \sqrt{A} \).

Now \( xyz \in A \).

Therefore \( xyz \in A \) and \( y, z \not\in A \), \( x \not\in \sqrt{A} \). It is a contradiction. Therefore \( X \subseteq \sqrt{A} \).

Let \( x, y, z \in T \) and \( xyz \in \sqrt{A} \). Suppose that \( y, z \not\in \sqrt{A} \).

Now \( xyz \in \sqrt{A} \Rightarrow (xyz)^m \in A \Rightarrow x^m y^m z^m \in A \) for some odd natural number \( m \).

Since \( y, z \in \sqrt{A} \), \( y^m, z^m \in A \). Now \( x^m y^m z^m \in A \Rightarrow x^m \in \sqrt{A} \Rightarrow x \in \sqrt{A} = \sqrt{A} \).

\( \sqrt{A} \) is a completely prime ideal and hence \( \sqrt{A} \) is a prime ideal. Therefore A is a left primary ideal. Similarly A is a lateral primary ideal and right primary ideal.

Hence A is a primary ideal.

NOTE 3.1.13 : In an arbitrary ternary semigroup a left primary ideal is not necessarily a lateral primary ideal or a right primary ideal.
EXAMPLE 3.1.14: In example 3.1.2, D is a left primary ideal but neither lateral primary ideal nor right primary ideal.

THEOREM 3.1.15: Let T be a ternary semigroup with identity and let M be the unique maximal ideal in T. If $\sqrt{A} = M$ for some ideal A in T, then A is a primary ideal.

Proof: Let $<x>y<z> \subseteq A$ and $y, z \notin A$. If $x \notin \sqrt{A}$ then $<x>y \not\subseteq \sqrt{A} = M$. Since M is the union of all proper ideals in T, we have $<x>y = T \Rightarrow y, z \in <x>y$ and hence $<y> = <x>y < z> \subseteq A$. It is a contradiction. Therefore $x \in \sqrt{A}$. Clearly $\sqrt{A} = M$ is a prime ideal. Thus A is left primary. By symmetry it follows that A is lateral primary and right primary. Therefore A is a primary ideal.

NOTE 3.1.16: If T has no identity, then the theorem 3.1.15 is not true, even if the ternary semigroup has a unique maximal ideal.

EXAMPLE 3.1.17: In example 3.1.4, $\sqrt{<a>} = M$ where $M = \{a, b\}$ is the unique maximal ideal. But $<a>$ is not a primary ideal.

THEOREM 3.1.18: If A is a semiprime ideal of a ternary semigroup T, then the following are equivalent.
1. A is a prime ideal.
2. A is a primary ideal.
3. A is a left primary ideal.
4. A is a lateral primary ideal.
5. A is a right primary ideal.
6. A is a semiprimary ideal.

Proof: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) and (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are clear.

(6) $\Rightarrow$(1): Suppose that A is a semiprimary ideal. Then $\sqrt{A}$ is a prime ideal. Since A is semiprime, A is the intersection of all prime ideals of T containing A. Therefore $A = \sqrt{A}$ is a prime ideal.

THEOREM 3.1.19: A ternary semigroup T is semiprimary iff prime ideals of T form a chain under set inclusion.
**Proof**: Suppose that $T$ is a semiprimary semigroup. Let $A$, $B$ and $C$ be three prime ideals of $T$. Now $\sqrt{A \cap B \cap C} = \sqrt{A} \cap \sqrt{B} \cap \sqrt{C} = A \cap B \cap C$. Therefore $A \cap B \cap C$ is semiprime. By theorem 3.1.18, since $T$ is a semiprimary ternary semigroup it follows that $A \cap B \cap C$ is prime. Suppose that $A \not\subseteq B$, $A \not\subseteq C$, $B \not\subseteq A$, $B \not\subseteq C$ and $C \not\subseteq A$, $C \not\subseteq B$. Then there exists $x \in A \setminus B$, $x \in A \setminus C$, $y \in B \setminus A$, $y \in B \setminus C$ and $z \in C \setminus A$, $z \in C \setminus B$. Now $\langle x \rangle \langle y \rangle \langle z \rangle \subseteq A \cap B \cap C$ and $x, y, z \not\in A \cap B \cap C$.

It is a contradiction. Therefore prime ideals of $T$ form a chain.

Conversely suppose that prime ideals of $T$ form a chain under set inclusion. For every ideal $A$, $\sqrt{A} = \bigcap P_\alpha$, where intersection is over all prime ideals $P_\alpha$ containing $A$ yields $\sqrt{A} = P_\alpha$ for some $\alpha$, so that $A$ is a semiprimary ideal. Therefore $T$ is a semiprimary ternary semigroup.

### 3.2. TERNARY SEMIGROUPS IN WHICH PRIME IDEALS ARE MAXIMAL:

In this section, the terms left cancellative, lateral cancellative, right cancellative, cancellative, invertible elements and ternary group are introduce and we study the structure of cancellative quasi commutative ternary semigroups. In fact we prove that if $T$ is a cancellative quasi commutative ternary semigroup, then (1) $T$ is a primary ternary semigroup, (2) proper prime ideals in $T$ are maximal, (3) semiprimary ideals in $T$ are primary, are equivalent. We obtain a characterization for semipseudo symmetric ternary semigroups with identity in which proper prime ideals are maximal and also we characterize semipseudo symmetric semigroups without identity in which proper prime ideals are maximal and globally idempotent principal ideals from a chain. Further we characterize quasi commutative ternary semigroups containing cancellable elements in which proper prime ideals are maximal.

We now introduce the terms left cancellative ternary semigroup, lateral cancellative ternary semigroup, right cancellative ternary semigroup, cancellative ternary semigroup, invertible and ternary group.

**DEFINITION 3.2.1**: A ternary semigroup $T$ is said to be **left cancellative** if for all $a, b, x, y \in T$, $abx = aby \Rightarrow x = y$. 

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DEFINITION 3.2.2: A ternary semigroup $T$ is said to be \textit{laterally cancellative} if for all $a, b, x, y \in T$, $axb = ayb \Rightarrow x = y$.

DEFINITION 3.2.3: A ternary semigroup $T$ is said to be \textit{right cancellative} if for all $a, b, x, y \in T$, $xab = yab \Rightarrow x = y$.

DEFINITION 3.2.4: A ternary semigroup $T$ is said to be \textit{cancellative} if $T$ is left cancellative, right cancellative and laterally cancellative.

DEFINITION 3.2.5: An element $a$ of a ternary semigroup $T$ is said to be \textit{invertible} in $T$ if there exists an element $b$ in $T$ such that $abx = bax = xab = xba = x$ for all $x \in T$.

DEFINITION 3.2.6: A ternary semigroup $T$ is said to be a \textit{ternary group} if for $a, b, c \in T$, the equations $abx = c, axb = c$ and $xab = c$ have solutions in $T$.

THEOREM 3.2.7: Let $T$ be a ternary semigroup with identity. If (non-zero, assume this $T$ has zero) proper prime ideals in $T$ are maximal, then $T$ is a primary ternary semigroup.

\textit{Proof}: Since $T$ contains identity, $T$ has a unique maximal ideal $M$, which is the union of all proper ideals in $T$. If $A$ is a (nonzero) proper ideal in $T$, then $\sqrt{A} = M$ and hence by theorem 3.1.15, $A$ is a primary ideal. If $T$ has zero and if $< 0 >$ is a prime ideal, then $< 0 >$ is primary and hence $T$ is primary. If $< 0 >$ is not a prime ideal, then $\sqrt{< 0 >} = M$ and hence by theorem 3.1.15, $< 0 >$ is a primary ideal. Therefore $T$ is a primary ternary semigroup.

NOTE 3.2.8: If the ternary semigroup $T$ has no identity, then from example 3.1.2, we remark that theorem 3.2.7, is not true even if the ternary semigroup has a unique maximal ideal. The converse of the theorem 3.2.7, is not true even if the ternary semigroup is commutative.

EXAMPLE 3.2.9: Let $T = \{a, b, 1\}$ be the ternary semigroup under the multiplication given in the following table.

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$a$</td>
<td>$a$</td>
<td>$a$</td>
</tr>
<tr>
<td>$b$</td>
<td>$a$</td>
<td>$b$</td>
<td>$b$</td>
</tr>
<tr>
<td>1</td>
<td>$a$</td>
<td>$b$</td>
<td>1</td>
</tr>
</tbody>
</table>
Now T is a primary ternary semigroup in which the prime ideal $<a>$ is not a maximal ideal.

**THEOREM 3.2.10**: Let T be a right cancellative quasi commutative ternary semigroup. If T is a primary ternary semigroup or a ternary semigroup in which semiprimary ideals are primary, then for any primary ideal Q, $\sqrt{Q}$ is non maximal implies $Q = \sqrt{Q}$ is prime.

**Proof**: Since $\sqrt{Q}$ is non maximal, there exists an ideal A in T such that $\sqrt{Q} \subseteq A \subseteq T$. Let $a \in A \setminus \sqrt{Q}$ and $b, c \in \sqrt{Q}$. Now $Q \subseteq Q \cup <abc> \subseteq \sqrt{Q}$. This implies by theorem 1.4.47, $\sqrt{Q} \subseteq \sqrt{(Q \cup <abc>)} \subseteq \sqrt{(Q)} = \sqrt{Q}$. Hence $\sqrt{(Q \cup <abc>)} = \sqrt{Q}$. Thus by hypothesis $Q \cup <abc>$ is a primary ideal. Let $s, t \in T \setminus A$. Then for some odd natural number $n$, $asbtc = s^atbtc = s^abct \in Q \cup <abc>$. Since $a \notin \sqrt{Q} = \sqrt{(Q \cup <abc>)}$ and $Q \cup <abc>$ is a primary ideal, $stx \in Q \cup <abc>$. If $stx \in <abc>$ then $stx = rabtc$ for some $r \in T^1$ and hence by right cancellative property, we have $s = ra \in A$, a contradiction. Thus $asbtc \in Q$, which implies, since $s \notin Q$, $btc \in Q$ and hence $Q = \sqrt{Q}$. Therefore $Q = \sqrt{Q}$ and so Q is prime.

**THEOREM 3.2.11**: Let T be a right cancellative quasi commutative ternary semigroup. If T is either a primary ternary semigroup of a ternary semigroup in which semiprimary ideals are primary, then proper prime ideals in T are maximal.

**Proof**: First we show that if P is a minimal prime ideal containing a principal ideal $<d>$, then P is a maximal ideal. Suppose P is not a maximal ideal.

Write $M = TP$ and $A = \{x \in T : xmn \in <d> \text{ for some } m, n \in M\}$.

Let $x \in A, s, t \in T$. $x \in A \Rightarrow xmn \in <d> \Rightarrow xmn = stdt$ for some $s_1, t_1 \in T$.

Now $stxmn = st(xmn) = st(s_1t_1) = (st(s_1)t_1) \in <d> \Rightarrow stx \in A$, similarly $sx t \in A$ and $xst \in A$. Therefore A is an ideal of T.

If $x \in A$, then $xmn \in <d> \subseteq P$. Since P is prime ideal and hence $x \in P$. So $A \subseteq P$.

Let $b \in P$ and suppose $N = \{b^kmn : m, n \in M \text{ and } k \text{ is a nonnegative odd integer}\}$.

If $b^kmn, b^pq, b^ruv \in N$ for $m, n, p, q, u, v \in M$ and $k, s, r$ are nonnegative odd integers. Then $(b^kmn)(b^pq)(b^ruv) = b^{k+s+r}mnqpvu \in N$.

Therefore N is a ternary subsemigroup of T containing M properly.

If $b \in P \Rightarrow bmn \in P \Rightarrow bmn \notin M$ and hence $bmn \in N$ and $bmn \notin M$. 

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Since \( P \) is a minimal prime ideal containing \(<d>\), \( M \) is a maximal ternary subsemigroup not meeting \(<d>\). Since \( N \) contains \( M \) properly, we have \( N \cap <d> \neq \emptyset \).

So there exists an odd natural number \( k \) such that \( b^k mn \in <d> \Rightarrow b^k \in A \Rightarrow b \in \sqrt{A} \).

Since \( P \) is prime, by theorem 1.4.27, \( P \) is semiprime and by theorem 1.4.49, \( P = \sqrt{P} \).

Therefore \( P \subseteq \sqrt{A} \Rightarrow P \subseteq \sqrt{A} \subseteq \sqrt{P} = P \). So \( P = \sqrt{A} \).

By hypothesis \( A \) is a primary ideal. Since \( P \) is not a maximal ideal, we have by theorem 3.2.10 \( \sqrt{A} = A \Rightarrow P = A \). Since \( <d> \subseteq P \) and \( <d^3> \subseteq <d> \).

Let \( B = \{ y \in T : ymn \in <d^3> \text{ for some } m, n \in M \} \). As before, we have \( B = P \).

Since \( d \in P = A = B \), we have \( dmn = std^3 \) for some \( s, t \in T^1 \).

Since \( T \) is a quasi commutative ternary semigroup, \( dmn = m^pn = std^2 \) for some natural number \( p \). By right cancellative property, \( m^n = std^2 \), a contradiction.

Therefore \( P \) is maximal. Now if \( P \) is any proper prime ideal, then for any \( d \in P \), \( <d> \) is contained in a minimal prime ideal, which is maximal by the above and hence \( P \) is a maximal ideal.

**COROLLARY 3.2.12:** If \( T \) is a cancellative commutative ternary semigroup such that either \( T \) is a primary ternary semigroup or in \( T \) an ideal \( A \) is primary if and only if \( \sqrt{A} \) is a prime ideal, then the proper prime ideals in \( T \) are maximal.

**Proof:** The proof of this corollary is a direct consequence of theorem 3.2.11.

**THEOREM 3.2.13:** Let \( T \) be a right cancellative quasi commutative ternary semigroup with identity. Then the following are equivalent.

1) Proper prime ideals in \( T \) are maximal.
2) \( T \) is a primary ternary semigroup.
3) Semiprimary ideals in \( T \) are primary.
4) If \( x, y \) and \( z \) are not units in \( T \), then there exists odd natural numbers \( n, m \) and \( p \) such that \( x^n = yzs \), \( y^m = xzt \) and \( z^p = xyu \) for some \( s, t, u \in T \).

**Proof:** Combining theorem 3.2.7, and 3.2.11, we have (1), (2) and (3) are equivalent.

1) \( \Rightarrow \) 4): Assume (1). Since \( T \) contains identity, \( T \) has a unique maximal ideal \( M \), which is the only prime ideal in \( T \). If \( x, y \) and \( z \) are not units.

If \( <x> \nsubseteq M \) then \( <x> = T \Rightarrow 1 \in <x> \Rightarrow x \) is a unit, a contradiction and hence \( x \in M \), similarly \( y, z \in M \). Therefore \( \sqrt{<x>} = \sqrt{<y>} = \sqrt{<z>} = M \)
\[ y, z \in \sqrt{\langle x \rangle}, \ x, z \in \sqrt{\langle y \rangle} \text{ and } x, y \in \sqrt{\langle z \rangle} \Rightarrow x^n = yzs, \ y^m = xzt \text{ and } z^p = xyu \text{ for some } s, t, u \in T. \]

4) \( \Rightarrow \) 2) : Let A be any ideal in T and \( xyz \in A \). Suppose that \( x, y, z \) are not units in T. Let \( y, z \notin A \), then \( x^n = yzs \Rightarrow x^{n+2} = xxyzs \in A \). Therefore \( x \in \sqrt{A} \).

Therefore A is left primary. Similarly A is lateral primary and right primary. Therefore T is primary ternary semigroup.

**NOTE 3.2.14 :** If T has 0, then the theorem 3.2.13 is true by assuming nonzero proper prime ideals are maximal.

**THEOREM 3.2.15 :** Let T be a right cancellative quasi commutative ternary semigroup not containing identity. Then the following are equivalent.

1) T is a primary ternary semigroup
2) Semiprimary ideals in T are primary
3) T has no proper prime ideals.
4) If \( x, y \in T \), then there exists odd natural numbers \( n, m \) and \( p \) such that \( x^n = yzs \), \( y^m = xzt \) and \( z^p = xyu \) for some \( s, t, u \in T \).

**Proof :** (1) \( \Rightarrow \) (2) Since T is primary ternary semigroup, then its every ideal is primary. Therefore semiprimary ideal is also primary.

(2) \( \Rightarrow \) (3) : Assume (2). By theorem 3.2.11, proper prime ideals of T are maximal and hence if P is any prime ideal, then P is maximal. Let \( a, b, c \in T \cap P \). Suppose \( abc \notin T \cap P \)
\[ \Rightarrow abc \in P \Rightarrow \text{either } a \in P \text{ or } b \in P \text{ or } c \in P, \text{ a contradiction. Therefore } abc \in T \cap P. \]
Clearly \( T \cap P \) satisfies associative property. Therefore \( T \cap P \) is ternary semigroup. Let \( a, b \in T \cap P \).
Then \( aaT \notin P \) and hence \( P \cup aaT = T \Rightarrow b \in aaT \Rightarrow b = aax \) for some \( x \in T \). If \( x \in P \), then \( b \in P \), a contradiction. Therefore \( aax = b \) has a solution in \( T \cap P \). Similarly \( yaa = b \) has a solution in \( T \cap P \) and hence \( T \cap P \) is a ternary group. Let \( e \) be the identity of the group \( T \cap P \).
Now \( e \) is an idempotent in T and since T is a right cancellative ternary semigroup, then \( e \) is a left identity and lateral identity of T. Since T is a quasi commutative ternary semigroup, idempotents in T are commute and hence \( e \) is the identity of T, a contradiction, since T has no identity. Therefore T has no proper prime ideals.

3) \( \Rightarrow \) 4) : Suppose T has no proper prime ideals. Then for any ideal A of T, \( \sqrt{A} = T \).

Let \( x, y, z \in T \). Now \( \sqrt{\langle x \rangle} = \sqrt{\langle y \rangle} = \sqrt{\langle z \rangle} = T \Rightarrow y, z \in \sqrt{\langle x \rangle} \). Therefore \( x, z \in \sqrt{\langle y \rangle} \) and
x, y ∈ √< z > ⇒ y^m, z^p ∈ < x >, x^n, z^p ∈ < y > and x^n, y^m ∈ < z > for some odd natural numbers n, m, p ⇒ x^n = yzs, y^m = xzt and z^p = xyu for some s, t, u ∈ T.

4) ⇒ 1): Let A be any ideal of T. Let xyz ∈ A, Suppose that x, y, z are not units in T, then x^n = yzs ⇒ x^{n+2} = xxyzs ∈ A ⇒ x ∈ √A. Therefore A is left primary. Since T is quasi commutative ternary semigroup and hence A is lateral primary and right primary. Therefore A is primary and hence T is a primary ternary semigroup. This completes the proof of the theorem.

THEOREM 3.2.16: Let T be a right cancellative quasi commutative ternary semigroup. Then the following are equivalent.

1) T is a primary ternary semigroup.
2) Semiprimary ideals in T are primary.
3) Proper prime ideals in T are maximal.

Proof: The proof of this theorem is a direct consequence of theorem 3.2.13, and 3.2.15.

COROLLARY 3.2.17: Let T be a cancellative commutative ternary semigroup. Then T is a primary ternary semigroup if and only if proper prime ideals in T are maximal. Furthermore T has no idempotents except identity, if it exists.

Proof: The proof of this corollary is a direct consequence of theorem 3.2.16.

THEOREM 3.2.18: Let T be a semipseudo symmetric ternary semigroup with identity. Then the following are equivalent.

1) Proper prime ideals in T are maximal.
2) T is either a simple ternary semigroup and so Archimedean ternary semigroup or T has a unique prime ideal P such that T is a 0-simple extension of the Archimedean ternary subsemigroup P.

In either case T is a primary ternary semigroup and T has at most one globally idempotent principal ideal.

Proof: (1) ⇒ (2): Suppose proper prime ideals in T are maximal. If T is a simple ternary semigroup, then clearly T is an Archimedean ternary semigroup. If T is not a simple ternary semigroup, then T has a unique maximal ideal P, which is also the unique prime ideal. Since P is a maximal ideal in T, we have T/P = T\{P\} is a 0-simple ternary semigroup.

Let a, b, c ∈ P. Since P is the prime ideal, then its intersection is also prime and hence
\[\sqrt{a} = \sqrt{b} = \sqrt{c} = P.\] So by theorem 1.5.31, \(a^3 \subseteq b\). This implies \(a^{3+2} \in PbP\). So \(P\) is an Archimedean ternary subsemigroup of \(T\).

(2) \(\Rightarrow\) (1) : Assume 2), **Case-1** : Suppose \(T\) is simple. Therefore \(T\) has no proper prime ideals and hence there exist no proper ideal of \(T\) containing \(P \Rightarrow P\) is maximal. Therefore 1) is true.

**Case-2** : Suppose \(T\) is not simple. Then \(T\) has unique proper prime ideal \(P\) such that \(T\) is a 0-simple extension of \(P\). Therefore \(T/P\) is 0-simple. By theorem 3.2.7, \(T\) is a primary ternary semigroup. Suppose \(<a>, <b>\) and \(<c>\) be three proper globally idempotent principal ideals. Then \(\sqrt{a} = \sqrt{b} = \sqrt{c} = P\). So by theorem 1.5.31 \(a^3 = b\). Since \(<a>\) is globally idempotent, \(<a> \subseteq <b>\). Similarly we can show that \(<b> \subseteq <a>\). Therefore \(<a> = <b>\). Similarly we can show that \(<b> = <c>\) and hence \(<a> = <b> = <c>\).

**THEOREM 3.2.19** : Let \(T\) be a semipseudo symmetric ternary semigroup without identity. Then the following are equivalent.

1) Proper prime ideals in \(T\) are maximal and globally idempotent principal ideals form a chain.

2) \(T\) is an Archimedean ternary semigroup or there exists a unique prime ideal \(P\) in \(T\) and \(T\) is 0-simple extension of the Archimedean ternary subsemigroup \(P\).

3) Proper prime ideals in \(T\) are maximal and \(T\) has atmost two distinct globally idempotent principal ideals with one of its radical is \(T\) itself.

**Proof** : (1) \(\Rightarrow\) (2) : If \(T\) has no proper prime ideals, then by theorem 1.5.38, \(T\) is an Archimedean ternary semigroup. Suppose \(T\) has proper prime ideals. Let \(M\) and \(N\) be two proper prime ideals in \(T\). By assumption \(M\) and \(N\) are maximal ideals in \(T\) and if \(a \in T\mathbb{N} \Rightarrow a \notin M \Rightarrow a^3 \notin M \Rightarrow M \cup <a^3> = T \Rightarrow a \in <a^3> \Rightarrow a\) is semisimple and hence every element in \(T\mathbb{N}\) is semisimple. Similarly every element in \(T\setminus M\) is semisimple. Let \(a \in T\setminus M\) and \(b \in T\mathbb{N}\). Now \(a\) and \(b\) are semisimple elements and hence \(<a>\) and \(<b>\) are globally idempotent principal ideals. By hypothesis either \(<a> \subseteq <b>\) are \(<b> \subseteq <a>\). Suppose \(<a> \subseteq <b>\). If \(b \in M\), then \(a \in M\), a contradiction.

So \(b \in T\setminus M\) and \(a, b \in T\setminus M \Rightarrow M \cup <a> = M \cup <b> = T \Rightarrow <a> = <b>\). Similarly we can show that if \(<b> \subseteq <a>\), then also \(<a> = <b>\). From this we can conclude that \(T\setminus M = T\mathbb{N}\) and hence \(M = N\). Thus \(T\) has a unique prime ideal. By an argument similar to theorem 3.2.18, We can prove that \(P\) is an Archimedean ternary subsemigroup of \(T\).
(2) ⇒ (3) : If T is an Archimedean ternary semigroup, then clearly by theorem 1.5.38, T has no proper prime ideals. Let a > and b > be two globally idempotent principal ideals. Now since T has no proper prime ideals, we have \( \sqrt{a} = \sqrt{b} = T \). By theorem 1.5.31, \( a^3 \subseteq b > \) and \( b^3 \subseteq a > \). Thus we have \( a > \subseteq b > \) and \( b > \subseteq a > \). So \( a > = b > \). Suppose T has a unique prime ideal P such that T is a 0-simple extension of the Archimedean ternary subsemigroup P. Since T/P is a 0-simple ternary semigroup, we have P is a maximal ideal. Now for every \( a, b \in T \), we have \( \sqrt{a} = \sqrt{b} = T \). Let a > and b > be two globally idempotent principal ideals and \( a, b \in P \). Now \( \sqrt{a} = \sqrt{b} = P \) and hence \( a > = b > \). Thus T has at most two proper globally idempotent principal ideals one of it’s radical is T itself.

(3) ⇒ (1) : Let a >, b > are two globally idempotent principal ideals in T. Let \( \sqrt{b} = T \Rightarrow a \in \sqrt{b} \Rightarrow a > \subseteq \sqrt{b} \Rightarrow a > \subseteq b > \Rightarrow a > \subseteq b > \). Therefore globally idempotent principal ideals form a chain.

**Theorem 3.2.20** : Let T be a semipseudo symmetric ternary semigroup with \( T \neq T^3 \). Then T is a primary ternary semigroup in which proper prime ideals in T are maximal if and only if T is an Archimedean ternary semigroup.

**Proof** : Let T be an Archimedean ternary semigroup. Then by theorem 1.5.38, T has no proper prime ideals. Hence it is trivially true that proper prime ideals are maximal. Let A be any ideal in T such that \( x > y > z > \) and \( y, z \notin A \). Since T is an Archimedean ternary semigroup \( x^3 \in TyTzT \Rightarrow x^3 \in y > z > \) and \( x^2 \in x > \Rightarrow x^{x+2} \in x > y > z > \). Now \( x^{x+2} \in x > y > z > \subseteq A \). So by theorem 1.5.31, \( x \in \sqrt{A} \). Thus A is left primary. Similarly we can show that A is lateral primary as well as right primary. Therefore T is a primary ternary semigroup in which proper prime ideals are maximal.

Conversely suppose that T is a primary ternary semigroup in which proper prime ideals are maximal. Now T is a semiprimary ternary semigroup and hence by theorem 3.2.19, prime ideal in T form a chain. Let P and Q be two proper prime ideals. Therefore P, Q are maximal. Since prime ideals form a chain, then \( P \subseteq Q \) or \( Q \subseteq P \). Therefore P = Q and hence T has a unique proper prime ideal which is also the unique maximal ideal. Now every element of T/P is semisimple and hence T/P \( \subseteq T^3 \). Let a, b \( \in T/P \) and \( x \in P \). If \( a > b > x > \neq x > \), then since T is a primary ternary semigroup and \( x \notin a > b > x > \), since prime ideal is unique, we have \( a \in \sqrt{(a > b > x > P} = P \). a
contradiction. So $< a > < b > < x > = < x >$ for all $x \in P$ and hence by theorem 1.5.38, $T$ is an Archimedean ternary semigroup.

3.3. TERNARY SEMIGROUPS IN WHICH PRIMARY IDEALS ARE PRIME AND MAXIMAL:

In this section, it is proved that $T$ be a ternary semigroup containing 0 and identity with the maximal ideal $M$. Then every nonzero primary ideal is prime as well as maximal if and only if $T/M$ is a 0-simple ternary semigroup with either (1) $M = (T \setminus M) a (T \setminus M) b (T \setminus M) \cup \{0\}, a, b \in M$ and $< a >^3 = 0, < b >^3 = 0$ or (2) $M$ is a 0-simple ternary semigroup. Further it is also proved that $T$ be a ternary semigroup containing identity and not containing 0. Then every primary ideal is prime as well as maximal if and only if $T$ is either a simple ternary semigroup or a 0-simple extension of a simple ternary semigroup. It is proved that $T$ be a ternary semigroup containing 0 and identity with the maximal ideal $M$ and suppose every nonzero primary ideal is prime. Then $T/M$ is a 0-simple ternary semigroup such that either (1) $M = (T \setminus M) a (T \setminus M) a (T \setminus M) \cup \{0\}, a \in M$ and $< a >^3 = 0$ or (2) $M^n = M$ for every odd natural number $n$. Further it is also proved that $T$ be a ternary semigroup containing identity and not containing 0 in which primary ideals are prime. Then $T$ is a 0-simple ternary semigroup extension of a globally idempotent ternary semigroup.

We begin with the following.

THEOREM 3.3.1: Let $T$ be a ternary semigroup containing 0 and identity with the maximal ideal $M$. Then every nonzero primary ideal is prime as well as maximal if and only if $T/M$ is a 0-simple ternary semigroup with either

1. $M = (T \setminus M) a (T \setminus M) b (T \setminus M) \cup \{0\}, a, b \in M$ and $< a >^3 = 0, < b >^3 = 0$ or
2. $M$ is a 0-simple ternary semigroup.

Proof: Suppose every nonzero primary ideal is prime and maximal. Since nonzero prime ideals are maximal, by theorem 3.2.7, $T$ is a primary ternary semigroup. If $< 0 >$ is the maximal ideal of $T$, then the proof of this theorem is trivial. Suppose $T$ has nonzero maximal ideal $M$. Since $T$ is a primary ternary semigroup and every nonzero primary ideal is maximal, we have $M$ is the only nonzero proper ideal in $T$. Since $M$ is a maximal ideal, $T/M$ is a 0-simple ternary semigroup. Now for every nonzero $a, b \in M$, $< a > = M$ and $< b > = M$. 

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Since $M^3$ is an ideal contained in $M$, either $M^3 = 0$ or $M^3 = M$.

If $M^3 = 0$, then for all $a, b, c \in M$, $<a> <b> <c> = 0$ and $<a>^3 = 0$, $<b>^3 = 0$ for all $a, b \in M$. Since for all nonzero $a, b, c \in M$, $<a> = <b> = <c> = M$, we have $c = fagbh$ for some $f, g, h \in T$. If $f$ or $g$ or $h \in M$, then by the above $c = 0$, a contradiction. So $f, g, h \in TM$.

Therefore $(T \setminus M) a (T \setminus M) b (T \setminus M) \cup \{0\}, a, b \in M$ and $<a>^3 = 0$, $<b>^3 = 0$.

If $M^3 = M$, then for every nonzero $a, b \in M$, we have $MaMbM = MTaTMbTM = M^5 = M$.

Therefore $M$ is a 0-simple ternary semigroup.

Conversely if $T/M$ is a 0-simple ternary semigroup with either $M = (T \setminus M) a (T \setminus M) b (T \setminus M)$ such that $a, b \in M$ and $<a>^3 = 0$, $<b>^3 = 0$ or $M$ is a 0-simple ternary semigroup, then clearly $M = <0>$ and $T$ has no other ideals, or

if $M = <0>$. Since $T/M$ is 0-simple the $<0>$ is maximal ideal of $T$.

Therefore $T$ has no other nonzero ideals.

Suppose $M \neq <0>$. Since $T/M$ is 0-simple then $M$ is a maximal ideal of $T$.

Let $A$ is any nonzero proper ideal. Therefore $A \subseteq M$. $a \in A \Rightarrow a \in M \Rightarrow <a> \subseteq M$.

$M = (T \setminus M) a (T \setminus M) \subseteq TaT \subseteq <a> \Rightarrow M \subseteq <a>$. Therefore $<a> = M$ and hence $M$ is the only nonzero ideal in $T$. Thus we have the conclusion.

**COROLLARY 3.3.2**: Let $T$ be a ternary semigroup containing identity and not containing 0. Then every primary ideal is prime as well as maximal if and only if $T$ is either a simple ternary semigroup or a 0-simple extension of a simple ternary semigroup.

**Proof**: If $T$ does not contain 0, then the case $M^3 = 0$ in the theorem 3.3.1, does not arise. Therefore the proof of this corollary is a direct consequence of theorem 3.3.1.

**THEOREM 3.3.3**: Let $T$ be a ternary semigroup containing 0 and identity with the maximal ideal $M$. Suppose that every nonzero primary ideal is prime. Then $T/M$ is a 0-simple ternary semigroup such that either

1) $M = (T \setminus M) a (T \setminus M) a (T \setminus M) \cup \{0\}, a \in M$ and $<a>^3 = 0$ or
2) $M^n = M$ for every odd natural number $n$.

**Proof**: Suppose every non zero primary ideal is prime. If $M^3 = 0$. Let $P$ be the any prime ideal. If $0 \in P \Rightarrow M^3 \subseteq P \Rightarrow M \subseteq P$. Clearly $P \subseteq M$. Therefore $M = P$ and hence $M$ is the unique prime ideal in $T$. Now $\sqrt{<a>} = M$ for every nonzero $a \in M$ and thus $<a>$ is
primary by theorem 3.1.15, Then by hypothesis \(< a >\) is prime and hence \(< a > = M\).
Therefore the conclusion follows as in the proof of theorem 3.3.1. Let \(M^3 \neq 0\) and \(P\) be any proper prime ideal containing \(M^3\). Therefore \(M^3 \subseteq P \Rightarrow M \subseteq P\). Clearly \(P \subseteq M\).
Therefore \(M = P\) and hence \(\sqrt{M^3} = M\). By theorem 3.1.15, \(M^3\) is a primary ideal and hence \(M^3\) is a prime ideal by hypothesis. Thus \(M = M^3\) and hence \(M = M^n\) for every odd natural number \(n\).

**COROLLARY 3.3.4 :** Let \(T\) be a ternary semigroup containing identity and not containing \(0\) in which primary ideals are prime. Then \(T\) is a 0-simple ternary semigroup extension of a globally idempotent ternary semigroup.

**Proof :** The proof of this corollary is a direct consequence of theorem 3.3.3.