Chapter - III

THEORY AND METHODOLOGY
THEORY AND METHODOLOGY

3.1 In the ‘Review of Literature’ the evolution of Data Envelopment Analysis (DEA) is presented. Radial efficiency measures, their properties and their numerical estimation by means of solving appropriate Linear Programming Problems and by products of these linear programming problems are also furnished.

The radial efficiency measures have their own limitations.

- They do not distinguish weak efficiency from strong efficiency. A decision making unit that is weak efficient also attains 100 percent radial efficiency while strong efficient DMU also attains the same efficiency score.

![Figure 3.1](image)

In the above figure we have an input level set $L(u_o)$ whose isoquant is constituted by the line segments AB, BC and CD. The DMUs E, F and P are inefficient. The radial efficiency of P is $\frac{OO}{OP}$.
\[0 < \frac{OQ}{OP} \leq 1\]

If inputs \((x_1, x_2)\) are radially with drawn, the DMU attains 100 percent efficiency score at \(Q\). At \(Q\) the DMU \(P\) is only weak efficient.

The same output \(u_0\) can be produced by choosing \(B\) rather than \(Q\).

\[ (x_Q, u_Q) \in \text{weak efficient } L(u_0). \]
\[ (x_B, u_B) \in \text{Eff } L(u_0) \]
\[ \text{Eff } L(u_0) \subseteq W \text{Eff } L(u_0) \subseteq L(u_0) \]

Instead of picking the bench marks from \(W\text{Eff } L(u_0)\), we shall pick the same from \(\text{Eff } L(u_0)\).

To overcome the problem slacks are introduced into the optimization problem. A target DMU with an efficiency score of 100 percent and all slacks assuming '0' value is said to be efficient.

**Slack Based Input Efficiency Problem**:

\[
\begin{align*}
\text{Min } & \lambda - \sum_{i=1}^{n} s_i^+ - \sum_{r=1}^{s} s_r^- \\
\text{subject to } & \sum_{j=1}^{m} \lambda_j x_{ij} + s_i^+ = x_{i0} \lambda, \quad i = 1, 2, \ldots, m. \\
& \sum_{j=1}^{m} \lambda_j u_{ij} - s_r^- = u_{r0}, \quad r = 1, 2, \ldots, s. \quad (3.1.1) \\
& \sum_{j=1}^{m} \lambda_j = 1, \\
& \lambda_j \geq 0.
\end{align*}
\]
Slack based output efficiency problem:

\[
\begin{align*}
\text{Max } & \quad \theta - \sum_{i=1}^{n} s^*_i - \sum_{i=1}^{r} s^*_i \\
\text{subject to } & \quad \sum_{j=1}^{n} \lambda_j x_{ij} + s^*_i = x_{io}, \quad i = 1, 2, \ldots, m. \\
& \quad \sum_{i=1}^{n} \lambda_i u_{ij} - s^*_r = u_{io}, \quad r = 1, 2, \ldots, s. \\
& \quad \sum_{j=1}^{n} \lambda_j = 1 \quad \text{------- (3.1.2)} \\
& \quad \lambda_j \geq 0.
\end{align*}
\]

In input orientation \(DMU_0\) is said to be efficient if and only if,

\[
\begin{align*}
\text{Min } & \quad \lambda = 1 \\
& \quad s^*_i = 0, \quad \forall i \\
& \quad s^*_r = 0, \quad \forall r.
\end{align*}
\]

Similarly, in output orientation \(DMU_0\) is said to be efficient if and only if,

\[
\begin{align*}
\text{max } & \quad \theta = 1 \\
& \quad s^*_i = 0, \quad \forall i \\
& \quad s^*_r = 0, \quad \forall r.
\end{align*}
\]

3.2 NON-RADIAL MEASURES:

Fare et al. (1985) proposed a linear programming method to estimate factor minimal cost, \(Q(u,p)\).

\[
Q(u_0, p) = \sum_{i=1}^{n} p_i x_i
\]

subject to

\[
\begin{align*}
\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{ij}, & \quad i = 1, 2, \ldots, m. \quad \text{------- (3.2.1)} \\
\sum_{j=1}^{n} \lambda_j u_{ij} \geq u_{ij}, & \quad r = 1, 2, \ldots, s. \\
\lambda_j & \geq 0.
\end{align*}
\]

\[\text{---}
\]

* Fare et al. (1985), op. cit.,
In a similar way output maximal revenue can be estimated solving the following problem:

\[ R(x, r) = \max \sum_{k=1}^{k=n} \lambda_i x_{ij} \]

subject to \[ \sum_{j=1}^{J} \lambda_j x_{ij} \leq x_{ij}, \quad i = 1, 2, \ldots, n \] \[ \sum_{j=1}^{J} \lambda_j u_{ij} \geq u_{ij}, \quad k = 1, 2, \ldots, s \]

\[ \lambda_j \geq 0. \]

3.3 The difference between output maximal revenue and factor minimal cost should give us maximum profit. If factor minimal cost and output maximal revenue are equal, the potential profit equals to zero.

Potential profit can be obtained solving the following linear programming problem:

\[ \max (ru - px) \]

subject to \[ (x, u) \in G \]

where \( G \) is the production possibility set.

In terms of DEA constraints the above optimization problem can be expressed as,

\[ \pi(r, p) = \max \left( \sum_{k=1}^{k=n} \lambda_i x_{ij} - \sum_{j=1}^{J} \lambda_j u_{ij} \right) \]

subject to \[ \sum_{j=1}^{J} \lambda_j x_{ij} \leq x_{ij}, \quad i = 1, 2, \ldots, m \] \[ \sum_{j=1}^{J} \lambda_j u_{ij} \geq u_{ij}, \quad r = 1, 2, \ldots, s. \]

\[ \lambda_j \geq 0. \]
Theorem (1):

\[ \pi(r, p) = R(x_0, r) - Q(u_0, p) \]  

(3.3.2)

Proof:

\[ R(x_0, r) - Q(u_0, p) = \max \sum_{k=1}^{n} \mu_k x_k - \min \sum_{i=1}^{m} p_i x_i \]

subject to

\[ \sum_{j=1}^{n} \lambda_j x_j \leq x_i, \quad i = 1, 2, \ldots, \ldots, m. \]

\[ \sum_{j=1}^{n} \lambda_j u_j \geq u_k, \quad k = 1, 2, \ldots, \ldots, s. \]

Equivalently, the constraints set may be expressed as,

\[ \sum_{j=1}^{n} \lambda_j x_j \leq \min \{x_i, x_{i0}\} \]

\[ \sum_{j=1}^{n} \lambda_j u_j \geq \max \{u_k, u_{k0}\} \]

Since \( x_i \leq x_{i0}, \forall i \)

\( u_k \geq u_{k0}, \forall k \)

the above constraints become,

\[ \sum_{j=1}^{n} \lambda_j x_j \leq x_i, \forall i \]

\[ \sum_{j=1}^{n} \lambda_j u_j \geq u_k, \forall k \]  

(3.3.3)

\[ \lambda_j \geq 0 \]

Consequently, we have,

\[ \pi(r, p) = R(x_0, r) - Q(u_0, p) \]  

(3.3.4)
In the above figure G is the production possibility set. Its frontier admits variable returns to scale. Along the segments AB returns to scale are increasing. At the point B returns to scale are constant. Along the line segments BC and CD returns to Scale are decreasing. The decision making unit P is inefficient. It uses input $x_0$ and produces output $u_0$. The line segment that passes through $p$ represents observed profit.

$$\pi(p, r_0, x_0, u_0) = ru_0 - px_0$$

Profit is maximized at B and the ratio,

$$\frac{\pi(p, r, x_0, u_0)}{\pi(p_0, r_0)}$$

measures profit efficiency.

$$0 \leq \left( \frac{\pi(p, r, x_0, u_0)}{\pi(p, r)} \right) \leq 1. \quad \text{(3.3.5)}$$

### 3.4 NON-DISCRETIONARY VARIABLES – ENVIRONMENTAL EFFICIENCY- RADIAL MEASURES:

Banker and Morey (1986) formulated a DEA model to incorporate external non-discretionary input into the DEA model. Their formulation is nothing but the linear programming problem (3.3.4). We write this problem as,
\[ \lambda(BM, EXO) = \text{Min} \lambda \]

subject to \[ \sum_{j=1}^{s} \lambda_j x_{ij} \leq \lambda x_{io}, \quad i = 1, 2, \ldots, m. \]

\[ \sum_{j=1}^{s} \lambda_j u_r \geq u_{ro}, \quad r = 1, 2, \ldots, s. \]  \hspace{1cm} (3.4.1)

\[ \sum_{j=1}^{s} \lambda_j u_s \leq u_{so}. \]

This problem assumes risk is endogenous.

Every feasible solution of (3.4.1) is a feasible solution of CCR (1984) problem. Consequently, we have,

\[ \lambda(CCR) \leq \lambda(BM, EXO) \]

Problem (3.4.1) treats the undesirable output \( u_b \) as exogenous. We call \( u_b \) as external non-discretionary factor. If \( u_b \) is internal non-discretionary factor, Banker and Moorey made the following formulation:

\[ \lambda(BM, ENDO) = \text{Min} \lambda \]

subject to \[ \sum_{j=1}^{s} \lambda_j x_{ij} \leq \lambda x_{io}, \quad i = 1, 2, \ldots, m. \]

\[ \sum_{j=1}^{s} \lambda_j u_r \geq \lambda u_{ro}, \quad r = 1, 2, \ldots, s. \]  \hspace{1cm} (3.4.2)

\[ \sum_{j=1}^{s} \lambda_j u_s \leq \sum_{j=1}^{s} \lambda_j u_{so}. \]

Every feasible solution of (3.4.2) is a feasible solution of CCR problem, consequently, we have following:

\[ \lambda(CCR) \leq \lambda(BM, ENDO) \quad \text{------- (3.4.3)} \]

In BM (1986) approach a DMU operating in an external or internal environment is compared with all the DMUs operating in equal, superior and inferior environment.
John Ruggiero* (1996) formulated a mathematical programming problem where a DMU is compared with DMUs operating in equal and inferior environments. For example, if a commercial bank branch is located in a place with certain extent of parking area, then it should be compared with bank branches with equal or smaller extent of parking area.

John Ruggiero formulated the following mathematical programming problem:

$$\lambda(JR) = \text{Min} \ \lambda$$
subject to
$$\sum_{j \in S} \lambda_j x_{ij} \leq \lambda x_{i_0}$$

$$\sum_{j \in S} \lambda_j u_{ij} \geq \lambda u_{i_0}$$

So = \{j: z_j \leq z_0\}

$$\lambda_j \geq 0.$$

This problem allows only one external non-discretionary factor. The reference set is formulated by such DMUs whose non-discretionary inputs are smaller than or equal to the non-discretionary input of DMU. While BM approach takes into consideration all DMUs to compare with DMUo, the JR approach considers a subset of all the DMUs and compare them with DMUo. From the dimensionality problem of DEA, the BM efficiency score can not exceed the JR efficiency.

$$\lambda(BM, EXO) \leq \lambda(JR)$$

Unlike the BM approach, in JR approach the reference set varies for one DMU to another.

In the above figure we have nested frontier production functions. The frontier production function determined by the DMUs A, B, C and D is the Banker and Moorey production frontier. It envelops all the sample points. There are two JR frontiers one determined by the DMUs E, F, G and H and the other determined by I, J, K and L.

The producer who operates at M is technically inefficient. He employs input $x_0$ and produces output $u_0$. Relative to M DMUs I, J, K, N and P operate under same or inferior environment. The referent DMUs for DMU M are I, J, K, L and the technical efficiency of DMU M is, $\lambda(JR)$.

Further contraction of input places DMU M on BM frontier. Consequently the input technical efficiency is,

$$\lambda(BM)$$

Clearly, $\lambda(BM) \leq \lambda(JR)$.

Thus, the BM approach under estimates the input technical efficiency of DMUs.

The JR (1996) approach allows only one external non-discretionary factor. Further, while we attempt to measure input technical efficiency of DMUs like P, a
few DMUs are there to formulate the corresponding frontier. Ruggiero* (1998) formulation is an extension of Ruggiero (1996) model. It considers multiple non-discretionary inputs which lead to a two stage method. In first stage a regression equation is estimated to combine the non-discretionary inputs into a single index, which is used to estimate input technical efficiency implementing Ruggiero (1996) model.

The underlying production possibility set of Ruggiero (1996) formulation, for a given $z$, satisfied the properties of convexity, Monotonicity, inclusion and minimum extrapolation. Convexity is not imposed on the ND input.

If larger values of an ND factor implies then it is viewed as an input of better environment. If smaller values of an ND factor imply superior environment then the ND factor is viewed as an output.

In the context of commercial banks the Ruggiero formulation (1996) takes the form:

$$\lambda(JR) = \text{Min} \lambda$$

subject to $\sum_{j \in I_0} \lambda_j x_{ij} \leq \lambda x_{io}$  \hspace{1cm} \text{(3.4.6)}

$$\sum_{j \in I_o} \lambda_j u_{rj} \geq u_{ro}$$

$$S_o = \{ j : u_{oj} \geq u_{bo} \text{ and a percent of DMUs for which } u_{oj} \leq u_{bo} \}$$

This follows from Ruggiero **(2004) formulations:

If we impose convexity on the problems (3.4.6) we get BCC and Ruggiero combined formulation. We solve the following problem.


\[ \lambda(JR, BCC) = \text{Min}\lambda. \]

subject to
\[ \sum_{j \in s_0} \lambda_j x_{y_j} \leq \lambda x_w \]
\[ \sum_{j \in s_0} \lambda_j u_{j} \geq u_{w_j} \]
\[ \sum_{j \in s_0} \lambda_j = 1 \]

where \( s_0 = \{ j : u_{y_j} \geq u_{w_j} \} \)

Every Feasible solution of (3.4.7) is a feasible solution of (3.4.6). Consequently, we have the following inequality satisfied.

\[ \lambda(JR) \leq \lambda(JR, BCC) \]
\[ \lambda(CCR) = \text{Min}\lambda \]

subject to
\[ \sum_{j=1}^{s} \lambda_j x_{j} \leq x_w, \quad i = 1,2, \ldots, m. \]
\[ \sum_{j=1}^{s} \lambda_j u_{j} \geq u_{w_j}, \quad j = 1,2, \ldots, n. \]
\[ \lambda_i \geq 0. \]

Clearly we have,
\[ \lambda(CCR) \leq \lambda(JR) \leq \lambda(JR, BCC). \]

\[ \lambda(CCR) = \begin{bmatrix} \lambda(CCR) \\ \lambda(JR) \end{bmatrix} \begin{bmatrix} \lambda(JR) \\ \lambda(JR, BCC) \end{bmatrix} \]

where \( \frac{\lambda(CCR)}{\lambda(JR)} \) measures input risk efficiency.

\[ \begin{bmatrix} \lambda(JR) \\ \lambda(JR, BCC) \end{bmatrix} \] measures input scale efficiency

\( \lambda(JR, BCC) \) measures input pure technical efficiency.

* T.Subramanyam and CS Reddy (2008), op.cit.
3.5 NON-DISCRETIONARY VARIABLES-ENVIRONMENTAL EFFICIENCY-DIRECTIONAL MEASUREMENT:

This study extends the work of T. Subramanyam (2010) to arrive at additive decomposition of overall directional technical efficiency. Unlike the radial efficiency measures the directional distance function introduced by Chambers et al. (1998) contract inputs and expands outputs to arrive at directional efficiency score. It can be shown that the input and output radial efficiency measures are particular cases of the Directional Distance Functions. The measure is defined as follows.

\[ \tilde{D}_G(x_u; f_s, f_u) = \max \{ \lambda : (x - \lambda f_s, u + \lambda f_u) \in G \} \]

- \( G \) is the production possibility set.
- \( x, u \) are input and output vectors.
- \( f_s, f_u \) are directional vectors of input and output vectors.

To better understand the above optimization problem consider the following diagram:

![Diagram](image)

Figure (3.4)

In the above figure input is measured along horizontal axis and output along vertical axis. The straight line that emanates from horizontal axis is a production frontier that admits constant returns to scale. The area bound by it from above is the production possibility set \( G \). The producer who operates at \( p(x_u, u_u) \) is inefficient.
In the above figure input is measured along horizontal axis and output along vertical axis. The straight line that emanates from horizontal axis is a production frontier that admits constant returns to scale. The area bound by it from above is the production possibility set $G$. The producer who operates at $p(x_0, u_0)$ is inefficient. Inputs are contracted and outputs are expanded simultaneously in the direction of the directional vector $(f_s, f_u)$. At $Q$ producer achieves directional efficiency.

- $Max \lambda = \bar{D}_G(x_0, u_0; f_s, f_u)$
- Inputs target: $x_0 - \bar{D}_G(x_0, u_0; f_s, f_u) f_s$
- Outputs target: $u_0 + \bar{D}_G(x_0, u_0; f_s, f_u) f_u$

The directional efficiency can be computed solving the following linear programming problem:

$$\bar{D}(x_0, u_0; f_s, f_u) = Max \lambda$$

subject to

$$\sum_{j=1}^{s} \lambda_j x_{ij} \leq x_{i0} - \lambda f_s$$  \hspace{1cm} \text{(3.5.1)}

$$\sum_{j=1}^{s} \lambda_j u_{rj} \geq u_{r0} + \lambda f_u$$

The above problem admits constant returns to scale.

Banker and Morey *(1986)* proposed a linear programming problem to account for exogenous, non-discretionary factors.

Their approach leads to the following linear programming problem:

$$Min \theta$$

subject to

$$\sum_{j=1}^{s} \lambda_j x_{ij} \leq \theta x_{i0}, \hspace{0.5cm} i = 1, 2, \ldots, m.$$  \hspace{1cm} \text{(3.5.2)}

$$\sum_{j=1}^{s} \lambda_j u_{rj} \geq u_{r0}, \hspace{0.5cm} r = 1, 2, \ldots, s.$$  \hspace{1cm} \text{(3.5.2)}

$$\sum_{j=1}^{s} \lambda_j z_{kj} \leq z_{k0}, \hspace{0.5cm} k = 1, 2, \ldots, q.$$  \hspace{1cm} \text{(3.5.2)}

- The above formulation is a linear programming problem.
- Outputs are exogeneous.

* Banker and Morey (1986), op.cit.
• q- non discretionary factors.
• Non-discretionary factors are exogeneous.
• The problem admits only constant returns to scale.

3.6 The exogenous non-discretionary factors problem can be addressed in Directional Distance frame work also. If there are q-non- discretionary inputs that are exogenous to production, following Banker and Morey(1986) we propose the following linear programming problem:

\[ \text{Max } \theta \]
\[ \text{subject to } \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{i0} - \theta f_s, \quad i = 1, 2, \ldots, m \]
\[ \sum_{j=1}^{n} \lambda_j u_{ij} \geq u_{i0} + \theta f_s, \quad r = 1, 2, \ldots, s \quad \text{(3.6.1)} \]
\[ \sum_{j=1}^{n} \lambda_j z_{ij} \leq z_{i0}, \quad k = 1, 2, \ldots, q. \]

\[ \text{Max } \lambda = \widetilde{D}_G \left( x_0, u_0, z_0; f_s, f_s / \text{CRS} \right) \]

Theorem (2):
\[ \widetilde{D}_G \left( x_0, u_0; f_s, f_s / \text{CRS} \right) \geq \widetilde{D}_G \left( x_0, u_0, z_0; f_s, f_s / \text{CRS} \right) \]

Proof: Consider the following constraint sets,

1. \( G_i = \left\{ (\lambda, \theta) : \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{i0} - \theta f_s, \sum_{j=1}^{n} \lambda_j u_{ij} \geq u_{i0} + \theta f_s \right\} \)

2. \( G_2 = \left\{ (\lambda, \theta) : \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{i0} - \theta f_s, \sum_{j=1}^{n} \lambda_j u_{ij} \geq u_{i0} + \theta f_s, \sum_{j=1}^{n} \lambda_j z_{ij} \leq z_{i0} \right\} \)

\( (\lambda, \theta) \in G_2 \Rightarrow (\lambda, \theta) \in G_i \)
\( \Rightarrow G_2 \subseteq G_i \)

\[ \text{Max } \{ \theta : (x_0 - \theta f_s, u_0 + \theta f_s) \in G_i \} \geq \text{Max } \{ \theta : (x_0 - \theta f_s, u_0 + \theta f_s, z_0) \in G_2 \} \]

\[ \widetilde{D}_G (x_0, u_0; f_s, f_s / \text{CRS}) \geq \widetilde{D}_G (x_0, u_0, z_0; f_s, f_s / \text{CRS}) \quad \text{(3.6.2)} \]
EXOGENOUS ENVIRONMENTAL EFFICIENCY INDICATOR:

The difference $\tilde{D}_G(x_0, u_0; f_x, f_u / CRS) - \tilde{D}_G(x_0, u_0, z_0; f_x, f_u / CRS)$ measures directional exogenous environment efficiency. \hspace{1cm} (3.6.3)

Theorem(3):

$\tilde{D}_G(x_0, u_0, z_0; f_x, f_u, f_z / CRS) \leq \tilde{D}_G(x_0, u_0, z_0; f_x, f_u, 0 / CRS)$

Proof: Consider the following constraint sets,

$G_2 = \left\{ (\lambda, \theta) : \sum \lambda_j x_j \leq x_{z0} - \theta f_x, \sum \lambda_j u_j \geq u_{z0} + \theta f_u, \sum \lambda_j z_j \leq z_{z0} \right\}$

and

$G_3 = \left\{ (\lambda, \theta) : \sum \lambda_j x_j \leq x_{z0} - \theta f_x, \sum \lambda_j u_j \geq u_{z0} + \theta f_u, \sum \lambda_j z_j \leq z_{z0} - g, \theta \right\}$

$\tilde{D}_G(x_0, u_0, z_0; f_x, f_u, f_z / CRS) \leq \tilde{D}_G(x_0, u_0, z_0; f_x, f_u, 0 / CRS)$ \hspace{1cm} (3.6.4)

Endogenous Environment Efficiency:

The following difference measures endogeneous environment efficiency:

$\tilde{D}_G(x_0, u_0, z_0; f_x, f_u, f_z / CRS) \leq \tilde{D}_G(x_0, u_0, z_0; f_x, f_u, f_z / CRS)$ \hspace{1cm} (3.6.5)

3.7. DIRECTIONAL SCALE EFFICIENCY:

Banker, Charnes and Cooper (BCC, 1984) had shown that by appending convexity constraint $\sum \lambda_j = 1$ to the CCR (1978) problem, the scale difference can be removed from the CCR efficiency scores.
The ratio $\frac{\lambda(CCR)}{\lambda(BCC)}$ measures input radial scale efficiency.

In a similar manner Directional Scale Efficiency can be measured considering the following optimization problem:

$$\text{Max } \theta$$

subject to

$$\sum_{j=1}^{n} \lambda_j x_j \leq x_{t0} - \theta f_s$$

$$\sum_{j=1}^{n} \lambda_j u_{j} \geq u_{t0} + \theta f_u$$

---(3.7.1)

$$\sum_{j=1}^{n} \lambda_j z_j \leq z_{t0} - \theta f_z$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

Here, $\text{Max } \theta = \tilde{D}_G(x_0, u_0, z_0; f_x, f_u, f_z / VRS)$

- VRS refers to variable returns to scale.

- The constraint $\sum_{j=1}^{n} \lambda_j = 1$ models variable returns to scale.

Theorem (4):

$$\tilde{D}_G(x_0, u_0, z_0; f_x, f_u, f_z / CRS) \geq \tilde{D}_G(x_0, u_0, z_0; f_x, f_u, f_z / VRS)$$

----- (3.7.2)

Proof: Consider the constraint sets,

$$G_3 = \left\{ (\lambda, \theta) : \sum_{j=1}^{n} \lambda_j x_j \leq x_{t0} - f_x, \sum_{j=1}^{n} \lambda_j u_j \geq u_{t0} + \theta f_u, \sum_{j=1}^{n} \lambda_j z_j \leq z_{t0} - \theta f_z, \sum_{j=1}^{n} \lambda_j = 1 \right\}$$

$$G_4 = \left\{ (\lambda, \theta) : \sum_{j=1}^{n} \lambda_j x_j \leq x_{t0} - \theta f_x, \sum_{j=1}^{n} \lambda_j u_j \geq u_{t0} + \theta f_u, \sum_{j=1}^{n} \lambda_j z_j \leq z_{t0} - \theta f_z, \sum_{j=1}^{n} \lambda_j = 1 \right\}$$

$$(\lambda, \theta) \in G_4 \Rightarrow (\lambda, \theta) \in G_3$$

$$\Rightarrow G_4 \subseteq G_3$$

Consequently, we have,

$$\tilde{D}_G(x_0, u_0, z_0; f_x, f_u, f_z / CRS) \geq \tilde{D}_G(x_0, u_0, z_0; f_x, f_u, f_z / VRS)$$
The following difference measures Directional Scale Efficiency:

\[ \bar{\bar{D}}_G(x_0, u_0, z_0; f_x, f_u; f_z / CR S) - \bar{\bar{D}}_G(x_0, u_0, z_0; f_x, f_u, f_z / V R S) \]

**DIRECTIONAL PURE TECHNICAL EFFICIENCY:**

The BCC (1984) formulation measures pure technical efficiency. In a similar way Directional pure technical efficiency can be derived.

\[ \bar{\bar{D}}_G(x_0, u_0, z_0; f_x, f_u, f_z / V R S) \] measures directional pure technical efficiency.

**3.8 ADDITIVE DECOMPOSITION:**

**Theorem (5):**

Overall directional efficiency can be additively decomposed as follows:

[Overall directional efficiency] = [Directional Exogenous Environment Efficiency] + [Directional Endogenous Environment Efficiency] + [Directional Scale Efficiency] + [Directional Pure Technical Efficiency]

**Proof:** Combining the inequalities (3.6.3), (3.6.5) and (3.7.2) we have the following arrangement:

\[ \bar{\bar{D}}_G(x_0, u_0; f_x, f_u / CR S) \geq \bar{\bar{D}}_G(x_0, u_0, z_0; f_x, f_u, 0 / CR S) \]
\[ \geq \bar{\bar{D}}_G(x_0, u_0, z_0; f_x, f_u, f_z / CR S) \geq \bar{\bar{D}}_G(x_0, u_0, z_0; f_x, f_u, f_z / V R S) \] \hspace{1cm} (3.8.1)

The overall directional efficiency can be additively decomposed as,

\[ \bar{\bar{D}}_G(x_0, u_0; f_x, f_u / CR S) = \]
\[ [\bar{\bar{D}}_G(x_0, u_0; f_x, f_u / CR S) - \bar{\bar{D}}_G(x_0, u_0, z_0; f_x, f_u, 0 / CR S)] \]
\[ + [\bar{\bar{D}}_G(x_0, u_0, z_0; f_x, f_u, 0 / CR S) - \bar{\bar{D}}_G(x_0, u_0, z_0; f_x, f_u, f_z / CR S)] \]
\[ + [\bar{\bar{D}}_G(x_0, u_0, z_0; f_x, f_u, f_z / CR S) - \bar{\bar{D}}_G(x_0, u_0, z_0; f_x, f_u, f_z / V R S)] \]
\[ + [\bar{\bar{D}}_G(x_0, u_0, z_0; f_x, f_u, f_z / V R S)] \] \hspace{1cm} (3.8.2)
Overall Directional Environment Efficiency:
The difference given below measures overall directional environment efficiency:

\[ \vec{D}_G (x_0, u_0; f_x, f_u / CRS) - \vec{D}_G (x_0, u_0, z_0; f_x, f_u, f_z / CRS) \quad (3.8.3) \]

Nerlove's Profit Efficiency Indicator:

Nerlove's profit efficiency indicator may be expressed as follows:

\[ PEI = \frac{\pi_0 (p, r) - \pi_0 (x_0, u_0; p, r)}{pf_x + rf_u} \quad (3.8.4) \]

- \( \pi_0 (p, r) \) is maximal profit of \( DMU_0 \).
- \( \pi_0 (x_0, u_0; p, r) \) is observed profit of \( DMU_0 \).
- \( p \) is vector of input prices.
- \( r \) is vector of output prices.
- \( f_x, f_u \) are directional vector of inputs and outputs respectively.
- \( pf_x + rf_u \) is a deflator that transforms profit efficiency difference into a number free from units of measurement.

Nerlove Profit Efficiency Indicator Vs Overall Directional Technical Efficiency**:

For a given set of input prices \( p \) and output prices \( r \), we always have,

\[ \pi (p, r) \geq ru - px \quad \text{for all } x, \]

\[ \pi (p, r) \geq r (u + \tilde{D}_f f_u) - p (x - \tilde{D}_f f_x) \]

\[ = ru - px + r \tilde{D}_f f_u + p \tilde{D}_f f_x \]

\[ = ru - px + \tilde{D}_f (rf_u + pf_x)(rf_u + pf_x) \]

\[ \frac{\pi (p, r) - (ru - px)}{rf_u + pf_x} \geq \tilde{D}_f (x_0, u_0; f_x, f_u / CRS) \quad (3.8.5) \]


Properties of Directional Distance Function:

(i) \( \tilde{D}_G (x - \alpha g_x, u + \alpha g_u) = \tilde{D}_G (x, u; g_x, g_u) - \alpha, \alpha \in \mathbb{R} \)

(ii) \( \tilde{D}_o (x, u; \lambda f_x, \lambda f_u) = \lambda \tilde{D}_o (x, u; f_x, f_u), \lambda > 0 \).

(iii) \( x' \geq x \Rightarrow \tilde{D}_o (x', u; f_x, f_u) \geq \tilde{D}_o (x, u; f_x, f_u) \).

(iv) \( u' \geq u \Rightarrow \tilde{D}_o (x, u'; f_x, f_u) \leq \tilde{D}_o (x, u; f_x, f_u) \).

(v) If the production technology admits constant returns to scale, we have

\[
\tilde{D}_o (x, u; f_x, f_u) = \tilde{D}_o (x, u; f_x, f_u) = 0
\]

A decision making unit is said to be direction efficient, if

\[
\tilde{D}_o (x, u; f_x, f_u) = 0
\]

DIRECTIONAL ALLOCATIVE EFFICIENCY:

We have seen that,

\[
\left[ \frac{\pi(p, r) - (ru - px)}{rf_x + pf_u} \right] \geq \tilde{D}_t (x_0, u_0; f_x, f_u / CRS) \quad \text{(3.8.6)}
\]

The difference between the left hand side expression and the right hand side expression measures Directional Allocative Efficiency.

\[
AE = \left[ \frac{\pi(p, r) - (ru - px)}{rf_x + pf_u} \right] - \tilde{D}_t (x_0, u_0; f_x, f_u / CRS) \quad \text{(3.8.7)}
\]

It can be shown that under certain conditions, the directional distance function reduces to the radial input and output technical efficiency measures.

We have,

\[
\tilde{D}_G (x, u; g_x, g_u) = \text{Max} \{ \theta : (x - \theta f_x, y + \theta f_u) \in G \},
\]

Let \( g_x = 0 \), then we have

\[
\tilde{D}_o (x, u; f_u) = \text{Max} \{ \theta : (x, u + \theta f_u) \in G \}
\]

\[
= \tilde{D}_o (x, u; f_u)
\]

We choose the directional output vector \( f_u \) such that it is equal to the observed output vector \( u \),

\[56\]
\[ f^* = u \]
\[ \tilde{D}_o(x, u; f^*) = \text{Max}\{ \theta : (x, u(1 + \theta)) \in G \} \]
\[ = \text{Max}\{ (1 + \theta) : x, u(1 + \theta) \in G \} - 1 \]
\[ = \text{Max}\{ \delta : (x, \delta u) \in G \} - 1 \]
\[ = [D_o(x, u) \in G]^{-1} - 1 \]

where \( D_o \) is radial output distance function.

Similarly, it can be shown that
\[ \tilde{D}_i(x, u; x) = 1 - \frac{1}{D_i(u, x)} \]

where \( D_i(u, x) \) is the radial input distance function defined on the production possibility set \( G \).

3.9 DEA – CHOICE OF INPUTS AND OUTPUTS:

The first step in Data Envelopment Analysis is modeling production units participating in production. By choosing inputs and outputs suitably we wish to capture the characteristic features of a production unit, there by arriving at a model. However, choice of too many inputs and outputs to model production units leads to loss of discriminatory power of DEA, in the sense that the proportion of 100 percent efficient decision making units will increase. Consequently, efficient DMUs can not be ranked.

Consider the production possibility set \( G \), defined such that,
\[ G = \{(x, u) : x \text{ produces } u\} \]

Let \( L(u_o) \) be the input level set of \( G \).
\[ L(u_o) = \{x : (x, u_o) \in G\} \]
Let us suppose that, \( x_0 \in R_+^n \), \( u_0 \in R_+^s \) be the input and output vectors of \( DMU_0 \). We write,

\[
x_0 = x_0^{\text{m}}, \quad u_0 = u_0^{\text{i}}
\]

The super scripts stand for the number of elements of inputs and outputs. Suppose that an additional input is introduced into production so that the output vector can now be denoted by \( x_0^{\text{m+1}} \).

We have two optimization problems:

1. \( \theta^{m+1} = \text{Min } \theta \)

\[
\sum_{j=1}^{\tilde{m}} \lambda_j x_{ij} \leq \theta x_{in}, \quad i = 1, 2, \ldots, m + 1.
\]

\[
\sum_{j=1}^{\tilde{m}} \lambda_j u_{ij} \geq u_{in}, \quad r = 1, 2, \ldots, s. \quad \text{(3.9.1)}
\]

2. \( \theta^m = \text{Min } \theta \)

\[
\sum_{j=1}^{\tilde{m}} \lambda_j x_{ij} \leq \theta x_{in}, \quad i = 1, 2, \ldots, m.
\]

\[
\sum_{j=1}^{\tilde{m}} \lambda_j u_{ij} \geq u_{in}, \quad r = 1, 2, \ldots, s. \quad \text{(3.9.2)}
\]

Since the constraint set of first problem is more restricted than that of second problem, we have

\[
\theta^{m+1} \geq \theta^m
\]

\[
\theta^m = 1 \Rightarrow \theta^{m+1} = 1 \quad \text{(3.9.3)}
\]

Thus, every efficient DMU of the former problem is efficient of the later problem.
Under output orientation we have the following optimization problems:

\[ \delta^m = \text{Max} \delta \]

such that

\[ \sum_{j=1}^{\lambda} \lambda_j x_{ij} \leq x_{io}, \quad i = 1, 2, \ldots, m. \]

\[ \sum_{j=1}^{\lambda} \lambda_j u_j \geq u_m, \quad r = 1, 2, \ldots, s. \]  \hspace{1cm} (3.9.4)

\[ \delta^{m+1} = \text{Max} \delta \]

such that

\[ \sum_{j=1}^{\lambda} \lambda_j x_{ij} \leq x_{io}, \quad i = 1, 2, \ldots, m. \]

\[ \sum_{j=1}^{\lambda} \lambda_j u_j \geq u_m, \quad r = 1, 2, \ldots, s + 1. \]  \hspace{1cm} (3.9.5)

The later problem is more restricted than the former problem so that we have,

\[ 1 \leq \delta^{m+1} \leq \delta^m \]

\[ \delta^m = 1 \Rightarrow \delta^{m+1} = 1 \]

"If an additional input and / or output are introduced into DEA, it looses discriminatory power".

To improve the discriminatory power of DEA, one can increase sample size by incorporating data on some more DMUs, or past data on the same DMU if time series data are available.

We consider the following input efficiency problems:

\[ \text{Min} \theta = \theta(n) = \text{Max} \sum_{r=1}^{\lambda} \mu_r o \]

subject to

\[ \sum_{i=1}^{\lambda} u_i x_{io} = 1 \]

\[ \sum_{r=1}^{\lambda} \mu_r - \sum_{i=1}^{\lambda} u_i x_{iq} \leq 0 \quad j = 1, 2, \ldots, n. \]  \hspace{1cm} (3.9.6)
\[ \text{Min } \theta = \theta(n + 1) = \text{Max } \sum_{i=1}^{n} \mu_i u_i \]

subject to

\[ \sum_{i=1}^{n} u_i x_{i0} = 1 \]

\[ \sum_{i=1}^{n} \mu_i u_i - \sum_{i=1}^{n} u_i x_{ij} \leq 0 \quad j = 1, 2, \ldots, n, n + 1. \]

The later problem is more constrained than the former problem.

\[ \theta(n + 1) \leq \theta(n) \quad \text{(3.9.7)} \]

\[ \Rightarrow \text{ The discriminatory power of the later problem is more than that of the previous problem.} \]

In view of the above discussion it is understood that the researcher should exercise parsimony while DEA inputs and outputs are choosen. The selection shall be based on the objectives of the research.

3.10 INPUT COST EFFICIENCY

Farrell (1957) proposed a graphical method to evaluate input cost efficiency. Failure to operate on the isoquant of an input level set implies the presence of input technical in efficiency. If a production unit operates on isoquant at that point the input cost may not be minimum. Thus, we seek for an input combination on the production isoquant to find cost minimum combination of inputs. On the isoquant departure from minimum cost point leads to allocative inefficiently.

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In the figure above, $x_1$ and $x_2$ are inputs measured along horizontal and vertical axes respectively, $PP^1$ is the input cost line.

The producer who operates at $P$ is inefficient.

Cost at $Q$ is not minimal, factor minimal cost occurs at $S$. Cost at $S$ is same as cost at $R$. Farrell’s cost efficiency is measured by the ratio,

$$\frac{OQ}{OP} = \frac{Q(u_0,p)}{\sum_{i=1}^{n} p_i x_{io}}$$

The numerator is factor minimal cost. Fare et. al., *(1985)* extended CCR (1978) problem to find factor minimal cost.

$$Q(u_0, p) = \min\{px : x \in L(u_0)\}$$

The following are some of the properties of factor minimal cost function.

• $Q(u_0, p)$ is linear homogeneous in $p$.

\[
Q(u_0, \lambda p) = \min_x \{ \lambda px : x \in L(u_0) \} = \lambda \min_x \{ px : x \in L(u_0) \} = \lambda Q(u_0, p)
\]

• $Q(u_0, p)$ is non-decreasing function of $u$.

Let $u_1 \leq u_2$

$\Rightarrow L(u_2) \subseteq L(u_1)$

$\min \{ px : x \in L(u_1) \} \geq \min \{ px : x \in L(u_2) \}$

$Q(u_2, p) \geq Q(u_1, p)$

• $Q(u, p)$ is non-decreasing function of $u$.

$p_1 \leq p_2$

$p_1 x \leq p_2 x, \ \forall x \in L(u_0)$

$\min_x \{ p_1 x : x \in L(u_0) \} \leq \min_x \{ p_2 x : x \in L(u_0) \}$

$Q(u_0, p_1) \leq Q(u_0, p_2)$

$Q(u_0, p) = \min_{(x, \lambda)} px$

such that

\[
\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_i, i = 1, 2, \ldots, m
\]

\[
\sum_{j=1}^{n} \lambda_j u_{jr} \geq u_{ro}, r = 1, 2, \ldots, s
\]

$\lambda_j \geq 0$
Farrell's cost efficiency can be multiplicatively decomposed as follows:

\[
\frac{O(u_0, p)}{px_0} = \left[ \frac{OT}{OP} \right] = \left[ \frac{OT}{OR} \right] \left[ \frac{OR}{OQ} \right] \left[ \frac{OQ}{OP} \right]
\]

- \( \frac{OT}{OP} \) measures allocative efficiency.
- \( \frac{OR}{OQ} \) measures scale efficiency.
- \( \frac{OQ}{OP} \) measures pure technical efficiency.

3.11 MAXIMAL REVENUE:

For given input vector potential revenue can be obtained solving the following linear programming problem:
\[
\text{Max} \sum_{i=1}^{\hat{s}} q_i u_i,
\]

subject to \( \sum_{j=1}^{\hat{t}} \lambda_j x_{ij} \leq x_{i0}, \quad i = 1, 2, \ldots, m \).
\[
\sum_{j=1}^{\hat{t}} \lambda_j \mu_j \geq u, \quad \lambda_j \geq 0.
\]

Equivalently, let
\[
R(x_0, q) = \text{max} \{ q u : u \in p(x_0) \}
\]

Properties:

- The maximal revenue function is linear homogeneous in input prices.

\[
R(x_0, \delta q) = \text{Max} \{ (\delta q) u : u \in p(x_0) \}
\]
\[
= \delta \text{Max} \{ q u : u \in p(x_0) \}
\]
\[
= \delta R(x_0, q)
\]

- The maximal revenue function is non-decreasing in \( x \):

Let \( x_j \leq x_i \Rightarrow p(x_j) \subseteq p(x_i) \)
\[
\text{Max} \{ q u : u \in p(x_j) \} \leq \text{Max} \{ q u : u \in p(x_i) \}
\]
\[
R(x_j, q) \leq R(x_i, q)
\]

Maximal revenue function is non-decreasing in input vector.

- \( R(\lambda x_0, q_0) = \lambda R(x_0, q) \), only if returns to scale are constant.

\[
R(\lambda x_0, q_0) = \text{Max} \{ r u : u \in p(\lambda x_0) \}
\]
\[
= \text{Max} \{ r u : u \in \lambda p(x_0) \}
\]
\[
= \lambda \text{Max} \{ r \lambda^{-1} u : \lambda^{-1} u \in p(x_0) \}
\]
\[
= \lambda \text{Max} \{ r \hat{u} : \hat{u} \in p(x_0) \}
\]
\[
= \lambda R(x_0, q)
\]
3.12 SOME EMPIRICAL STUDIES OF COMMERCIAL BANKS:

Measurement of technical change can be extended to input approach where input distance functions are used in the place of output distance functions.

For a very long time Indian commercial banks were regulated by the Reserve Bank of India and the Central Government. Kumbhakar et. al (2008) analyzed the relationship between deregulation and total factor productivity growth. His study was based on generalized shadow cost function approach. Total factor productivity growth is decomposed into a technological change, a scale and a miscellaneous component. The study covered both pre and post deregulation considered the public and private sector banks.

Wheelock and Wilson (1999) used the Malmquist decomposition to examine US banking during the period from 1984 to 1993. During this period there was a considerable technological advancement in US banking. The authors used Malmquist indexes to measure total factor productivity and the consequent technical efficiency changes.

Tapas and Sinha (2004) implemented Malmquist indices to examine the technical efficiency of Indian Commercial books. The study covered the period 1998-99 to 2001-02. Mukherjee et.al (2001) assessed the total factor productivity of US commercial banks, the sample period being 1984-1990. The basic tool to measure technical change is Malmquist index. During this period overall productivity declined during the sample period. Stum and Williams (2004) examined the total factor productivity of Australian Banks. Two methods are implemented on DEA Malmquist approach and the other Stochastic Frontier Approach. The sample period was 1988-2001. Important findings were that foreign banks were found more efficient than domestic banks. Bank efficiency increased in post regulation period. Competition resulted from diversity in bank types was found important to prompt efficiency improvements. Mohan and Ray (2004) used the Tomquist Index numbers (Discrete version of Divisia Index) and DEA – Malmquist Total Factor productivity index to find total factor productivity growth of public, private and foreign sector banks with bank data covering the period 1992-2000. The finding was no significant difference in TFP growth for public private and foreign banks of India.
Galagedera and Edirisuriya (2005) assessed total factor productivity change of Indian commercial banks covering the period 1995-2002. The basic tool of the study was Malmquist TFP index. The findings were for all the banks put together no significant growth of TFP. Public sector banks were different from Private Banks in terms of TFP growth.

There are several studies world wide which assessed and analyzed the productive efficiency of commercial banks. Saha and Ravisankar (2000) measured technical efficiency of 25 Public sector banks, the tools being DEA. The sample period was from 1991-92 to 1994-95. It has been found that the Public sector banks of India together experienced an enhancement of efficiency during the sample period. Sathe (2003) used DEA to measure technical efficiency of public, private and foreign sector banks. The study revealed that the mean efficiency score of Indian banks compared well with the world mean efficiency score. The efficiency of Private sector banks as a group was found paradoxically lower than that of Public sector banks and foreign sector banks of India operating on Indian soil. The study recommended that the prevailing policy of reducing Non-Performing Assets, rationalization of staff and number of branches if continued might result in efficiency gains and promote Indian commercial banks internationals competitive. Chen Taipai (2002) used chance constrained DEA and Stochastic Frontier Analysis (SFA) to measure technical efficiency of 39 banks in Taiwan. A translog production frontier was estimated by SFA.

Kumar and Varma (2003) used DEA CCR (1978) model to study the technical efficiency variation among Indian Commercial banks. The conclusions were the public sector banks had the scope of producing 1.21 times as much output from the same inputs. Larger banks are more efficient than the small or medium banks. Bhattacharya et.al (1997) used DEA model to conclude that PSB of India had the highest efficiency followed by foreign and private sector banks Shanmugam et. al., (2001) used DEA and SFA approaches in a study of Indian Commercial banks. Overall mean technical efficiency varied between 52 and 58 percent. They found that deposit was the most dominating output of the banks. Mukherjee et. al., (2002) used DEA to study the performance of Indian Commercial banks. The sampling period of the study was 1996-99. The findings were that PSBs were more efficient than private
and foreign sector banks, PSBs performance was found better not only under self appraisal but also in peer appraisal.

Sharman and Gold (1985) studied the performance of 14 US Commercial banks using DEA. Partan (1987) also used DEA to study the performance of 35 Canadian banks. Oral and Yolalan (1990) studied the efficiency variation among 20 commercial banks of Turkey. They concluded that service efficient branches were most profitable units. Jahanshahloo et. al., (2003) examined efficiency of Iranian Commercial Banks using different efficiency methods and concluded that efficiency variations are method dependent. There is no consensus in the selection of DEA inputs and outputs. Deposits, Borrowings, Staff, Spread, Commission, Exchange, Brokerage; Capital, Lovable funds, Interest income in Space, Number of transactions; Rent, Number of Branches, Customers response; Number of accounts, Credit applications; Computer terminals; ATMs, Interest Costs, Non-interest costs, Branch location, Image of Bank, Product range, Network size; Investments, operating profit were the DEA inputs and outputs employed in the study of efficiency differences among commercial banks of different countries.