Chapter - III

Theory & Methodology
THEORY AND METHODOLOGY

3.0 INTRODUCTION:

The process of comparison of inputs and/or outputs of Decision Making Units (DMUs) with their envelopment surface counterparts is known as Data Envelopment Analysis (DEA).

- Data Envelopment Surface envelopes the inputs and outputs of all DMUs.
- Data Envelopment surface is known as Frontier production Function.
- Technically the most efficient DMUs, determine the Envelopment Frontier.
- In the proposed study a DMU is the total manufacturing sector of an Indian State.
- The constraints which constitute the envelopment frontier can be derived by an axiomatic approach (wide 3.1)

INPUT EFFICIENCY:

Failure to operate on production frontier leads to productive inefficiency, input or output oriented.

- An inefficient DMU that can not adjust its output should be able to reduce its inputs to gain input efficiency.
- The present study hypothesizes radial input reduction.
- Input efficiency measurement identifies five notions of efficiency.
Pure technical, scale, overall technical, allocative and overall productive are the five notions of input productive efficiencies (wide 3.3.)

Pure technical and overall technical efficiencies can be determined relative to variable returns to scale and constant returns to scale production frontiers.

The production possibility set that admits variable returns to scale can be symbolically expressed as,

$$GR^v = \left\{ \left( x, u \right) : \sum_{i=1}^{N} \lambda_i x_i \leq x, \sum_{i=1}^{N} \lambda_i u_i \geq u, \sum_{i=1}^{N} \lambda_i = 1, \lambda_i \geq 0 \right\}$$

The production possibility set that admits constant returns to scale takes the form,

$$GR^k = \left\{ \left( x, u \right) : \sum_{i=1}^{N} \lambda_i x_i \leq x, \sum_{i=1}^{N} \lambda_i u_i \geq u, \lambda_i \geq 0 \right\}$$

Pure technical efficiency can be measured relative to VRTS Frontier,

$$\lambda^v = \text{Min}\left\{ \lambda : (\lambda x_0, u_0) \in GR^v \right\}$$

Over all technical efficiency can be measured relative to CRTS frontier,

$$\lambda^k = \text{Min}\left\{ \lambda : (\lambda x_0, u_0) \in GR^k \right\}$$

$$GR^v \subseteq GR^k \Rightarrow \lambda^v \geq \lambda^k \ (\text{wide 3.1 (b)}) .$$

Scale efficiency is a derived measure,

$$SE = \frac{OTE}{PTE} = \frac{\lambda^k}{\lambda^v}, \text{(wide 3.2)}$$

The remaining two notions of input efficiency, viz., input allocative and overall productive efficiencies are measured relative to cost frontier.
Overall productive efficiency (input oriented) measure requires a knowledge of factor minimal cost.

Factor minimal cost is the minimum cost of the inputs incurred to produce a target output.

Departure of factor minimal cost from the observed cost leads to overall productive inefficiency. (wide (3.2))

The ratio of factor minimal cost to observed cost measures overall productive efficiency of the target DMU.

The radial measure of input allocative efficiency is a derived measure.

Input allocative efficiency can be interpreted as the ratio of factor minimal cost to the cost of overall technically efficient input vector. (wide (3.3)).

The measures proposed can be estimated solving appropriate linear programming problems.

(3.1.13) leads to input pure technical efficiency.

(3.4.3) leads to input overall technical efficiency.

(3.2.7) leads to factor minimal cost and hence to input overall and allocative efficiencies.

Input oriented productive efficiencies can be assessed using a parametric frontier production and cost functions. One of the popular, widely applied, but a restricted production function is the Cobb-Douglas production frontier.
Output elasticities with respect to inputs are constant.

- Elasticity of substitution between any pair of inputs is unity.
- The Cobb-Douglas production function does not allow complementarity between inputs.
- Any departure from one or more of the above restrictions leads to the specification error.
- The CD production frontier cannot handle multiple outputs case.

Assuming that the underlying production frontier is of Cobb-Douglas type, parametric expressions for input pure and overall technical efficiencies are derived.

- Pure technical (input oriented) efficiency measure is,

\[ \lambda^v = \left( \frac{\mu_0}{\prod_{i=1}^{n} \lambda_i} \right)^{\frac{1}{\theta}} \]

where \( \sum_{i=1}^{n} \alpha_i = \theta \)

\( 0 \leq \alpha_i \leq 1 \)

- Overall technical (input oriented) efficiency measure is,

\[ \lambda^t = \left( \frac{\mu_0}{\prod_{i=1}^{n} \lambda_i} \right) \]

\( \sum_{i=1}^{n} \alpha_i = 1 \)

\( 0 \leq \alpha_i \leq 1 \)
Input oriented scale efficiency measure is,

\[ \lambda^* = \frac{\mu_0}{\prod_{i=1}^{m} x_i^0} \] (wide 3.5)

Sometimes one or more of the inputs are such that the production manager do not have control on them. In such cases it is desirable to find technical efficiency of the inputs over which the DMU has control.

A linear programming problem is proposed to measure single input efficiency (wide (3.4.4)).

Single input technical efficiency measures are constructed for the Cobb-Douglas frontier. (wide (3.5.2)).

To implement the Cobb-Douglas measure of pure technical, overall technical and scale efficiencies a knowledge of its parameters is desirable.

Following C.P. Timmer the frontier parameters can be obtained by linear programming approach. (wide (2.3)).

To measure cost efficiency of Cobb-Douglas technology factor minimal cost is desired.

Exploiting the duality between cost and production function, the factor minimal cost is derived.

If production function is Cobb-Douglas its dual cost function is also Cobb-Douglas (Principle of self duality).

Factor minimal cost function of Cobb-Douglas technology,
\[ Q(u, p) = A u^f \prod_{i} p_i^0 \] (wide (3.1.8.9))

- The ratio of factor minimal cost \( Q(u, p) \) to the observed cost leads to a parametric estimate of overall productive efficiency.

**OUTPUT EFFICIENCY:**

An inefficient DMU that can not adjust its inputs, looks for further output augmentation, in order to gain output oriented productive efficiency.

- The current study hypothesises radial output augmentation.

- Output efficiency measurement identifies five notions of efficiency.

- Output pure technical, overall technical, scale efficiencies are measured relative to the frontier production function that admits constant or variable returns to scale. (wide (3.8)).

- Output pure technical efficiency can be measured relative to VRTS frontier

\[ \delta^* = \max_{\delta} \{ \delta : (x_0, \delta u_o) \in GR^* \} \]

- Output overall technical efficiency can be measured relative to CRS frontier

\[ \delta^t = \max_{\delta} \{ \delta : (x_0, \delta u_o) \in GR^t \} \]

- \( GR^* \subseteq GR^t \Rightarrow \delta^* \leq \delta^t \) (wide (3.7))

- The output scale efficiency is a derived measure

\[ SE = \frac{OTE}{PTE} = \frac{\delta^t}{\delta^*}, \] (wide (3.7))
The notions of output overall productive and allocative efficiencies are related to maximum revenue frontier, $\Pi(r, x_0)$, where $x_0$ is input vector of target DMU and $r$ is the output price vector.

- The output overall productivity measurement requires an estimate of potential revenue.
- Potential revenue is maximum revenue that the input vector of target DMU can generate to gain overall productive efficiency.
- Departure of potential revenue from the observed revenue leads to overall productive inefficiency.
- The ratio of potential revenue to observed revenue measures output overall productive efficiency.
- Output allocative efficiency measure is a derived measure.
- The output allocative efficiency measure can be interpreted as the ratio of maximal revenue to the revenue that can be attained from the overall technical efficient output vector. (wide (3.8))

For the Cobb-Douglas production structure expressions for output overall technical and pure technical efficiency are obtained.

- The output overall technical efficiency measure implied by the Cobb-Douglas production structure is,

$$
\delta^* = \frac{A\prod_{i}^{n} x_i^{\alpha_i}}{u}, \quad \sum_{i=0}^{n} \alpha_i = 1, \quad 0 \leq \alpha_i \leq 1
$$

77
The output pure technical efficiency measure of Cobb-Douglas production structure is,

$$\delta^* = \frac{\prod_{i=1}^{n} x_i^{\alpha_i}}{u}, \quad 0 \leq \alpha_i \leq 1 \text{ (wide (3.9))}$$

The output efficiency measures defined above can be estimated formulating and solving suitable linear programming problems.

- (3.7.6) leads to output pure technical efficiency measure.
- (3.7.4) leads to output overall technical efficiency measure.
- (3.8.2) leads to maximum revenue.

**RELATIONSHIP BETWEEN INPUT AND OUTPUT MEASURES:**

Under certain conditions input and output technical efficiency measures are related to each other.

- If returns to scale are constant, the overall input and output measures are inversely related.

$$\lambda^k = (\delta^k)^{-1} \quad \text{(wide (3.8))}$$

- For Cobb-Douglas production structure,

$$\lambda^{-\theta} = \delta, \text{ where } \theta = \sum_{i=1}^{n} \alpha_i$$

- $\theta$ measures returns to scale.

- $\theta = 1 \Rightarrow$ Returns to scale are constant.

- If returns to scale are constant, for Cobb-Douglas production structure,

$$\lambda = \delta^{-1} \text{ (wide (3.9))}$$
Input efficiency measures can be derived relative to input level sets, while output efficiency measures can be obtained relative to output sets.

- GR refers to GRAPH or the production possibility set.
  \[ GR = \{(x, u): x \text{ produces } u\} \]

- Graph and input level sets are related as follows:
  \[ GR(x, u_o) = L(u_o) \]

- Graph and output level sets are related as follows:
  \[ GR(x_o, u) = P(x_o) \]

- Input and output level sets are dualistically related.
  \[ L(u) = \{x: u \in P(x)\} \]
  \[ P(x) = \{u: x \in L(u)\} \]

3.1:

In 1957 Michael Farrell introduced a technique with which the efficiency of a production activity could be measured and decomposed into technical and allocative components. The pioneering work of Farrell focussed attention on the concept of productive efficiency and the consequences of its recognition for the modelling of production process. The standard of efficiency used by Farrell was the frontier unit isoquant, a concept which involves the production and cost frontiers. Farrell associated deviations from the frontier isoquant with technical inefficiency and deviations from the cost minimizing input ratios with allocative inefficiency.
Kopp (1981), drawing on the advances made in the construction and estimation of frontier models, has demonstrated that Farrell’s original technique can be generalized from the frontier unit isoquant to non-homothetic frontier production functions. The methods of efficiency measurement discussed by Kopp could then be applied to the derived production frontier and the necessity of estimating the production function directly avoided. In the Schmidt and Lovell (1977) model the firm is permitted to be both technically and allocatively efficient. To measure both forms of inefficiency Schmidt and Lovell estimate a joint system of a frontier production function and the associated firm order conditions for a cost minimum. They developed a method to estimate technical and allocative efficiencies of different firms by considering the duality between stochastic production frontier and cost functions.

The technical efficiency in the manufacturing sector has generally been estimated by a frontier production function approach. The major studies in this direction include Lovell and Schmidt (1977), Forsund (1980), Lovell and Schmidt (1980), Greene and Mayes (1991), Ramaswamy (1990), Bhavani (1991), Agarwal and Goldar (1992). The primary purpose of this chapter is to construct and solve linear programming problems to estimate the technical, scale, allocative and overall productive efficiencies of manufacturing sectors.
Data Envelopment Analysis (DEA), a non parametric approach can be viewed as an axiomatic approach, with the axioms as convexity, inefficiency and minimum extrapolation. Production units which mix vector valued inputs subject to a variation of production techniques in order to produce a vector valued outputs those are called the Decision Making Units (DMU).

Let \((x_j, u_j)\) be the input vector employed and output vector produced by the \(j^{th}\) DMU, \(j = 1, 2, \ldots n\), then

(a) CONVEXITY: The convex set generated by \((x_j, u_j)\) is

\[
S = \{(x, u) : x = \sum_{j=1}^{n} \lambda_j x_j, u = \sum_{j=1}^{n} \lambda_j u_j, \lambda_j \geq 0, \sum_{j=1}^{n} \lambda_j = 1\} \quad \cdots (3.1.1)
\]

![Figure (3.1.1)](image)

(b) INEFFICIENCY: If \(T\) is the production possibility set, then

(i) \((\overline{x}, \overline{u}) \in T, x \geq \overline{x} \Rightarrow (x, u) \in T\) \quad \cdots (3.1.2)

If \(\overline{x}\) produces \(\overline{u}\), then any input vector \(x\) larger than \(\overline{x}\) can also produce \(\overline{u}\).
The outcome of this assumption is augmentation of \( Q \) to \( S \).

(ii) \( u \leq \bar{u} \Rightarrow (\bar{x}, u) \in T \)  

..... (3.1.3)

If \( \bar{x} \) produces \( \bar{u} \), then it can produce any output vector less than \( \bar{u} \). This inefficiency axiom augments \( R \) to \( SUQ \).

(c) **MINIMUM EXTRAPOLATION**: The production possibility set \( T \) is the intersection of all \( T_a \) containing the observations, \( (x_j, u_j) \), \( j = 1, 2, \ldots, n \).

Thus, we have,

\[ T = SUQ R \]

The piece-wise linear production frontier generated by the observed data and the structure given by the axioms (a), (b) and (c) could be depicted on a two dimensional graph sheet as shown below:

![Figure (3.1.2)](image)

The piece-wise linear production frontier is generated by the production units that operate at A and B.
x produce u ⇔ \((x, u) \in T\)

we can find \((\bar{x}, \bar{u}) \in S\) such that \(\bar{x} \leq x, u \geq \bar{u}\)

But \((\bar{x}, \bar{u}) = (\sum \lambda_i x_i, \sum \lambda_i u_i)\)

where \(\lambda_i \geq 0, i = 1, 2, \ldots, n\)

\(\sum \lambda_i = 1\)

Consequently, we obtain,

\(\sum \lambda_i x_i \leq x, \sum \lambda_i u_i \geq u\)

Thus, the production possibility set is characterized by,

\[ T = \{(x, u) : \sum \lambda_i x_i \leq x, \sum \lambda_i u_i \geq u, \lambda_i \geq 0, \sum \lambda_i = 1\} \]  

(3.1.4)

Then, 'T' admits variable returns to the scale.

Input sets of the production technology may be defined as,

\[ L(u_0) = \{x : x \text{ produces } u_0\} \]

\[ = \{x : \sum \lambda_i x_i \leq x, \sum \lambda_i u_i \geq u_0, \lambda_i \geq 0, \sum \lambda_i = 1\} \]  

(3.1.5)
In terms of piece-wise linear technology a typical input set may be expressed as a diagram shown below:

![Diagram](image)

Figure (3.1.4)

The production units which operate at A, B, C, D and E determine the isoquant of \( L(u_0) \).

Another axiom of DEA is the property of ray unboundedness.

![Diagram](image)

Figure (3.1.5)
RAY UNBOUNDEDNESS:

\((x,u) \in T \Rightarrow \lambda(x,u) \in T\), \(\forall \lambda > 0\)

we have \((x_i,u_i) \in T\), \(i = 1,2,\ldots,n\)

\[\Rightarrow \lambda(x_i,u_i) \in T\], \(i = 1,2,\ldots,n\), \(\lambda > 0\)

Consequently, we obtain,

\[\sum \lambda_i (\lambda x_i) \leq x\]

\[\sum \lambda_i (\lambda u_i) \geq u\]

\[\sum (\lambda \lambda_i) x_i \leq x\]

\[\sum (\lambda \lambda_i) u_i \geq u\]

\[\Rightarrow \sum \delta_i x_i \leq x\]

\[\sum \delta_i u_i \geq u\]

\[\sum \delta_i = \sum \lambda \lambda_i = \lambda \sum \lambda_i = \lambda\]

\[\sum \delta_i = \lambda\]

Thus \(\delta_i \geq 0\) does not satisfy the condition \(\sum \delta_i = 1\)

The production possibility set induced by the ray unboundedness assumption is shown in the above figure.

\[T = \{(x,u): \sum \delta_i x_i \leq x, \sum \delta_i u_i \geq u, \delta_i \geq 0\}\]

\[\ldots (3.1.6)\]

'T' admits constant returns to the scale. The constant returns to the scale frontier that is determined by the production units operating at A and B.
Returns to the scale property which is a property of the production frontier, i.e., production surface property. To enquire the nature of returns to scale, constant, decreasing or increasing of an inefficient unit either the output or input is projected onto the frontier and returns to the scale could be estimated.

There are three approaches to measure the technical efficiency of a production unit in a competitive environment, by non-parametric data envelopment analysis.

(a) INPUT APPROACH:

The production frontier which passes through the origin admits constant returns to the scale. The empirical frontier is determined by the production units A and B. For example, the production unit $D(u_0, x_0)$ is inefficient. If the producer who is scale efficient can reduce his inputs to the
level $\lambda x_0$ then he can still produce and achieve a technical efficiency. The value of $\lambda_0$ can be obtained by solving the following optimization problem.

$$\lambda_0 = \text{Min} \lambda$$

such that $\lambda x_0 \in L(u_0)$

In figure (3.1.7.) we have a production frontier with one input and one output. By reducing the input vector $x_0$ radially to the level $\lambda_0 x_0$, the producer attains the input technical efficiency. As seen in the above figure
(3.1.8), any attempt to reduce \( x_0 \) beyond \( \lambda x_0 \) makes the input vector to produce a smaller output than \( u_0 \).

\[ L^c(u_0) \] is the input set of all inputs capable of producing the output \( u_0 \) under constant returns to the scale.

In terms of the empirical production, the problem may be stated as,

\[
\begin{align*}
\text{Min } \lambda \\
\text{such that } & \sum_{i=1}^{k} \lambda_i x_i \leq \lambda x_0 \\
& \sum_{i=1}^{k} \lambda_i u_i \geq u_0 \\
& \lambda_i \geq 0
\end{align*}
\]

...... (3.1.7)

where \( x_i \in R^+_n \) is the input vector of \( i^{th} \) production unit

\( u_i \in R^+_n \) is the output vector of \( i^{th} \) production unit

\( \lambda_i \geq 0 \) are intensity parameters

The parameters \( x_0 \) and \( u_0 \) are the input and output vectors of the production unit whose efficiency has been under evaluation. A few points are found to be worth noting:

(i) For each production unit, to evaluate technical efficiency, one linear programming problem has to be solved. Thus, as many linear programming problems are to be solved as there are production units.

(ii) \( \text{Min } \lambda = 1 \) may imply a weak efficiency.
In figure (3.1.9), the line segments AB, BC, CD and DE constitute the isoquant of the input level set $L^k(u_0)$. The segments AB and CD form weak efficient points and the segments BC and CD constitute efficient points of the production isoquant. Since the production frontier is found to be a piece-wise linear, the isoquant of $L^k(u_0)$ will also be a piece-wise linear.

The producer who operates at P is technically inefficient. To achieve technical efficiency of this unit, the input vector at P is radially reduced to Q and input at P is to be compared with input at Q. At Q the production unit achieves weak technical efficiency only.

Somehow, if P is compared with any point of the segment of BC, the technical efficiency can be estimated. This is possible by inserting slacks appropriately. Consequently, the problem could be stated as
\[ \text{Min } Z = \theta - \epsilon \left[ \sum_{j=1}^{K} \lambda_j u_j - w_\theta + \sum_{j=1}^{K} s_\theta \right] \]

\[ \sum_{j=1}^{K} \lambda_j u_j - w_\theta = u_0 \]

\[ \sum_{j=1}^{K} \lambda_j x_j + s_\theta = x_0 \]

\[ \lambda_j \geq 0, \ j = 1, 2, \ldots \]

Where \( u_0 \) and \( x_0 \) are the input and output vectors of the production unit whose efficiency is under evaluation. The factors \( w_\theta \) and \( s_\theta \) are the vectors of slack variables and \( \epsilon > 0 \) is non-archimedian quantity.

If all slacks vanish, then the potential input vector could be compared with an efficient input vector.

Returns to the scale can be modelled through a homogeneous production function. Let \( u = f(x) \) be a linear homogeneous production frontier and \( \lambda \) be larger than one. Then,

\[ f(\lambda x) = \lambda^\theta f(x) \]

\( \theta > 1 \) implies increasing returns to scale

\( \theta = 1 \) implies constant returns to scale

\( \theta < 1 \) implies decreasing returns to scale

Associated with a well behaved production frontier there is its dual cost frontier.
If \( Q(u,p) \) refers to the factor minimal cost frontier, then \( \left[ \frac{\partial\ln Q}{\partial \ln U} \right] \) is called elasticity of scale which estimates the returns to the scale through the dual cost frontier. For example, the Cobb-Douglas production and cost frontiers are given by

\[
    u = A \prod_{i=1}^{s} x_i^\alpha_i
    \quad \text{..... (3.1.8)}
\]

\[
    Q(u,p) = \bar{A} u^\theta \prod_{i=1}^{s} p_i^{\alpha_i}
    \quad \text{..... (3.1.9)}
\]

where \( \theta = \sum_{i=1}^{s} \alpha_i \)

\[
    \ln Q(u,p) = \ln \bar{A} + \frac{1}{\theta} \ln u + \sum_{i=1}^{s} \frac{\alpha_i}{\theta} \ln p_i
\]

\[
    \frac{\partial \ln Q(u,p)}{\partial \ln u} = \frac{1}{\theta}
\]

\[
    \frac{\partial \ln u}{\partial \ln Q} = \theta = \sum_{i=1}^{s} \alpha_i
\]

Returns to the scale can be modelled into non-parametric frontier production function and hence into the linear programming problems postulated.
The empirical production frontiers with one input and one output production technology as shown in the figure (3.1.10). The line segments joining A and B, B and C, C and D model production technology admitting the variable returning to the scale. The concept of returns to the scale is a production surface property. Production units operate on the line segment joining A and B admit increasing returns to the scale. The production unit that operates at B admits constant returns to the scale. The line segments joining B and C, C and D correspond to the decreasing returns to the scale.

The line segments joining the points O and B, B and C, C and D determine the non-increasing returns to the scale frontier. The line segment joining O and B and the radial extension of the same is the constant returns to scale frontier. The production units operating at E and F are inefficient.

The producer who operates at A is technically inefficient. By horizontal projection onto the surface of VRTS frontier the pure technical efficiency of $A(x_0, u_0)$ can be assessed. By further projection onto CRTS
frontier overall technical efficiency can be estimated. Such a frontier is shown in the above figure (3.1.11)

\[ \lambda^t = \left[ D^t(u_0, k_0) \right]^{-1} \]

\[ \lambda^r = \left[ D^r(u_0, x_0) \right]^{-1} \]

\[ \lambda^t \leq \lambda^r \Leftrightarrow \left[ D^r(u_0, x_0) \right]^{-1} \leq \left[ D^r(u_0, x_0) \right]^{-1} \]

\[ \Leftrightarrow D^t(u_0, x_0) \geq D^r(u_0, x_0) \]  

...... (3.1.10)

Where \( D^t(u_0, x_0) \) and \( D^r(u_0, x_0) \) are Shephard's input distance function associated with CRTS and VRTS frontiers respectively. However, the underlying returns to the scale of \( A(x_0, u_0) \) are increasing.

The production unit that operates at \( A \) whose input and output are \( x_0 \) and \( u_0 \) respectively is found to be technically inefficient. By horizontal projection of \( x_0 \) onto VRTS frontier pure technical efficiency can be achieved. By further projection onto CRTS frontier overall technical efficiency could be achieved. Returns to the scale of \( A \) are decreasing.
\( \lambda^* \leq \lambda \)

(b) INPUT TECHNICAL EFFICIENCY - CONSTANT RETURNS TO SCALE:

\[ \text{Max } \Pi = \lambda \]

such that

\[
\begin{align*}
\sum_{i=1}^{n} \lambda_i x_i &\leq \lambda x_0 \\
\sum_{i=1}^{n} \lambda_i u_i &\geq u_0 \\
\lambda &> 0
\end{align*}
\]

\[ \ldots \text{(3.1.11)} \]

The above LP problem admits constant returns to scale.

(c) INPUT TECHNICAL EFFICIENCY - NON-INCREASING RETURNS TO SCALE:

The empirical production frontier that admits non-increasing returns to scale in one input and one output technology may be expressed as shown in the following figure (3.1.13).

![Diagram](image)

Figure (3.1.13)
The production units which operate at A, B and C determine the non-increasing returns to scale frontier. The production units E and D are inefficient relative to non-increasing returns to scale empirical frontier. To estimate technical efficiency under this technology one solves the Linear programming problem:

$$\text{Min } \Pi = \lambda$$

such that \( \lambda x_0 \in L^D(u_0) \)

Equivalently, \( \text{Min } \Pi = \lambda \)

\[
\begin{align*}
\sum \lambda_i x_i &\leq \lambda x_0 \\
\text{such that } &\sum \lambda_i u_i \geq u_0 \\
\lambda_i &\geq 0 \\
\sum \lambda_i &\leq 1
\end{align*}
\] ...... (3.1.12)

The empirical frontier that admits constant, increasing and decreasing returns to scale may be expressed as shown in the figure (3.1.14).

The variable returns to scale frontier is determined by the decision making units A, B, C and D and production unit is inefficient. The linear
programming problem that identifies the constant, increases and decreases and returns to the scale are,

\[ \text{Min } \Pi = \lambda \]

such that \( \lambda x_0 \in L^* (u_0) \)

Equivalently, we solve,

\[ \text{Min } \Pi = \lambda \]

such that \( \sum \lambda_i x_i \leq \lambda x_0 \)

\( \sum \lambda_i u_i \geq u_0 \)

\( \lambda_i \geq 0 \)

\( \sum \lambda_i = 1 \) \[ \text{..... (3.1.13)} \]

3.2 DECOMPOSITION OF OVER ALL TECHNICAL EFFICIENCY:

We have seen that, \( \lambda^t \leq \lambda^* \), consequently the ratio \( \frac{\lambda^t}{\lambda^*} \) defines scale efficiency. Thus, we define scale efficiency as

\[ SE = \frac{\lambda^t}{\lambda^*} \] \[ \text{..... (3.2.1)} \]

Let us define, overall technical efficiency and pure technical efficiency respectively as follows:

\[ \text{OTE} = \lambda^* \]

\[ \text{PTE} = \lambda^* \]

Thus, \( \text{OTE} = \text{PTE} \times \text{SE} \)

In terms of Shephard's distance functions,

\[ \text{OTE} = [D^t (u_0, x_0)]^{-1} \]
\[ \text{PTE} = \left[ D^*(u_0, v_0) \right]^{-1} \]

\[ \text{SE} = \left[ D^+(u_0, x_0) \right]^{-1} \]

\[ \left[ D^+(u_0, x_0) \right]^{-1} \]

\[ \frac{D^*(u_0, x_0)}{D^+(u_0, x_0)} \]

\[ \ldots \text{(3.2.2)} \]

In terms of input level sets \( L(u) \), the efficiency concepts discussed above can be understood by means of the following diagram:

![Diagram](image)

**Figure (3.2.1)**

**FACTOR MINIMAL COST:**

Associated with every production frontier in primal space there is a dual cost function, which is called the factor minimal cost.

For example, the factor minimal cost function of the Cobb-Douglas frontier production function is also known as Cobb-Douglas. This is called a self duality.

Cobb-Douglas Production frontier:
\[ u_c = A \prod_{i=1}^{n} x_i^{\alpha_i} \] \hspace{1cm} (3.2.3)

Dual cost frontier:

\[ Q(u, p) = A \nu^{\alpha} \prod_{i=1}^{n} x_i^{\alpha_i} \] \hspace{1cm} (3.2.4)

where \( \alpha = \sum_{i=1}^{n} \alpha_i, \ 0 \leq \alpha_i \leq 1 \)

Another such example is the Zellner–Revanker's variable returns to scale production function.

\[ uw^{\alpha} = A \prod_{i=1}^{n} x_i^{\alpha_i} \] \hspace{1cm} (3.2.5)

If \( \theta = 0 \), this production frontier reduces to the Cobb-Douglas production function:

Elasticity of scale: \( \theta(u) = \frac{1}{1+\theta u} \)

Since the elasticity of scale is a function of output, it varies with a variation in \( u \).

The factor minimal cost function associated with the VRTS parametric frontier is,

\[ Q(u, p) = uw^{\alpha} A \prod_{i=1}^{n} x_i^{\alpha_i} \] \hspace{1cm} (3.2.6)

The factor minimal cost can be estimated for an empirical DEA frontier's solving the following optimization problem:

\[ Q(u, p) = \text{Min } px \]
such that $x \in \mathcal{B}(u)$

Equivalently, we express the problem as follows:

$$Q(u, p) = \min px$$

such that

$$\sum \lambda_i x_i \leq x$$

$$\sum \lambda_i u_i \geq u_0$$

$$\lambda_i \geq 0$$

..... (3.2.7)

Where $u_0$ is the output vector of the production unit whose efficiency is under evaluation, P is the vector of input prices.

![Diagram](image)

Figure (3.2.2)

The producer who operates at P is inefficient. By reducing his inputs radially to Q he can achieve an overall technical efficiency. The cost at Q is more than the cost at S. Thus by reallocating his inputs from Q to S, the producer can achieve a minimum cost to produce output $u_0$. S is a cost efficient input point in the input space. However, cost at S is equal to cost at R. The overall productive efficiency or cost efficiency can be gauged by comparing the cost at P with the cost at S.
3.3 OVERALL PRODUCTIVE EFFICIENCY:

From the figure (3.2.2.), the overall productive efficiency is given by

\[ OPE = \frac{OR}{OP} = \frac{Q(u_0, y)}{p x_0} \]  

..... (3.3.1)

Where \( u_0 \) is the output vector of the production unit whose efficiency is under evaluation.

ALLOCATIVE EFFICIENCY: It is a derived measure being obtained by comparing the input cost at \( Q \) with the input cost at \( S \). Thus, we have,

\[ AE = \frac{OR}{OQ} \]

\[ \frac{\text{Factor minimal cost}}{\text{Cost of input vector at } Q} \]  

..... (3.2.3)

DECOMPOSITION OF AN OVERALL PRODUCTIVE EFFICIENCY:

The overall productive efficiency can be decomposed into the product of an overall technical efficiency and allocative efficiencies, in turn the overall technical efficiency can be decomposed into the product of pure technical and allocative efficiencies.

![Figure 3.3.1](image-url)
\[ OPE = \frac{OT QQ OR}{OP OT QQ} \]

\[ = PTE x SE x AE \]

\[ \ldots (3.3.3) \]

Thus, overall productive efficiency could be the product of pure technical, scale and allocative efficiencies.

3.4 SINGLE INPUT TECHNICAL EFFICIENCY:

A producer often fails to control certain inputs. In such cases reduction of input(s) is not possible, therefore those are exogeneously fixed.

![Graph](image.png)

The producer who operates at P is an input technical inefficient. The producer who combines the inputs \((x_1^p, x_2^p)\) produces the scalar output \(u_0\). \(x_2^p\) is exogeneously fixed input. Reduction of input \(x_1^p\) is possible. By reducing this input, the point P is projected onto Q that falls on the isoquant of \(L(u_0)\), thereby the producer attains a single input \((x_1)\) technical efficiency. He solves the following optimization problem:

\[ \text{Min } \lambda \]

such that \( (\lambda x_1^p, x_2^p) \in L(u_0) \)

\[ \ldots (3.4.1) \]
If the underlying production frontier is a piece-wise linear and satisfies the DEA postulates, equivalently we solve the following linear programming problem:

\[
\begin{align*}
\text{Min } & \lambda \\
\text{such that } & \lambda_1 x_{11} + \lambda_2 x_{12} + \ldots + \lambda_k x_{1k} \leq \lambda x_i^p \\
& \lambda_1 x_{21} + \lambda_2 x_{22} + \ldots + \lambda_k x_{2k} \leq x_i^q \\
& \lambda_1 u_1 + \lambda_2 u_2 + \ldots + \lambda_k u_k \geq u_0 \\
& \lambda_i \geq 0, i = 1, 2, \ldots, k \\
& \sum_{i=1}^{k} \lambda_i = 1
\end{align*}
\]  

... (3.4.2)

**RETURNS TO SCALE:**

Imposing constant returns to the scale single input efficiency that can be determined by solving the optimization problem:

\[
\begin{align*}
\text{Min } & \lambda \\
\text{such that } & \sum_{j=1}^{k} \lambda_j x_{1j} \leq \lambda x_{10} \\
& \sum_{j=1}^{k} \lambda_j x_{2j} \leq x_{20} \\
& \sum_{j=1}^{k} \lambda_j u_j \geq u_0 \\
& \lambda_j \geq 0, j = 1, 2, \ldots, k
\end{align*}
\]  

...... (3.4.3)

\(x_{10}, x_{20}\) and \(u_0\) are the inputs and output of the production unit whose efficiency is under evaluation.
In the above figure (3.4.2.), \( L(u_k) \) and \( L^s u_k \) admit variable and constant returns to scale. \( x_k^s \) is exogeneously fixed input.

Input technical efficiency under constant returns to scale = \( \lambda^c \)

Input technical efficiency under variable returns to scale = \( \lambda^v \)

\( \lambda^t \leq \lambda^v \)

Input technical efficiency under non-increasing returns to scale can be estimated by solving the following linear programming problem:

\[
\text{Min } \lambda \\
\text{such that } \sum_{j=1}^{k} \lambda_j x_{ij} \leq \lambda x_{i0} \\
\sum_{j=1}^{k} \lambda_j x_{2j} \leq x_{20} \\
\sum_{j=1}^{k} \lambda_j u_j \geq u_0 \\
\lambda_j \geq 0 \\
\sum_{j=1}^{k} \lambda_j \leq 1
\]

..... (3.4.4)

3.5 INPUT TECHNICAL EFFICIENCY - PARAMETRIC FRONTIER PRODUCTION FUNCTION:

One of the very widely used production frontier both in theoretical and empirical research is the Cobb – Douglas production function.

\( u \in R^+ \)

\( x \in R^+_n \)

\( u \leq \prod_{j=1}^{n} x_j^a \)

..... (3.5.1)
The input vector may be radially reduced, till the input technical efficiency is achieved:

\[ u = A \prod_{i=1}^{n} (\lambda x_i)^{\alpha_i}, 0 \leq \lambda \leq 1 \]

\[ = \lambda^{\sum_{i=1}^{n} \alpha_i} A \prod_{i=1}^{n} x_i^{\alpha_i} \]

\[ \frac{u}{A \prod_{i=1}^{n} x_i^{\alpha_i}} = \lambda^{\sum_{i=1}^{n} \alpha_i} = \lambda^{\theta} \]

where \( \theta = \sum \alpha_i \)

\[ \lambda^* = \frac{u}{A \prod_{i=1}^{n} x_i^{\alpha_i}} \]

A knowledge of \( A \) and \( \alpha_i \) makes the estimation of input technical efficiency possible.

If returns to scale are constant the input technical efficiency reduces to,

\[ \lambda^* = \frac{u}{A \prod_{i=1}^{n} x_i^{\alpha_i}}, \sum_{i=1}^{n} \alpha_i = 1 \]

**SINGLE INPUT EFFICIENCY:**

Let \( u \in R^+ \)

\( x \in R_+^* \)

Suppose, \( x_3 \) is exogenously fixed.
\[ u \leq Ax_1^a x_2^a \]

Reduce \( x_1 \) till an equality is achieved.

Consequently, we obtain

\[ u = Ax_1^a (Ax_2)^a \]

\[ = \lambda^e Ax_1^a x_2^e \]

\[ \lambda^e = \frac{1}{x_2^e} \]

\[ \lambda = \left[ \begin{array}{c} 1 \\ \frac{1}{x_2^e} \end{array} \right] \]

\[ \ldots (3.5.2) \]

3.6 EFFICIENT–WEAK EFFICIENT SUBSETS OF INPUT ISOQUANTS:

A producer who operates on an appropriate input isoquant is technically efficient. If an empirical production frontier satisfies DEA postulates the isoquant is constituted by an efficient and a weak efficient points.

![Diagram of isoquant](image)
From Figure (3.6.1), the line segments AB, BC, CD and DE constitute the isoquant of the input set $L(u_o)$. Further, the segments AB and DE forms weak efficient, BC and CD constitute efficient sub-sets of the input set $L(u_o)$. If a producer operates interior to the input level set, he is technically inefficient.

By radially reducing his input vector in the direction of origin the producer attains a technical efficiency. If the interior point is compared with a weak efficient point the producer is said to be a weak technical efficient.

![Figure (3.6.2)](image)

The figure (3.6.2) depicts that the producer who operates at P is technically inefficient. By radially reducing inputs he compares P with Q, consequently weak technical efficiency is achieved. However, the second input can further be reduced so that P is compared with R, where R belongs to an efficient sub-set of the input isoquant. Therefore, a weak technical
CROSS EFFICIENCY:

To estimate cross efficiency of the decision making production units the dual of the following problem is solved:

Primal: \( \text{Min } \lambda \)

such that \[
\begin{align*}
\sum_{i=1}^{r} \lambda_i x_i & \leq \lambda x_0 \\
\sum_{i=1}^{r} \lambda_i u_i & \geq u_0 \\
\lambda_i & \geq 0, s = 1, 2, 3, \ldots, r
\end{align*}
\]

..... (3.6.1)

Dual: Maximize \( \sum_{j=1}^{n} w_j u_{j0} \)

subject to \[
\begin{align*}
\sum_{i=1}^{n} v_i x_{i0} & = 1 \\
\sum_{j=1}^{n} w_j u_{j0} - \sum_{i=1}^{n} v_i x_{ik} & \leq 0 \\
\sum_{i=1}^{n} v_i x_{i0} & = 1 \\
w_j, v_i & \geq 0
\end{align*}
\]

..... (3.6.2)

The optimal solution, if exists, gives the DEA weights for the reference DMU. The inputs and outputs of other DMUs are also weighed by the weights of the reference a DMU.

The slacks of the constraints, \[
\sum_{j=1}^{n} w_j u_{j} - \sum_{i=1}^{n} v_i x_{ik} \leq 0 \] gives the cross efficiencies of other DMUs.
For the dual problem, some of the weights may turn out to be zero. To avoid the problem of zero multipliers, the non-negativity restrictions are reformulated as,

\[ w_i \geq \varepsilon \]
\[ v_i \geq \varepsilon \]

where \( \varepsilon \) is an infinitely small positive number.

As the model is run for each DMU separately, the set of weights \((w_i, v_i)\) will typically be different for the various DMUs. Thus, we solve as many LP problems as there are DMUs.

The optimization aims at according each DMU the best efficiency rating some of the weights may be assigned exceedingly small value. Such results can be criticised on several grounds.

(i) It may appear strange, that after a careful selection of inputs and outputs the weights assigned to some of the outputs and / or inputs are negligible.

(ii) The determination of weights, in a manner most favourable to an individual DMU by the unbounded DEA model can result in some cases in covering up of the most serious deficiencies such as low outputs and / or high inputs. This is in sharp contrast to the very purpose of launching an efficiency study.
(iii) In some cases, it may be considered an unacceptable that the same
factor is accorded widely with differing weights, while assessing
different units.

3.7 OUTPUT EFFICIENCY:

If a production unit is an inefficient, for a given input vector one
enquires for potential output augmentation. The output sets that are
characterized by certain axioms are basic for an efficiency measurement.

Let \( P(x) \) be the output level set, then

\[
P(x) = \{ u: x \text{ produces } u \} = \{ u: x \in L(u) \} \quad \text{.... (3.7.1)}
\]

The above equation explains the duality between the input sets.

The producer who operates at \( P \) employs an input level \( x_0 \) and
produces an output level \( u_0 \). The ordinate at \( x_0 \) is vertically projected onto
the piecewise linear frontier so that the potential output is,

\[
\delta u_0, \text{ where } \delta \geq 1
\]

Thus, '\( \delta \)' measures the output technical efficiency.
If the producer employs a single input and produces two outputs, a typical output level set is as follows:

![Diagram](image)

Figure (3.7.2)

P(x) is the collection of all output vectors which can be produced by an input vector x. The producer who operates at P, would be an output technical inefficient. By radial augmentation of outputs a composite DMU can be constructed which is Q that outperforms P. The technology of Q is either a linear or convex combination of the technologies of B and C.

To estimate the output technical efficiency we solve,

Maximize \( \delta \)

such that \( \delta u \in P(x) \) .... (3.7.2)

The line segments AB, BC and CD constitute isoquant of P(x). The line segments AB and BC form efficient and CD weak efficient subsets of isoquant of P(x).

In terms of a piece-wise linear technology, the output set may be expressed as,
\[ F^\Phi(x_0) = \left\{ u: \sum_{i=1}^n \lambda_i x_i \leq x_0, \sum_{i=1}^n \lambda_i u_i \geq u_0, \lambda_i \geq 0 \right\} \]  

(3.7.3)

The output level set \( F^\Phi(x_0) \) admits constant returns to scale. \( x_0 \) is a fixed input vector.

To estimate the output technical efficiency consistent with constant returns to scale we solve the following linear programming problem:

Maximize \( \delta \)

such that

\[
\begin{align*}
\sum_{i=1}^n \lambda_i x_i & \leq x_0 \\
\sum_{i=1}^n \lambda_i u_i & \geq \delta u_0 \\
\lambda & > 0
\end{align*}
\]

(3.7.4.)

The ray that passes through the origin is constant returns to scale frontier which is shown in the figure (3.7.3). The production technology employs, one input and produces one output. The producer who operates at \( P(x_0) \) becomes as a technically efficient. By projections the ordinate at \( x_0 \) vertically onto the CRS frontier production function a composite unit can
be constructed, viz., Q which outperforms Q. To assess output technical efficiency P is compared with Q. $\delta^t$ is an output technical efficiency consistent with constant returns to scale.

The output level set that admits variable returns to scale may be expressed, if the production is a piecewise linear as

$$P^r(x_0) = \left\{ u : \sum \lambda_i x_i \leq x_0, \sum \lambda_i u_i \geq u_0, \lambda_i \geq 0, \sum \lambda_i = 1 \right\} \quad \ldots \ldots (3.7.5)$$

With reference to $P^r(x_0)$ the output technical efficiency can be estimated solving the following optimization problem:

Maximize $\delta$

such that

$$\begin{align*}
\sum \lambda_i x_i &\leq x_0 \\
\sum \lambda_i u_i &\geq \delta u_0 \\
\lambda_i &\geq 0 \\
\sum \lambda_i &= 1
\end{align*} \quad \ldots \ldots (3.7.6)$$

let $\delta^*$ be the output technical efficiency that measures the relative to the variable returns to scale frontier.
From the above figure we notice that

\[ P^*(\mathbf{x}_0) \subseteq P^*(\mathbf{y}_0) \]

\[ \max \{ \delta : \delta \mathbf{x}_0 \in P^*(\mathbf{x}_0) \} \leq \max \{ \delta : \delta \mathbf{x}_0 \in P^*(\mathbf{y}_0) \} \]

\[ \Rightarrow \delta^* \leq \delta^d \]

In terms of output sets \( P^*(\mathbf{x}_0) \) and \( P^d(\mathbf{y}_0) \), we have the following figure:

![Figure (3.7.5)](image)

We shall call the output technical efficiency measured relative to \( P^*(\mathbf{x}_0) \) as pure output technical efficiency and that becomes as an overall output technical efficiency.

Output pure technical efficiency:

\[ \text{OPTE} \quad \frac{OQ}{OP} \]

Output overall technical efficiency

\[ \text{OTE} : \quad \frac{OR}{OP} \]

113
OUTPUT SCALE EFFICIENCY:

The overall output technical efficiency can be decomposed into the product of output pure technical and scale efficiencies. In terms of output sets, scale efficiency is obtained as the ratio,

\[ OSE = \frac{OR}{OQ} \]

Thus, we have the following decomposition,

\[ OTE = \frac{OQ}{OP} \cdot \frac{OR}{OQ} = (OPTE)(OSE) \]

3.8 RELATIONSHIP BETWEEN INPUT AND OUTPUT TECHNICAL EFFICIENCIES:

Let returns to scale that be a constant so that the relevant input and output sets are \( L^* (u) \) and \( P^*(x) \).

\[ \lambda^k = \text{Min} \{ \lambda : \lambda x \in L^* (u) \} \]

\[ = \text{Min} \{ \lambda : u \in P^* (\lambda x) \} \]

\[ = \text{Min} \left[ \lambda : \frac{u}{\lambda} \in P^* (x) \right] \]

\( u \in P^* (\lambda x) \Rightarrow \lambda x \) produces \( u \)

\( x \) produces \( \frac{u}{\lambda} \), since returns to scale are constant

\[ \frac{u}{\lambda} \in P^* (x) \]

\[ \lambda^k = \text{Min} \left[ \lambda : \frac{u}{\lambda} \in P^* (u) \right] \]
\[
\lambda^k = \frac{1}{\text{Max.} \left\{ \frac{1}{\lambda} : u \in P^k(u) \right\}}
\]

\[
\lambda^k = \frac{1}{\text{Max.} \left\{ \delta : \delta u \in P^k(u) \right\}}
\]

\[
\lambda^k = \frac{1}{\delta^k}
\]

\[\cdots (3.8.1)\]

Thus, input overall technical efficiency measures is an inverse of the output overall technical efficiency measure.

Input technical efficiency under an increasing and decreasing returns to scale is not reciprocally related with its counterpart of output technical efficiency.

**Maximum Revenue:**

For a given input vector revenue that can be maximized if we have the following optimization problem;

\[
R(x_0, r) = \text{Max} ru
\]

subject to

\[
\begin{align*}
\sum \lambda_i x_i &\leq x_0 \\
\sum \lambda_i u_i &\geq u \\
\lambda_i &\geq 0
\end{align*}
\]

\[\text{(3.8.2)}\]

where \( r \) is output price vector.

![Diagram](image-url)
\( x_0 \) is given as an input vector. The figure (3.8.1) explains that the producer who operates at \( P \) is output technical inefficient. A further radial augmentation of output upto \( Q \) on the output set isoquant leads to output technical efficiency. Since \( P^k(x_0) \) admits constant returns to scale, at \( Q \) overall output technical efficiency can be achieved. But \( Q \) does not give a revenue maximizing output vector. A re-allocation of outputs from \( Q \) to \( S \) gives a maximum revenue. But revenue at \( S \) is equal to the revenue at \( R \).

**REVENUE EFFICIENCY:**

The ratio of the maximum to observed revenue is the measure of revenue efficiency.

\[
\text{Revenue Efficiency} = \frac{R(x_0, r)}{ru_0} = \frac{OR}{OP}
\]

where \( R(x_0, r) \) : Maximum revenue

\( ru_0 \) : Realized Revenue

**OUTPUT ALLOCATIVE EFFICIENCY:**

On the output isoquant, failure to operate at revenue maximizing output vector leads to output allocative efficiency.

\[
OAE = \frac{OQ}{OR}
\]

Revenue efficiency can be decomposed into the product of overall output technical and allocative efficiency.

\[
RE = \frac{OQ \, OR}{OP \, OQ}
\]
\[ \text{RE = OTE \times OAE} \]

But, \( \text{OTE = OPTE \times OSE} \)

\implies \text{RE = OPTE \times OSE \times OAE} \]

where OPTE : Output Pure Technical Efficiency
OSE : Output Scale Efficiency
OAE : Output Allocative Efficiency

\text{CROSS EFFICIENCY:}

Cross efficiencies can be estimated for the various decision making units solving the dual of the following problem:

\textbf{Primal:}

Maximize \( \delta \)

subject to

\[ \sum_{j=1}^{r} \lambda_j x_j \leq x_0 \]

\[ \sum_{j=1}^{r} \lambda_j u_j \geq \delta u_0 \quad (3.8.3) \]

\( \lambda_j \geq 0 \)

\textbf{Dual:}

Minimize \( \sum_{i=1}^{n} v_i x_i \)

subject to

\[ \sum_{i=1}^{n} v_i x_i - \sum_{j=1}^{r} w_j u_j \leq 0, s = 1, 2, \ldots, r \]

\[ \sum_{j=1}^{r} w_j u_{j0} \leq 1 \quad (3.8.4) \]

\[ -\sum_{j=1}^{r} w_j u_{j0} \leq 1 \]

\( v_i, w_j \geq 0 \)
Equivalently, we solve,

Minimize \( \sum_{i=1}^{n} v_i x_{i0} \)

subject to \( \sum_{j=1}^{r} w_j u_j - \sum_{i=1}^{n} v_i x_{i0} \geq 0, s = 1, 2, ..., r \) 

\( \sum_{j=1}^{r} w_j u_j = 1 \)

\( v_i, w_j \geq 0 \) \hspace{1cm} (3.8.5)

The optimal solution of the above linear programming problem gives a simple efficiency of the DMU under evaluation. It also gives cross efficiencies for other DMUs.

3.9 PARAMETRIC PRODUCTION FUNCTIONS – OUTPUT EFFICIENCY:

Let \( u \) be the observed output and \( \hat{u} = \prod_{i=1}^{n} x_{i0}^{\alpha_i} \) be frontier output. If the producer is output technical inefficient, then we have,

\( u \leq \hat{u} \)

\( u \leq \prod_{i=1}^{n} x_{i0}^{\alpha_i} \)

Let \( \delta > 1 \) be the real number for which

\( \delta u = \prod_{i=1}^{n} x_{i0}^{\alpha_i} \)

\( \delta = \frac{\prod_{i=1}^{n} x_{i0}^{\alpha_i}}{u} \)
For the Cobb-Douglas production function, the input technical efficiency measure is such that

\[ u = \prod_{i=1}^{n} (\lambda x_i)^{\alpha_i} \]

\[ = \lambda^{\sum\alpha_i} \prod_{i=1}^{n} x_i^{\alpha_i} \]

\[ \lambda^{\sum\alpha_i} = \frac{\prod_{i=1}^{n} x_i^{\alpha_i}}{u} = \delta \]

Thus, the input technical efficiency and output technical efficiency are related by,

\[ \lambda^{-\theta} = \delta \]

where \( \theta = \sum_{i=1}^{n} \alpha_i \) is the degree of returns to scale.

\( \delta \) is reciprocal of \( \lambda \) if and only if \( \theta = 1 \), i.e., returns to scale are constant.

**THE TRANSLOG PRODUCTION FRONTIER:**

\[ \ln u \leq \alpha_0 + \sum_{i=1}^{n} \alpha_i \ln x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} \ln x_i \ln x_j \]

(3.9.1)

Let \( \lambda \) be the radial measure of an input technical efficiency.

We reduce \( \lambda \) till the above inequality becomes as an equality.

\[ \ln u = \alpha_0 + \sum_{i=1}^{n} \alpha_i \ln (\lambda x_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} \ln (\lambda x_i) \ln (\lambda x_j) \]

\[ = \alpha_0 + \sum_{i=1}^{n} \alpha_i \ln x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} \ln x_i \ln x_j + \ln \lambda \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} \ln x_i \]

119
\[ + \ln \lambda \sum_{j=1}^{n} \left( \sum_{i=1}^{n} \alpha_{y} \right) \ln x_j + \left( \ln \lambda \right)^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{y} \]

If the translog frontier linear homogeneous so that it admits constant returns to scale, then,

\[ \sum_{i=1}^{n} \alpha_{i} = 1, \sum_{i=1}^{n} \alpha_{y} = 0, \sum_{j=1}^{n} \alpha_{y} = 0 \]

consequently, \( \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{y} = 0 \)

The translog frontier reduces to

\[ \ln u = \alpha_{0} + \sum_{i=1}^{n} \alpha_{i} \ln x_{i} + \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{y} \ln x_{i} \ln x_{j} + \ln \lambda \]

\[ = \ln \hat{u} + \ln \lambda \]

\[ \Rightarrow u = \hat{u} \lambda \]  

\[ \quad \text{......... (3.9.2)} \]

\( \frac{\lambda}{\hat{u}} = \lambda \) is input technical efficiency measure that is consistent with constant returns to scale.

However, if returns to scale are not constant and if all the inputs are separable from the input technical efficiency measure, then

\[ \sum_{j=1}^{n} \alpha_{y} = 0, \sum_{i=1}^{n} \alpha_{y} = 0 \]

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{y} = 0 \]

The translog production function reduces to

\[ \ln u = \alpha_{0} + \sum_{i=1}^{n} \alpha_{i} \ln x_{i} + \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{y} \ln x_{i} + \sum_{i=1}^{n} \alpha_{i} \ln \lambda \]

120
\[
\ln u = \ln \hat{u} + \ln \lambda \sum_{i=1}^{n} \alpha_i.
\]

\[
u = \hat{u} \lambda \sum_{i=1}^{n} \alpha_i
\]

\[
u = \lambda \sum_{i=1}^{n} \alpha_i
\]

\[
\lambda = \left( \frac{\hat{u}}{u} \right)^{\frac{1}{\theta}}, \text{ where } \theta = \sum_{i=1}^{n} \alpha_i
\]

The output technical efficiency measure \(\delta\) satisfies the condition,

\[
\ln(\delta u) = \alpha_0 + \sum_{i=1}^{n} \alpha_i \ln x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \ln x_i \ln x_j
\]

\[
\ln \delta + \ln u = \ln \hat{u}
\]

\[
\ln \delta = \ln \hat{u} - \ln u = \ln \left( \frac{\hat{u}}{u} \right)
\]

\[
\delta = \frac{\hat{u}}{u} = \lambda^{-\theta}
\]

\[\text{..... (3.9.3)}\]

where \(\theta = \sum_{i=1}^{n} \alpha_i\)