Chapter – II

REVIEW OF LITERATURE
2.0 INTRODUCTION

The basic hypothesis is that individuals respond to incentives irrespective of the fact that they abide or violate law. There are costs and gains not only in legitimate but also in illegitimate activities. The costs of illegitimate activity if apprehended are punishment, consequently loosing freedom, income due to failure in participation of legitimate activity and wealth lost in terms of fines.

2.1 ROLE OF PROBABILITY OF CAPTURE AND LENGTH OF PUNISHMENT ON NUMBER OF CRIMES

Gary S. Becker* in his path breaking article hypothesizes apart from sociological and other factors, probability of capture and apprehension, and opportunities lost due to prison sentence influence the number of offences that one commits.

At aggregate level the social value of the gain to offenders is a function of number of offences they commit.

\[ G = G(0) \]

\[ G' = \frac{dG}{d0} > 0 \]

Offenders gain moves directly with the number of offences.

Harm done to the society by offenders is measured by the function,
Harm done to the society by offenders moves directly with the number of offences.

Net cost of damage to society may be measured by the difference,

\[ D(0) = H(0) - G(0) \]

If offenders receive diminishing marginal gains and cause increasing marginal harm from additional offences,

\[ D''(0) = H''(0) - G''(0) > 0. \]

The output of Criminal Justice system may be viewed as a function of manpower, materials and capital.

\[ A = f(m, r, c) \]

\( f \) is a production function summarizing the state of arts. The cost of the CJ's activity is,
C = C (A)

\[ C' = \frac{dC}{dA} > 0 \]

If P is the probability of conviction,

\[ A = pO \]

Consequently the activity cost function may be expressed as,

\[ C = C(p0) \]

\[ \frac{\partial C}{\partial p} = C''0 > 0 \]

\[ \frac{\partial^2 C}{\partial p^2} = C''0^2 > 0 \]

A more general specification of activity function may be expressed as follows:

\[ A = h (p, o, a) \]

- \( h \) may be viewed as a production function
- \( 'a' \) stands for arrests and other determinants of activity \( A \).

It is hypothesized that number of offences respond to probability of conviction and severity of punishment.

The aggregate supply of offences function may be postulated as,

\[ 0 = 0 (p, f, u) \]

- \( \frac{\partial 0}{\partial p} < 0 \)
- \( \frac{\partial 0}{\partial f} < 0 \)
p and f may be viewed as average probability of conviction and length of punishment respectively.

Mankind has invented a variety of ingenious punishments to inflict on convicted offenders.

Define, \( L = L(D, C, bf, 0) \)

- \( L \) measures social loss
- \( \frac{\partial L}{\partial D} > 0 \)
- \( \frac{\partial L}{\partial C} > 0 \)
- \( f^i = bf \)
- \( f^i \) is social cost
- \( b \) transforms the length or severity of punishment into social cost
- If \( f \) stands for fines, then \( b = 0 \)
- For all other convictions \( b > 0 \)
- \( \frac{\partial L}{\partial bf} > 0 \)

\[ L = D(0) + C(p, 0) + bf 0. \]

- \( D(0) \): Damages done to the society by offenders
- \( C(p, 0) \): Cost of criminal justice system
- \( p0 \): Offences punished
- \( f^i = bf \) is social loss per offence
- \( bpf0 \): Social loss from punishment
- \( p \) and \( f \) are the decision variables

The social loss function has to be minimized with respect to the decision variables \( p \) and \( f \).
First order optimality conditions:

\[
\frac{\partial L}{\partial f} = \frac{\partial D}{\partial f} \frac{\partial D}{\partial f} + \frac{\partial c}{\partial f} \frac{\partial D}{\partial f} + bpf \frac{\partial D}{\partial f} + bpf = 0
\]

\[D'0f + C'0f + bpf 0f + bpf = 0\]  
\[\ldots (2.1.1)\]

\[
\frac{\partial L}{\partial p} = \frac{\partial D}{\partial p} \frac{\partial D}{\partial p} + \frac{\partial c}{\partial p} \frac{\partial D}{\partial p} + \frac{\partial c}{\partial p} + bpf \frac{\partial D}{\partial p} + bpf = 0
\]

\[D'0p + C'0p + bpf 0p + bpf = 0\]  
\[\ldots (2.1.2)\]

Simplification of (2.1.1) and (2.1.2) leads to the followings:

\[D' + C' = -bpf \left( 1 - \frac{1}{\epsilon_f} \right)\]

\[D' + C' + C_p \frac{1}{0_p} = -bpf \left( 1 - \frac{1}{\epsilon_p} \right)\]

- \(\epsilon_f = \frac{-f}{0_f}\),

- \(\epsilon_f\) is elasticity of offences with respect to the length as punishment

- \(\epsilon_p = \frac{-p}{0_p}\)

- \(\epsilon_p\) is elasticity of offences with respect to the probability of conviction

\(D' + C'\): Marginal cost of increasing the number of offences, through a reduction in \(f\)

\(D' + C' + C_p \frac{1}{0_p}\): Marginal cost of increasing the number of offences through reduction in \(p\).
\[ D' + C' > 0 \text{ since } D' > 0, \ C' > 0 \]

\[ D' + C' + C_p \frac{1}{\theta_p} < D' + C', \text{ since } \theta_p = \frac{\partial \theta}{\partial p} > 0 \]

- \[ D''(0) > 0 \]
- \[ C''(0) > 0 \]
- \[ D' + C' \text{ increases with number of offences.} \]
- \[ D' + C' + C_p \frac{1}{\theta_p} \text{ increases with number of offences.} \]
- \[ \text{This expression will be negative if } C_p \text{ is sufficiently large.} \]

The right hand side expressions of the two equilibrium conditions can be interpreted as marginal revenues.

\[ -bpf \left( 1 - \frac{1}{\epsilon} \right) \]

- \[ -bpf0 \text{ can be interpreted as revenue''} \]
- \[ -bpf \text{ is average revenue} \]

Marginal revenue (attained through \( f \)):

\[ \]
Marginal revenue (attained through $g$):

\[-bp_0 - bp_f \frac{\partial \theta}{\partial f}\]

\[= - bp_0 - bp_f \theta_f\]

\[= 0_f \left[ -bp_0 \frac{1}{0_f} - bp_f \right]\]

\[= 0_f \left( -bp_f \right) \left[ 1 - \frac{1}{\varepsilon_f} \right]\]

Marginal revenue (attained through $g$):

\[= 0_p \left( -bp_f \right) \left[ 1 - \frac{1}{\varepsilon_p} \right] \quad \text{.....(2.1.4)}\]

- The expressions (2.1.3) and (2.1.4) become positive if the elasticities $\varepsilon_f$ and $\varepsilon_p$ are less than unity.
- Both the marginal revenues are equal if and only if $\varepsilon_f = \varepsilon_p$, which in turn implies that $0_p = 0_f$.
- $\varepsilon_p$ can exceed unity if $C_p$ is sufficiently large.

Under suitable conditions the marginal revenues and costs can be graphically expressed as follows:

![Graph showing marginal revenue (MR) and marginal cost (MC) curves with points A and B, and expressions for MC and MR.](Fig. 2.1.2)
A and B are equilibrium points. At equilibrium,

- \( MC_r > MC_p \)
- \( MR_r > MR_p \)
- \( MR_r > MR_p \) if and only if, \( \epsilon_p > \epsilon_f \)
- \( \epsilon_p > \epsilon_f \) implies offenders have risk preference, since they respond more to changes in \( p \) than \( f \).

The loss from offences is minimized if \( p \) and \( f \) are selected from those regions where offenders are risk preferrers. In order to insure that "crime does not pay", a rational public policy should choose \( p \) and \( f \) accordingly from those regions where offenders prefer risk.

**RISK NEUTRAL OFFENDERS POLICY OF CAPTURE AND PUNISHMENT**

- If supply of offences depended only on \( pf \) – the offenders were risk neutral.
- A reduction in \( p \) compensated by an equal increase in \( f \) would leave \( pf \), \( D(0) \), and \( bpf0 \) unchanged, but would reduce the loss.
- To minimize loss reduce \( p \) arbitrarily close to zero and set \( f \) sufficiently high.

**RISK AVOIDERS POLICY OF CAPTURE AND PUNISHMENT**

- Risk avoiders respond greatly to changes in probability of apprehension
- For risk avoiders reduce \( p \) close to zero
- Reduction $p$ requires reduction of activity cost and this would minimize loss.

**RISK PREFERERS - POLICY OF CAPTURE AND PUNISHMENTS**

- Select positive finite values of $p$ and $f$.
- There is no need that a rise in $f$ has to be compensated by an increase in $p$.
- Punishment should be severe.

2.2 **A LINEAR MODEL FOR CRIMINAL JUSTICE SYSTEM**

Alfret Blumstein and Richard Larson (1969) estimated linear models of total criminal justice system. Police, courts and correction constitutes the criminal justice system. Despite the fact that significant social, technological and managerial changes have occurred the CJS remained to be stable. The stability is mainly due to the independence of the individual components of the system. Nowhere it is observed that a single manager of a CJS with control over all its constituent part. It is desirable to study the interaction between the correction components of CJS. Modelling of CJS and estimation of them depend upon primary or secondary data or a combination of both of them.

---

Former offenders (recidivists) and those not previously so identified participate in criminal acts. The following flow chart depicts the CJS.

Table 2.2.1

Seven crime types are viewed

(i) Homicide,
(ii) Robbery,
(iii) Assault
(iv) Burglary
(v) Larceny
The authors believe the complex model constructed is a gross simplification of reality. Administrators of the CJS require projections of future workloads, costs, and manpower requirements. The following flowchart of court sub model gives an idea how inputs and outputs are generated for it:
\[ N_{a} = \text{Number of adult arrests who formally charged.} \]
\[ N_{m} = \text{input for the court} \]
\[ P_{b} = \text{Probability of Pre trial processing leads trial} \]
\[ P_{t} = \text{Probability that the trial is a …… trial} \]
\[ P_{b} = \text{Probability that the trial is a bench trial} \]
\[ P_{a} = \text{Probability of ……… ple} \]
\[ P_{b} = \text{Probability of conviction under index of crime } j = 1, 2, …… 7. \]
This model leads to computation of
- Man power requirement
- The costs involved
- Workloads and so on.

The model's feedback feature includes the probability of re-arrest as a decreasing function of age and a crime-switch matrix reflecting a successive crime distribution.

The various results of the model are,
- Cost distribution by crime type.
- Criminal career costs
- Estimates of the sensitivities of the costs
- Offender flows within the system.

2.3 A PROPERTY CRIME MODEL

David Lawrence Sjoquist* (1973) proposed a model to analyse property crimes. The work begins with the hypothesis that property crimes may be explainable, at least in part, by economic theory. Crimes selected for the study are robbery and burglary.

Under some conditions, criminals can be treated as rational economic beings, assumed to behave in the same economic manner as any other individual making economic decision under risk.

\[ t: \text{Time at disposal} \]

An individual performs two activities legal (work) and illegal (crimes against property). It is of interest to see how an individual allocates time between these activities?

This depends on gain and costs involved in these activities.

Participation in the two activities is expected to result in psychic and financial gains and costs.

The gain per unit of time from legal activity is measured by the individual wage rate.

\[ g_w : \text{wage rate (constant)} \]

\[ \bar{g}_w : \text{Total gain from legal activity} \]

\[ t_w : \text{Time allocated to legal activity.} \]

\[ \bar{g}_w : t_w g_w \]

Participation in illegal activity results in psychic and/or financial gain.

The psychic gain is measured by the money which the individual is willing to pay to obtain the psychic gain. The financial gain is measured directly by the dollar value to the criminal of the assets stolen.

- Gain from illegal activity is stochastic.
- The individual possesses subjective probability distribution.

Wage rate from illegal activity: \( g_c \) (individual psychic gain)

Time spent in illegal activity : \( t_c \)

Total gain from illegal activity : \( \bar{g}_c = t_c g_c \)

\[ \bar{t} = t_w + t_c \]
The financial and psychic costs, other than those associated with arrest, conviction and punishment are included in $g_c$.

Illegal activities also involve the possibility of costs resulting from arrest, conviction and punishment. These include public scandal, loss of freedom, the distaste of prison life, lawyer fees, possible reduction in potential income etc.

The expected cost of imprisonment depends upon the legal wage rate ($g_w$) and the time spent on illegal activity, $t_c$.

Total cost of illegal activity: $\bar{p}$

Quasi fixed cost component: $p^*$

Variable cost component: $\hat{p} = \hat{p}(g_w, t_c)$

$$\bar{p} = p^* + \hat{p}(g_w, t_c)$$

Assume, $\frac{\partial \hat{p}(g_w, t_c)}{\partial t_c} = p_c$ constant.

Assume probability of conviction and punishment, conditioned on arrest equals to one.

State (1): Arrested - $(\bar{g}_w + \bar{g}_c - \bar{p})$ - utility $U(\bar{g}_w + \bar{g}_c - \bar{p})$

State (2): Not arrested - $(\bar{g}_w + \bar{g}_c)$ - utility $U(\bar{g}_w + \bar{g}_c)$

Expected utility:

$$E(U) = (1-r)U(\bar{g}_w + \bar{g}_c) + r U(\bar{g}_w + \bar{g}_c - \bar{p})$$

where $r =$ probability of arrest
\( (1-r) \) = probability of no arrest

\[ E(U) = (1-r)U(t_w g_w + t_c g_c) + r \left[ t_w g_w + t_c g_c - (p^* + \bar{p}(g_w, t_c)) \right] \] \quad (2.3.1)

\[ \frac{\partial E(U)}{\partial t_w} = (1-r) \frac{\partial U(g_w + \bar{g}_c)}{\partial (g_w + \bar{g}_c)} + r \frac{\partial U(g_w + \bar{g}_c - \bar{p})}{\partial (g_w + \bar{g}_c - \bar{p})} \]

\[ = (1-r)U_1^1 (g_w - g_c) + r U_2^1 (g_w - g_c + p) = 0^* \] \quad (2.3.2)

The first order conditions for optimum are summarized as follows:

\[ \frac{g_w - g_c}{g_c - g_w - p} = \frac{(1-r)U_1^1}{r U_2^1} \] \quad (2.3.3)

- \( U_1^1 > 0 \); \( U_2^1 > 0 \)
- RHS is positive (equilibrium)
- To participate in crime a necessary condition is.
  \[ g_c > g_w \Rightarrow g_w - g_c < 0 \]
- RHS becomes positive, if \( g_c - g_w < p \)

The second order condition requires that.

\[ (1-r) U_1^{11} [g_c - g_w]^2 + r U_2^{11} [g_c - g_w - p]^2 < 0 \] \quad (2.3.4)

A sufficient condition for the above condition to hold is that the individual is risk averse.

i.e., \( U_1^{11}, U_2^{11} < 0 \)

\[ -\bar{p} = p^* - \bar{p}(g_w, t - t_c), \quad \frac{\partial}{\partial t_w} (-\bar{p}) = p \]
If an individual has strong enough risk preference, the indifference curves will be concave, in which case the individual will either specialize in work or crime.

How do people respond to changes in risk, wages and punishment?

For risk averse individual,
\[
\frac{dt_w}{dr} > 0
\]

If indifference curves are concave then a person who specialised in work will now specialize in crime, if the change in r is large enough.

For risk averse individual
\[
\frac{dt_w}{dg_c} < 0
\]
\[
\frac{dt_w}{dg_w} > 0
\]
\[
\frac{dt_w}{dp} > 0
\]

**Testing the model**

Time spent in criminal activity by ith individuals:
\[
t^i_c = f^i \left( r^i, p^i, g^i_w, g^i_c, x^i \right)
\]

where \( r \) : subjective probability

\( p \) : cost of imprisonment

\( g_w \) : wage rate (work)

\( g_c \) : rate of criminal activity per unit time

\( x^i \) : measures tastes
Cobb Douglas structure:

\[ t^i = \alpha_0 r^i d_{i1} r_{i2} g_{w} g_{c} p_{i3} x_{i4} \] ........(2.3.5)

Aggregate version of the model is as follows:

\[ \frac{TC}{I} = \alpha_0 r^{\alpha_1} (GW)^{\alpha_2} (GC)^{\alpha_3} p^{\alpha_4} x^{\alpha_5} \]

where \( TC \) : total time spent in illegal activity

\( I \) : community population.

GW, GC, p and X represent population means of \( t_c, g_w, g_c, p \) and \( x \).

A cross-sectional sample of 53 municipalities with 1960 populations of 25,000 to 2,00,000 was selected.

- Number of property crimes committed is proxy for time spent in illegal activity (N).

This in turn was measured by the total number of three types of property crimes recorded by the local police.

- Three separate measures of \( r \) used.

\( r(A) \): Probability of arrest

\( r(cA) \): Probability of conviction.

\( r(c/A) \): Probability of conviction given arrest.

\[ r(A) = \frac{\text{Number of arrests}}{\text{Number of reported crimes}} \]

\[ r(AC) = \frac{\text{No.of convictions}}{\text{No. of reported crimes}} \]
The net gain from legal activity, GW, was measured by the annual labour income to manufacturing workers in 1968.

An unemployed person has more time to allocate to illegal activity and since his current income is low, he would have a greater incentive to commit crimes. The same reasoning can be applied to a family with a low income.

Financial loss of public is replaced by retail sales per establishment reasoning that the size of the expected gain depends upon the size of the establishment a criminal chooses to rob or burglarize.

The length of the sentence served can be used to reflect the cost of illegal activity.

Since we had no measure of psychic costs we have ignored them.

With proxies the model is expressed as,

\[
\frac{N}{I} = \beta_0 r^{\beta_1} S^{\beta_2} Gw^{\beta_3} E^{\beta_4} NW^{\beta_5} Sx^{\beta_6} D^{\beta_7} I^{\beta_8} \phi
\]

**N** : No. of property crimes.

**I** : Size of population

**r** : Subjective probability of arrest

**S** : Average prison sentence served.

**E** : Retail sales per establishment robbed

**NW** : Percent of non-white.
SY : Mean school years completed
D : Population density
ϕ : Random variable following log normal distribution

- Log linearizing the multiple linear regression equation is estimated
- \( N = \) Reported property crimes: robbery + burglary + larceny over $50.
- \( r(A), r(Ac) & r(c/A) \) are used as proxy of \( r \).

The empirical results give, an increase in the probability of arrest and conviction and an increase in the cost of crime (punishment) both are found decrease in the number of major property crimes committed.

2.4 A MODEL FOR PARTICIPATION IN ILLEGAL ACTIVITIES

Ehrlich, I. (1973), formulates a theory of participation in illegitimate activities in association with the general theory of occupational choices by presenting the offender's choice problem as an optimization problem of allocation of offender's resources under uncertainty.

It is hypothesized that an individual can participate in two market activities:

---
i, an illegal activity,

l, a legal activity

W\_l (t\_i): returns due to legal activity,

W\_i (t\_i): returns due to illegal activity

Participation of illegal activity leads to one of the two states.

a: apprehension and punishment at the end of the period with a subjective probability p.

b: getting away with crime, its probability being 1-p.

If the offender succeeds, he reaps the entire value of the output of his illegitimate activity, net of the cost of the inputs foregone, W\_i (t\_i). However, if apprehended and punished his returns are reduced by an amount F\_i (t\_i).

The utility of any state of the world's may be expressed as,

\[ U_s = U (X_s, t_c), \]

s: a, b

X\_s : stock of composite market good

\( t_c \): amount of time devoted to consumption

U : indirect utility function

\[ X_b = W^l + W\_l (t\_i) + W\_i (t\_i) \]

\[ X_a = W^l + W\_i (t\_i) - F\_i (t\_i) + W\_i (t\_i) \]
where $w^j$: market value of individual’s assets

The events $X_s$ and $X_b$ occur with probabilities $p$ and $1 - p$

The expected utility function may be formulated as follows:

$$E_U(X_s, t_c) = (1-p) U(X_b, t_c) + p U(X_s, t_c)$$

$$= (1-p) U(X_b) + p U(X_s), t_c \text{ is suppressed}$$

Time constraints: $t_o = t_i + t_t + t_c$

$$t_i, t_t, t_c \geq 0$$

where $t_o$ is total available time spent on consumption, legitimate and illegitimate activities.

For a given $t_o$, $t_i$ and $t_t$ are decision variables of the individual.

Lagrangeon function:

$$Z = E U(X_s, t_c) + \lambda [t_o - t_i - t_t - t_c]$$

Non-negativity restrictions

$$t_i, t_t, t_c \geq 0.$$

The Kuhn-Tucker optimization (maximization of expected ‘utility conditionally’) conditions are as follows:

$$\frac{\partial Z}{\partial t_i} - \lambda = 0, \quad t_i \geq 0$$

$$\left(\frac{\partial E U}{\partial t_i} - \lambda\right)_{t_i} = 0, \quad K = i,$$

If the optimal solution is an interior point of the solution space, then it requires.
Rearranging terms we obtain

\[
\frac{\partial EU}{\partial X_i} = \lambda, \ K = i, 1
\]

\[
\frac{\partial EU}{\partial X_i} \frac{\partial X_i}{\partial EU} = 1
\]

\[
\frac{(1 - p)U'(X_s)w_i + pU'(X_s)(w_i - f_i)}{(1 - p)U'(X_s)w_i + pU'(X_s)w_i} = 1
\]

Rearranging terms we obtain

\[
\frac{W_i - W_j}{W_i - f_i - W_i} = \frac{pU'(X_s)}{(1 - p)U'(X_s)} \quad \text{.....(2.4.1)}
\]

The right hand side expression is the absolute slope of indifference curve*. The left hand side expression may be interpreted as the slope of opportunity boundary, for given \( t_o \) and \( t_c \).

\( W_i \): Marginal return due to illegal activity

\( W_l \): Marginal return due to legitimate activity

\( f_i \): Marginal penalty due to illegal activity if apprehended and punished.

The opportunity boundary may be expressed as, \( \phi(X_s, X_b) = C \)

Along the opportunity boundary, we have

\[
d\phi = \frac{\partial \phi}{\partial X_a} dX_a + \frac{\partial \phi}{\partial X_b} dX_b = 0
\]

\[
dEU = \frac{\partial EU}{\partial X_s} dX_s + \frac{\partial EU}{\partial X_b} dX_b \]

is the total derivative of expected utility function. Along the isoquant we have, \( dEU = 0 \)

\[
\Rightarrow \frac{dX_b}{dX_a} = \frac{pU'(X_s)}{(1 - p)U'(X_s)} \quad \text{.....(2.4.1)}
\]

\[
\frac{dX_b}{dX_a} < 0
\]
Combining (2.4.1) and (2.4.2) we obtain the offender's equilibrium condition.

With each pair \((t_i, t_i)\) there is a pair \((X_a, X_b)\). The locus of all such pairs yield the opportunity boundary.

The first order condition for optimum, if \(t_e\) is given,

\[- \frac{w_i - w_i}{w_i - f_i - w_i} = \frac{pU'(X_a)}{1 - pU'(X_a)}\]

The decision to engage in illegal activity is not an either/or choice, and offenders are free to combine a number of legitimate and illegitimate activities and switch occasionally from one to another during any period during their life time.

\(^*\) (i) \(t_i = 0 \Rightarrow t_i = t_e - t_i\)

In this case the offender is specialized in illegal activity.

\[X_a = W' + W_i(t_e - t_i) - F_i(t_e - t_i)\]

\[X_b = W' + W_i(t_e - t_i)\]
At B total time is spent on legitimate activity, consequently:

\[ X_a = X_b = W' + W_i (t_0 - t_e) \]

At A total time is spent on illegitimate activity, so that

\[ X_a = W'^l + W_i (t_0 - t_e) - F_i (t_0 - t_e) \]
\[ X_b = W'^l + W_i (t_0 - t_e) \]

At C the individual attains equilibrium, for which the indifference curve is convex and possesses only one point of contact with the opportunity boundary that is concave to the origin.

A necessary prerequisite for the equilibrium condition to hold is that the potential marginal penalty \( f_i \) should exceed the differential marginal return from illegal activity.

\[ f_i > W'_i - W_i' \]

33
This condition supports the saying that the 'evil of punishment must be made to exceed the advantage of offence'.

For several offences a concurrent imprisonment leads to create an incentive for offenders to specialize in legitimate activity, leading to operate at 'A'. The equilibrium condition is necessary and sufficient for a strict global maximum if the indifference are convex and opportunity boundaries are concave to the origin.

The second order condition for a local maximum is**, 
\[ \Delta = (1-p)U''(X_s)(w_i - w_l)^2 + pU''(X_s)(w_i - f, - w_l)^2 < 0 \]

A sufficient condition for entry into illegitimate activity is that the slope of opportunity boundary exceed the absolute slope of indifference curve at the position where total working time is spent on legitimate activity***, (point B). This requires in turn (i) \( w_i > w_l \), (ii) \( f_i > w_i - w_l \).

** Convexity of indifference curves requires diminishing marginal utilities of real wealth \( (X_s, X_b) \).

(i) \[ \frac{\partial EU}{\partial X_s} = p \frac{\partial U}{\partial X_s} > 0 \Leftrightarrow U'(X_s) > 0 \]

(ii) \[ \frac{\partial^2 EU}{\partial X^2} = pU'''(X_s) < 0 \Leftrightarrow U'''(X_s) \]

(iii) \[ \frac{\partial EU}{\partial X_b} = (1-p)U'(X_s) > 0 \Leftrightarrow U'(X_s) > 0 \]

\[ \frac{\partial^2 EU}{\partial X^2} = (1-p)U'''(X_s) < 0 \Leftrightarrow U'''(X_s) < 0 \]

*** Concavity of opportunity boundary requires diminishing marginal wages and 'constant or increasing marginal penalties'.

** It is assumed that \( f, W_i \) and \( W_l \) are constant.

\[ \frac{(w_i - w_l)}{w_i - f, - w_l} > \frac{P U'(X_s)}{(1-p)U'(X_s)} = \frac{p}{1-p} , \text{ since at B, } X_s = X_b \]

\[ \Rightarrow \frac{-(w_i - w_l)}{w_i - f, - w_l} > \frac{p}{1-p} \]
For risk avoiders or risk neutral persons, \( w_i > w_l \) is a necessary condition for entry into \( i \). If \( w_i < w_l \) the offenders specialize in legitimate activity.

Offenders are of three types (a) risk neutral, (b) risk avoider and (c) risk preferrer.

It can be shown, under certain conditions that a risk-neutral offender will spend more time in illegitimate activity relative to risk avoider, and a risk preferrer will spend more time there relative to both.*

If the individual is risk avoider, then it can be shown that for him expected marginal return from illegitimate activity is less than marginal return from legal activity.

\[
(1-p) w_i + p (w_i - f_l) < w_l
\]

If the offender is risk avoider, his utility function satisfies the properties that,

\[ U^1 > 0, \quad U^{11} < 0. \]

* (i) \( X_a \) is more attractive than \( X_b \) - incentive to spend more time in illegal activity. \( X_a \) is substituted for \( X_b \).

(ii) \( X_a \) is more attractive than \( X_b \) - incentive to spend more time in legal activity. \( X_a \) is substituted for \( X_b \).
For risk avoider, \( \frac{U'(X_x)}{U'(X_b)} < 1 \)
\[ \Rightarrow (1 - p)W_i + p(W_i - f_i) < W_i \]

For risk preferring or neutral offender,
\[ (1-p) W_i + p (W_i-f_i) \geq W_i \]

If the opportunity boundary is linear (\( p \) being constant), then risk preferrers would necessarily specialize in illegal activity.

\[ \Rightarrow \text{The broken curves represent concave indifference curves of risk preferrer.} \]
\[ \Rightarrow \text{At A the offender is in equilibrium.} \]

\* For risk avoider \( X_b \) is more attractive.
\[ U'(X_b) > U'(X_i) \]
\[ \Rightarrow \frac{U'(X_b)}{U'(X_i)} < 1 \]
\[ \Rightarrow \frac{(1-p)(w_i-w_f)}{p(w_i-f_i-w_f)} < 1 \]
\[ \Rightarrow (1-p)(w_f-w_i) < -p(w_i-f_i-w_f) \]
\[ \Rightarrow (1-p)w_f - (1-p)w_i + p(w_i-f_i-w_f) < 0 \]
\[ (1-p)w_i + p(w_i-f_i) < w_i \]
Equilibrium point gives a corner solution.

An increase in probability of apprehension and punishment (p), marginal cost of punishment (f) tends to reduce the incentive to participate in illegitimate activity because it increases the expected marginal cost of punishment.

If \( w_i, w_l \) and f are held constant, it can be shown that for a risk aversive offender a one percent increase in probability of capture and apprehension reduces the time spent in illegal activity.

\[
\Delta = \frac{\ln p}{dt_i} \left[ p(w, -w_l)U'(X_a) - p(w, -f, -w_l)U'(X_a) \right]
\]

\[
\frac{d t_i}{d \ln p} = \frac{1}{\Delta} \left[ p(w, -w_l)U'(X_a) + p(f, -w_l)U'(X_a) \right] < 0
\]

\[
\frac{d t_i}{d \ln p} < 0
\]

---

**State a:** probability p, marginal penalty f

**State b:** Probability (1-p), marginal penalty 0

Expected marginal cost of punishment: \( p f + (1-p)0 = pf \)

**"** From equilibrium condition we have,

\( (w, -w_l)(1-p)U'(X_a) = (f, + w_r w_l) p U'(X_a) \)

\[
\Rightarrow (1-P)(w, -w_l)U'(X_a) - (w, -w_l) U'(X_a)
\]

\[
\frac{dp}{dt_i} = -p (w, -f, -w_l) U''(X_a) - (w, -f, -w_l)U'(X_a) \frac{dp}{dt_i}
\]

\[
(1-P)(w, -w_l) U''(X_a) + p (w, -f, -w_l) U''(X_a) =
\]

\[
\frac{dp}{dt_i} \left[ -(w, -w_l)U'(x_a) + (w, -w_l)U'(x_a) \right]
\]

\[
\Delta = \frac{d \ln p}{dt_i} \left[ -p(w, -f, -w_l)U'(x_a) + p(w, -w_l)U'(x_a) \right]
\]

37
A one percent increase in all the penalty rates, \( (f_i) \), hence in the average rate, \( f = \frac{E}{t} \), reduces the optimal time spent in illegal activity, for an offender who is risk averse.

**Empirical model specification**

For time spent on illegal activity by jth individual, it is assumed that number of offences committed is a proxy which depends on the subjective probability of capture and apprehension, marginal penalty paid if captured and administered with punishment, marginal return from legal and illegal activities, rate of unemployment and other factors.

\[
U_q = \phi_q(p_q, f_q, w_q, w_l, u_q, \pi_q)
\]

where \( \phi_q \) is the supply of offences equation, \( \pi_q \) include factors such as family wealth, degree of self protection if caught.

An aggregate version of the micro supply of offences equation may be expressed as,

\[
Q = \phi(P, F, Y, Y, U, \pi)
\]

The aggregate supply of offences equation may be viewed as the cumulative distribution of a density function showing variations across persons.

Let offenders constitute a non-competing group that does not respond to incentives.

\[
\bar{S} = \frac{S}{N}
\]
where \( S \) : Number of offenders who do not respond to incentives.

\( N \) : Number of persons

\( \bar{S} \) is assumed to be a constant.

Let us assume that \( S \) and \( N \) grow at the same rate \( g \), consequently, we specify.

\[
\begin{align*}
N_t &= N_0 (1 + g)^t \\
S_t &= S_0 (1 + g)^t
\end{align*}
\]  
\( \ldots (2.4.3) \)

There is only one form of punishment, namely, imprisonment of \( T \) time periods. Consider the following identity:

\[
J_t = \sum_{r=1}^{T} p(S_{t-r} - J_{t-r}) 
\]  
\( \ldots (2.4.4) \)

where \( J_t \) is jail population at \( t \)

\( S_{t-r} - J_{t-r} \) Number of offenders in the society at \( t-r \)

\( p \) : Probability of capture and apprehension

\( p(S_{t-r} - J_{t-r}) \): Number of offenders sentenced to jail at the end of the time period \( t-r \).

Combining (2.4.3) and (2.4.4) we get the following linear difference equation*:

\[
J_t + p \sum_i J_{t-i} = p \sum_i S_{t-i}
\]

\[
\text{RHS} = p \sum_i S_{t-i} = p \sum_i S_i (1 + g)^{t-i} = S_i p \sum_i (1 + g)^{t-i}
\]

\[
J_t + p \sum_i S_{t-i} = S_t p \sum_i (1 + g)^{t-i}
\]

Divide throughout with \( N_t \).

\[
\frac{J_t}{N_t} + \frac{p \sum_i J_{t-i}}{N_t} = \frac{S_t}{N_t} p \sum_i (1 + g)^{t-i}
\]

\[
\frac{J_t}{N_t} + \frac{p \sum_i J_{t-i}}{N_t (1 + g)^t} = \frac{S_t}{N_t} p \sum_i (1 + g)^{t-i}
\]

\[
\frac{J_t}{N_t} + \frac{p \sum_i (1 + g)^t J_{t-i}}{N_t (1 + g)^t} = \frac{S_t}{N_t} p \sum_i (1 + g)^{t-i}
\]

\* \( J_t + p \sum_i J_{t-i} = p \sum_i S_{t-i} \)

\[ \text{RHS} = p \sum_i S_{t-i} = p \sum_i S_i (1 + g)^{t-i} = S_i p \sum_i (1 + g)^{t-i} \]

\[ J_t + p \sum_i S_{t-i} = S_t p \sum_i (1 + g)^{t-i} \]

\[ \text{Divide throughout with } N_t. \]

\[ \frac{J_t}{N_t} + \frac{p \sum_i J_{t-i}}{N_t} = \frac{S_t}{N_t} p \sum_i (1 + g)^{t-i} \]

\[ \frac{J_t}{N_t} + \frac{p \sum_i J_{t-i}}{N_t (1 + g)^t} = \frac{S_t}{N_t} p \sum_i (1 + g)^{t-i} \]

\[ \frac{J_t}{N_t} + \frac{p \sum_i (1 + g)^t J_{t-i}}{N_t (1 + g)^t} = \frac{S_t}{N_t} p \sum_i (1 + g)^{t-i} \]

39
The particular integral of the difference equation is,

\[
\bar{J} + p(1+g)^{-1} \bar{J}_{-1} + \cdots + p(1+g)^{-T} \bar{J}_{-T} = p \bar{S} \sum_{i=1}^{T} (1+g)^{-i}
\]

Number of offences per head, \( k = \frac{\bar{Q}}{N} \), takes the form.

\[
k = \frac{\bar{Q} \bar{S}}{1 + p \sum_{i=1}^{T} (1+g)^{-i}}
\]

In steady state the rate of offences committed is a negative function of \( PT \).

In steady state, \( g = 0 \), which implies that,

\[
k = \frac{\bar{Q} \bar{S}}{1 + pT}
\]

\( pT \) may be interpreted as expected length of imprisonment for an offender.

\[
\begin{align*}
\bar{J} + \bar{Q} \sum_{i=1}^{T} (1+g)^{-i} &= \bar{S} \sum_{i=1}^{T} (1+g)^{-i} \\
\bar{J} \left[1 + p \sum_{i=1}^{T} (1+g)^{-i}\right] &= \bar{S} \sum_{i=1}^{T} (1+g)^{-i} \\
\bar{J} &= \frac{\bar{S} \sum_{i=1}^{T} (1+g)^{-i}}{1 + p \sum_{i=1}^{T} (1+g)^{-i}} \\
K &= \frac{\alpha}{\theta N} = \frac{\bar{Q} \theta}{\theta N} = \bar{Q} \theta
\end{align*}
\]

where \( \theta = \frac{\bar{Q} \theta}{\theta N} = \bar{Q} \theta \)

Repeating \( \bar{J} \) by the particular integral of the linear difference equation, we obtain,

\[
K = \frac{\bar{S} \sum_{i=1}^{T} (1+g)^{-i}}{1 + p \sum_{i=1}^{T} (1+g)^{-i}}
\]

\[
K = \frac{\bar{Q} \bar{S}}{1 + p \sum_{i=1}^{T} (1+g)^{-i}}
\]
\[ X = \begin{cases} T \text{ if caught} \\ 0 \text{ if not caught} \end{cases} \]

\[ E(X) = T p (X = T) + 0 P (X = 0) = T p + 0 (1 - p) = pT \]

\[ \Rightarrow pT \text{ is expected length of imprisonment.} \]

As expected length of punishment increases, per capita offences decreases. The elasticity of rate of offences with respect to probability of imprisonment and severity of punishment would be the same approximately.

However, in short run \( p \) is believed to have greater preventive effect than an equal proportional increase in \( T \). Many criminologists believed that probability of punishment was of greater importance than the length of punishment.

While \( \epsilon_{kp} \) measures the deference effect on crime, \( \epsilon_{kt} \) measures the preventive effect.

A Cobb-Douglas version of the model was estimated and certain valuable conclusions are drawn.

### 2.5 MURDERS AND CAPITAL PUNISHMENT

Stephen et al. postulated a structural model to study murder behaviour and the Criminal Justice System. Capital punishment deters

---

If \( \xi, \bar{S} \) are constants, then

\[ \ln K = \ln (\xi \bar{S}) - \ln (1 + pT) \]

\[ \frac{\partial \ln K}{\partial p} = \frac{1}{1 + pT} \]

\[ \epsilon_p = \frac{\partial \ln K}{\partial \ln p} = -\frac{pT}{1 + pT} < 0 \]

\[ \epsilon_{tr} = -\frac{\partial \ln K}{\partial \ln T} = -\frac{pT}{1 + pT} < 0 \]

\[ \epsilon_{tr} = \epsilon_{tr} \]

individuals from committing murder, the hypothesis of which was tested by Isaac Ehrlich, and the hypothesis was found accepted. A large body of literature critical of Ehrlich's study has been developed. However, little has been given to the identification of his murder supply equation.

Stephen et al. postulated a structural model to study murder behaviour and the Criminal Justice System. The article primarily appears to be a critic of Ehrlich's article on murder behaviour. The authors combine information on Criminal Justice System and estimate the model.

**Murder Supply Equation**

\[
\frac{Q}{N} = a_{11} + a_{12} \log P + a_{13} \log P_{pa} + a_{14} \log P_{pc} \\
+ a_{15} \log A_1 + a_{16} \log L + a_{17} \log Y_p + a_{18} \log U + a_{19} T \ldots (2.5.1)
\]

where

- \( Q \) : No. of murders
- \( N \) : Civilian Population
- \( P_a \) : Probability of apprehension
- \( P_{pc} \) : Probability of conviction given apprehension
- \( P_{pe} \) : Probability of execution given apprehension
- \( A_1 \) : Population proportion in the age group 14 - 24.
- \( L \) : Population proportion in the labour force
- \( Y_p \) : Permanent income per capita
- \( U \) : Percent labour force unemployed
- \( T \) : Chronological time
The endogenous variable is number of murders for unit population. Probability of apprehension \((P_a)\), probability of conviction given apprehension and probability of execution given apprehension are supposed to be inversely related to murders per unit population.

Unemployment as measured by \(U\), percent labour force unemployed is expected to move directly with \(\frac{Q}{N}\).

\[
\log P_a = a_{21} + a_{22} \log Q + a_{23} \log P_{OL} + a_{24} \log NW
\]

...(2.5.2)

where \(P_{OL}\) : resources available to police only
\(NW\) : proportion of non-white population

- More are police resources, more is the chance of apprehension.
- Number of murders influence probability of apprehension
- Causation exists for \(Q\) and \(P_a\) from both sides.

\[
\log P_{c/a} = a_{31} + a_{32} \log A + a_{33} \log \hat{R} + a_{34} \log NW
\]

...(2.5.3)

Where \(A\) : arrests for observed murders,
\(\hat{R}\) : resources available to the criminal justice system for arresting, convicting and execution of capital murders

- Resources available to the CJS are expected to move in the same direction with probability of conviction given apprehension.
- \(a_{33} > 0\).

\[
\log P_{e/c} = a_{41} + a_{42} \log C + a_{43} \log R + a_{44} CL + a_{45}
\]
$T \log C_{-1} + a_{46} \log A_4.$ \hspace{1cm} \ldots \ldots (2.5.4)$

where

$C$ : Convictions for observed murders.

$CL$ : Dummy variable representing class action appeals and
and taking value for 1965 - 69

$C_{-1}$ : Convictions for observed murders with one period lag

$A_4$ : Proportion of the population within the 14-19 age
group

$\log R = a_{51} + a_{52} \log Q + a_{53} \log Y_P + a_{54} \log N + a_{55}$

$\log A_2 + a_{56} \log NW + a_{57} \log A_3 + a_{58} \log Q_{-1} + a_{59}$

$\log Q_{-2} + a_{5,10} \log QC_{-1} + a_{5,11} OC_{-2}$ \hspace{1cm} \ldots \ldots (2.5.5)$

- More murders require more resources are to be allocated to C.J.S

  $a_{52} > 0$

- Increasing population requires an increase of resource allocation to C.J.S

  $a_{54} > 0$

- Larger lagged murders require greater fund allocation to CJS.

  $a_{58}, a_{59} > 0$

- Lagged violent crimes and resource allocation to CJS move in the same
direction.

  $a_{5,10} > 0, a_{5,11} > 0.$

$\log POL = a_{61} + a_{62} \log Q + a_{63} \log Y_P + a_{64} \log N + a_{65}$

$\log A_2 + a_{66} \log NW + a_{67} \log A_3 + a_{68} \log Q_{-1} + a_{69}$

$\log Q_{-2} + a_{6,10} \log OC_{-1} + a_{6,11} \log OC_{-2}$ \hspace{1cm} \ldots \ldots (2.5.6)$
• Q and POL move in the same direction
  \[ a_{62} > 0 \]

• Police expenditure and population move in the same direction
  \[ a_{64} > 0 \]

• Increased lagged murders and violent crimes require greater allocation of funds to police.
  \[ a_{66} > 0, \ a_{69} > 0, \ a_{6,10} > 0, \ a_{6,11} > 0. \]

\[ Q = N(Q/N) \quad \ldots(2.5.7) \]

\[ A = QP_a \quad \ldots(2.5.8) \]

\[ C = QP_a^pP_{cv} = AP_{cv} \quad \ldots(2.5.9) \]

The above three equations are identities.

Equation (2.5.1) is a murder supply function. CJS activities are represented by the equations (2.5.2), (2.5.3) and (2.5.4). Equation (2.5.5) and (2.5.6) represent resource allocation to CJS. Equations (2.5.7), (2.5.8) and (2.5.9) are identities.

The simultaneous equation model is estimated by two stage least squares method. To estimate and test the model the authors have

\* Q: Number of murders
\* P_a: Probability of apprehension
\* QP_a: Number apprehended
\* A = QP_a
\* C: Number convicted
\* P [Conviction] = P [Convicted / Apprehension] P [Apprehension]
  \[ = P_{cv} \cdot P_r \]
\* C = Q P [Conviction]
  \[ = Q P_{cv} \cdot P_r \]
constructed a data base with internally consistent definitions. Revised versions of Ehrlich data were used in the study.

2.6 BRIEF REVIEW ON DEA

To measure the efficiency of a component of Criminal Justice System, viz., police we wish to know what is efficiency? In a path breaking article Farell introduced three notions of efficiency, (i) technical efficiency (ii) cost efficiency and (iii) allocative efficiency. The reference technology for this purpose is unit output isoquant. In traditional economics an isoquant is defined as the collection of all input vectors which can produce a scalar valued output. Since production is always assumed efficient, a production function, if exists, may be expressed as,

\[ u = f(x) \]

where \( u \in R^+ \)

\[ x \in R^n_+ \]

In this context an isoquant is expressed as,

\[ u_0 = f(x), \text{ where } u_0 \text{ is a scalar valued} \]

![Isoquant](Fig.2.6.1)
However, Farell introduced the notion of inefficiency.

- Any input vector that falls on an isoquant is efficient
- If an input vector is inefficient, it falls above the isoquant
- Inefficiency introduced in this form is called input technical efficiency.

Farell assumed that,

- Production is linear homogeneous so that

\[
0 = f(x_1, x_2) \Rightarrow \\
1 = f\left(\frac{x_1}{u}, \frac{x_2}{u}\right)
\]

- The left hand side gives unit output

\[
\frac{x_1}{u} \text{ and } \frac{x_2}{u} \text{ refer to input usages required to produce one unit of output.}
\]

\[
f\left(\frac{x_1}{u}, \frac{x_2}{u}\right) = 1 \Rightarrow \text{Producer is technically efficient}
\]

- \[
f\left(\frac{x_1}{u}, \frac{x_2}{u}\right) < 1 \Rightarrow \text{producer is technically inefficient}
\]

![Farell Isoquant](image)

**Fig. 2.6.2**
More generally, an input set may be defined as,

\[ L(u) = \{ x : x \text{ produces } u \} \]

In figure (2.6.2) the shaded region consists of inefficient input vectors. The condition of linear homogeneity of production function, implies constant returns to scale.

For a linear homogeneous production function, we have,

\[ f(\lambda x) = \lambda f(x) \]

If all inputs are increased by \( \lambda \), output can also be increased by \( \lambda \). In terms of input level sets \( L(u) \) the property of linear homogeneity leads to,

\[ L(\lambda u) = \lambda L(u) \]

- **Farell's efficiency measurement** is radial, in the sense that the inefficient unit's inputs are radially reduced in the direction of origin till the input vector satisfies feasibility condition.

\[
L(\lambda u) = \{ x : x \text{ produces } \lambda u \} = \left\{ x : \frac{x}{\lambda} \text{ produces } u \right\} = \lambda \left\{ \frac{x}{\lambda} : \frac{x}{\lambda} \text{ produces } u \right\} = \lambda \left\{ \frac{x}{\lambda} \text{ produces } u \right\}, \text{ where } \frac{x}{\lambda} = \bar{x} = \frac{x}{\lambda} = \lambda L(u) \\
L(\lambda u) = \lambda L(u) 
\]
Farrell’s input technical efficiency is, $\delta$

$$0 \leq \delta \leq 1.$$  

Departure of input vector in the direction of origin makes the input vector infeasible.

**Farrell’s cost efficiency measurement**

Any unit output required input combination that falls on an isoquant is technically efficient. But, it may not be cost efficient in the sense that the cost of some other input vector on an isoquant gives lesser cost as such the former technically efficient unit turns out to be cost inefficient.
Let the factor minimal cost function be denoted by, \( Q(l, p) = p \left( \frac{x}{u} \right) \)
where \( Q(l, p) \) is factor minimal cost to produce unit output, \( p \) is price vector of input vector \( \frac{x}{u} \).

\[
p\left( \frac{x}{u} \right) = Q(l, p)
\]

For a linear homogenous production function the associated factor minimal cost function is such that:

\[
Q(u, p) = u Q(l, p)\]

\[
\frac{x}{u}
\]

\[
p\left( \frac{x}{u} \right) = Q(p) D
\]

\[
\text{Fig.2.6.4}
\]

\[
* Q(l, p) = \min_{x \in L(l)} \left\{ px \left( \frac{x}{u} \right) : \left( \frac{x}{u} \right) \in L(l) \right\}
\]

\[
= \min_{x \in L(u)} \left\{ px \left( \frac{x}{u} \right) : x \in L(l) \right\}
\]

If RTS are constant,

\[
Q(l, p) = \min_{x \in L(u)} \left\{ px \left( \frac{x}{u} \right) : x \in L(l) \right\}
\]

\[
= \frac{1}{u} \min_{x \in L(u)} \left\{ px : x \in L(l) \right\}
\]

\[
= \frac{1}{u} \min_{x \in L(u)} \left\{ px : x \in L(u) \right\}
\]

\[
= \frac{1}{u} Q(u, p)
\]
SS' is isocost line, whose mathematical representation is,

\[ p \left( \frac{x}{u} \right) = Q(l, p) \]

\[ px = uQ(l, p) \]

\[ B \left( \frac{x_1^b}{u}, \frac{x_2^b}{u} \right) \] is technically efficient, whose cost is.

\[ \frac{x_1^b}{u} + p \left( \frac{x_2^b}{u} \right) = p \left( \frac{x^b}{u} \right) \]

- Cost at C is \( p \left( \frac{x}{u} \right) \).

- But cost at C and cost at D are equal.

- Cost at D is lesser than cost at B, implying that cost at C is smaller than cost at B.

**Farell's cost efficiency** is defined as

\[ CE = \frac{Q(l, p)}{p \left( \frac{x^A}{u} \right)} \]

Alternatively, **Farell's efficiency measures** are as follows:

**Technical efficiency:** \( \frac{OB}{OA} (= \delta) \)

**Cost efficiency:** \( \frac{OD}{OA} = \frac{Q(l, p)}{p \left( \frac{x^A}{u} \right)} \)

**Farell's allocative efficiency:**

\[ \Rightarrow Q(u, p) = uQ(l, p) \]
Failure to allocate inputs at C instead of B leads to allocative inefficiency

- Cost at C: $Q(1, p)$
- Cost at B: $p \left( \frac{x^b}{u} \right)$

Allocative efficiency:
$$\frac{Q(1, p)}{p \left( \frac{x^b}{u} \right)} = \frac{uQ(1, p)}{px^a}$$

$$0 \leq \frac{uQ(1, p)}{px^a} \leq 1$$

Alternatively, allocative efficiency is.

$$AE = \frac{OD}{OB}$$

$$0 \leq AE \leq 1$$

Decomposition of cost efficiency:

Farell's cost efficiency can be decomposed into the product of technical and allocative efficiencies:

$$CE = TE \times AE$$

$$\frac{OD}{OA} = \frac{OB}{OA} \times \frac{OD}{OB}$$

SHEPARD'S INPUT DISTANCE FUNCTION

R.W. Shephard *(1970) has introduced input distance function which measures input technically efficiency. His input technical efficiency measure is reciprocal of Farell technical efficiency measure.

Shephard defines input sets as follows:

\[ L(u) = \{ x : x \text{ produces } u \} \]

These input sets are basis for subsequent studies.

Shephard imposes the following structural properties on \( L(u) \):

1. \( L(0) = \mathbb{R}_+^n \)
   - Null output vector can be produced by every input vector \( x \in \mathbb{R}_+^n \).

2. \( 0 \not\in L(u), u > 0 \)
   - Null input vector cannot produce positive output.

3. \( x \in L(u) \), \( x' \geq x \Rightarrow x' \in L(u) \)
   - If \( x \) produces \( u \), then any input vector larger than \( x \) can produce \( u \).
   - This happens due to inefficiency.
   - This property is called as free disposability of inputs.

4. \( u_2 \geq u_1 \geq 0 \Rightarrow L(u_2) \subset L(u_1) \)
   - If \( x \) can produce \( u_2 \), then due to inefficiency \( x \) can produce \( u_1 \), an output vector smaller than \( u_2 \).
   - This condition is called free disposability of outputs.

The conditions (3) and (4) were referred in subsequent studies of efficiency as inefficiency axioms; But Shephard named them as free disposability of inputs and outputs.

5. \( \bigcap_{0 \leq u_0 \leq 0} L(u) = L(u_0), u_0 > 0 \)

6. \( \bigcap_{0 \leq u_0 \leq 0} L(u) = L(u_0), u_0 > 0 \)
This property is imposed in order to guarantee the existence of production function. If \( \phi(x) \) stands for a production function, then:
\[
\phi(x) = \max\{u : x \in L(u)\}
\]

(7) \( \bigcap_{u \in [0, \infty)} L(u) = \phi \)

where \( \phi \) is empty set

No bounded input vector can produce an infinite rate of output.

(8) \( L(u) \) is closed for all \( u \) such that, \( u \in [0, \infty) \)

(9) \( L(u) \) is convex for all \( u \in [0, \infty) \)

\[
x \in L(u), \ y \in L(u), \ \theta \in [0, 1] \implies (1-\theta)x + \theta y \in L(u)
\]

The input vector \((1-\theta)x + \theta y\) may be interpreted as an operation of the technology a fraction \((1-\theta)\) of some unit time interval with the input vector \(x\) and a fraction \(\theta\) with \(y\), assuring the output rate \(u\).

**PRODUCTION FUNCTION**

Shephard defines the production function \( \phi(x) \) as follows:
\[
\phi(x) = \text{Max}\{x : x \in L(u), \ 0 \leq u < \infty\}
\]

\( \triangleright \) \( \phi(x) \) is given the traditional meaning that it is maximum output producible by \( x \in R^* \)

\[\begin{align*}
x &\in L(u), \ \forall \ u \text{ such that } 0 \leq u \leq \bar{u} \\
u > \bar{u} &\implies x \notin L(u) \\
\implies x &\in \bigcap_{u \in \bar{u}} L(u) = L(\bar{u}), \text{ by virtue of axiom (6)} \\
\implies x &\in L(\bar{u}) \\
\implies \phi(x) &\equiv \bar{u} \\
\implies \phi(x) &\text{ exists}
\end{align*}\]
The structure of $L(u)$ imposes special restrictions on $\phi(x)$.

(1) $\phi(0) = 0$

Null input vector produces null output

(2) $x' \geq x \Rightarrow \phi(x') \geq \phi(x)$

Let $u \in \{u : x \in L(u)\}$

$\Rightarrow x \in L(u)$, since $x \leq x'$;

$\Rightarrow x' \in L(u)$

$\Rightarrow u \in \{u : x' \in L(u)\}$

$\{u : x \in L(u)\} \subseteq \{u : x' \in L(u)\}$

$max \{u : x \in L(u)\} \leq max \{u : x' \in L(u)\}$

$\phi(x) \leq \phi(x')$

![Diagram](image-url)
\[ x = (x_1, x_2) \]
\[ x' = (x'_1, x'_2) \]

\[ \{ u : x \in L(u) \} = \{ u_1, u_2, u_3 \} \]

\[ \{ u : x' \in L(u) \} = \{ u_1, u_2, u_3, u_4 \} \]

\[ \{ u : x \in L(u) \} \subseteq \{ u : x \in L(u) \} \]

\[ \text{Max} \{ u : x \in L(u) \} \leq \text{Max} \{ u : x' \in L(u) \} \]

\[ \phi(x) \leq \phi(x') \]

(3) \( P(x) \) is upper semi-continuous on its domain

(4) \( \phi(x) \) is quasi-concave on its domain.

**Homogenous production functions**

A production function consistent with the above properties is homogeneous of degree one, then it is super additive, continuous and concave.