CHAPTER – II

Review of Literature

REVIEW OF LITERATURE

2.1 Productive efficiency measurement dates back to Farrell (1957)* who outlined a graphic procedure based on unit output isoquant. Two inputs x_1 and x_2 were combined to produce one output.

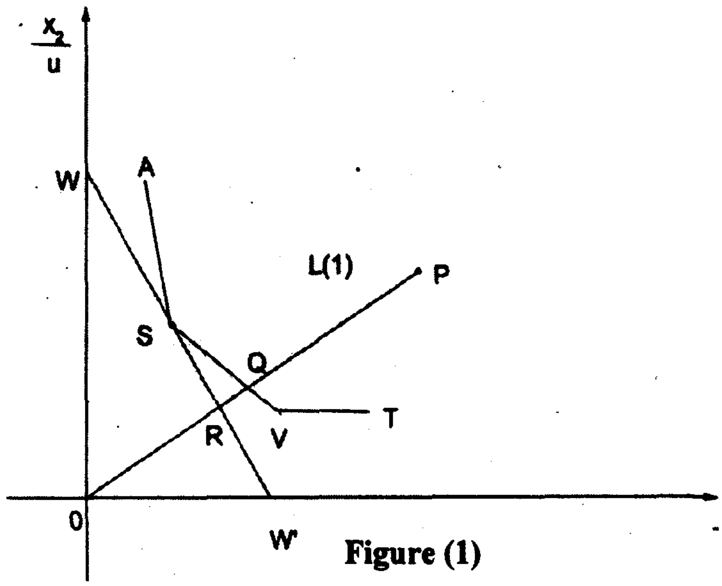

Figure (1)

$\frac{x_1}{u}$ and $\frac{x_2}{u}$ are input requirements to produce unit output measured along horizontal and vertical axes. L(1) is the production possibility set that consists of all input pairs required to produce one unit of output. The line segments AS, SV and VT constitute input isoquant which serves as reference set for inefficient decision making units.

The producer who operates at P is inefficient. By radially retarding inputs at P and the producer can attain efficiency when he reaches the point Q on the isoquant of L(1).

$$\text{Input technical efficiency : ITE} = \frac{0Q}{0P}$$

WW' is the price line that is tangent to the isoquant at S. The inputs at S minimize total cost of production. Cost at S is cost at R so that input cost efficiency can be estimated by the ratio $\frac{0R}{0P}$.

* Farrell (1957), Op.cit.

$$\text{Input cost efficiency} = \frac{0R}{0P}.$$

This ratio can be decomposed into two multiplicative factors, vize technical and allocative efficiencies.

$$\frac{0R}{0P} = \frac{0R}{0Q} \times \frac{0Q}{0P}$$

2.2 At any point of time several production units combining similar inputs and producing similar outputs, compete with each other in production. Their input vectors and output vectors generate a production possibility set, expressed as follows:

$$P = \{ (x, u) : x \text{ produces } u \}$$

The production possibility set determined by the CCR model is found based on certain axioms.

Let (x_j, u_j), $j = 1,2,\ldots.n$; be the inputs combined and outputs produced by j^{th} production unit.

$$x_j \in R_m^+$$

$$u_j \in R_s^+$$

A.1 $(x_j, u_j) \in P \Rightarrow \left(\sum_{j=1}^{n} \lambda_j x_j, \sum_{j=1}^{n} \lambda_j u_j \right) \in P$

where $\lambda_j \geq 0, \sum_{j=1}^{n} \lambda_j = 1$

A.2 $(x_j, u_j) \in P \Rightarrow (\Sigma \lambda_j x_j, \Sigma \lambda_j u_j) \in P \Rightarrow \lambda (\Sigma \lambda_j x_j, \Sigma \lambda_j u_j) \in P$

$\Rightarrow \lambda (\Sigma \delta_j x_j, \Sigma \delta_j u_j) \in P$

A.3 $(\bar{x}, \bar{u}) \in P$

$x \geq \bar{x}, \quad u \leq \bar{u}$

$\Rightarrow (x,u) \in P$

Since we can express,

$\bar{x} = \Sigma\delta_j x_j$

$\bar{u} = \Sigma\delta_j u_j$

we have, $\Sigma\delta_j x_j \le x$, $\Sigma\delta_j u_j \ge u$

Given (x, u) we can find δ_j such that the above inequalities are satisfied.

A.1, A.2 and A.3 are called the convexity, ray unboundedness and inefficiency axioms.

A.4 The production possibility set is the intersection of all such sets which contain (x_j, u_j), j = 1,2,....n.

The CCR production possibility set can be expressed as,

$$P = \left\{(x,u) \; : \; \sum_{j=1}^{n}\lambda_j \; x_j \; \le \; x, \; \sum_{j=1}^{n}\lambda_j \; u_j \; \ge \; u\right\}$$

Thus, given (x, u) we can find λ_j such that

$$\Sigma\lambda_j x_j \le x, \; \Sigma\lambda_j u_j \ge u, \; \lambda_j \ge 0$$

In CCR* frame work to compute input technical efficiency we solve the following linear programming problem:

Min λ

such that $\sum_{j=1}^{n}\lambda_j \; x_{ij} \le \lambda x_{i0}\,, \quad i=1,2,....m$ ---- (2.2.1)

$$\sum_{j=1}^{n}\lambda_j \; u_{rj} \ge u_{r0}\,, \quad r=1,2,....s$$

$$\lambda_j \ge 0$$

Let $\lambda(CCR) = Min \; \lambda$

$0 \le \lambda(CCR) \le 1$

* Charness et al. (1978), Op.cit.

The CCR (1978) problem was extended by Banker, Charness and Cooper (BCC, 1984)[*] to account for variable returns to scale. The formulation was as follows:

Min λ

$$\text{such that } \sum_{j=1} \lambda_j \, x_{ij} \le \lambda \, x_{i0} \qquad (2.2.2)$$

$$\sum_{j=1}^{n} \lambda_j \, u_{rj} \ge u_{r0}$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \ge 0$$

$$Min\ \lambda = \lambda(BCC)$$

Since the later problem is more constrained we have,

$$\lambda\,(BCC) \ge \lambda\,(CCR) \qquad \text{----} \qquad (2.2.3)$$

The ratio, $\dfrac{\lambda(CCR)}{\lambda(BCC)}$ measures input scale efficiency.

$$\underbrace{\lambda\,(CCR)}_{\substack{\text{Input over all} \\ \text{technical efficiency}}} = \underbrace{\left[\frac{\lambda\,(CCR)}{\lambda\,(BCC)}\right]}_{\text{Input scale efficiency}} \underbrace{\left[\lambda\,(BCC)\right]}_{\substack{\text{Input pure technical} \\ \text{efficiency}}} \qquad \text{----} \qquad (2.2.4)$$

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[*] Banker et al. (1984), Op.cit.

2.3 Fare et al. (1990)* extended the work of Bankar, Cooper and Charnes and formulated linear programming problems to estimate factor minimal cost. Their formulation is as follows:

$$Q(u,p) = Min\ px$$

such that $\sum_{j=1}^{n} \lambda_j x_{ij} \le x_i,\ i = 1,2,\ldots m$

$$\sum_{j=1}^{n} \lambda_j u_{rj} \ge u_{r0},\ j = 1, 2, \ldots n$$

$$\lambda_j \ge 0$$

The above problem fails to identify variable returns to scale.

Properties of Q (u, p)

P.1 $Q(u, \lambda p) = \lambda Q(u, p)$

The factor minimal cost is linear homogenous in input prices.

P.2 $p \ge p' \Rightarrow Q(u, p) \ge Q(u, p')$

P.3 $u_1 \ge u_2 \Rightarrow Q(u_1, p) \ge Q(u_2, p)$

Input cost efficiency : $\frac{Q(u,p)}{px_0}$

where x_0 is observed input vector.

2.4 Directional Distance Functions

Chamber et al. (1998)** proposed Directional Distance Function (DDF) to measure productive efficiency. This is not a radial measure like CCR (1978) and BCC (1984). The DDF is consistent with additive decomposition.

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* Fare et al. (1990), Op.cit.

** Chamber et al. (1998), Op.cit.

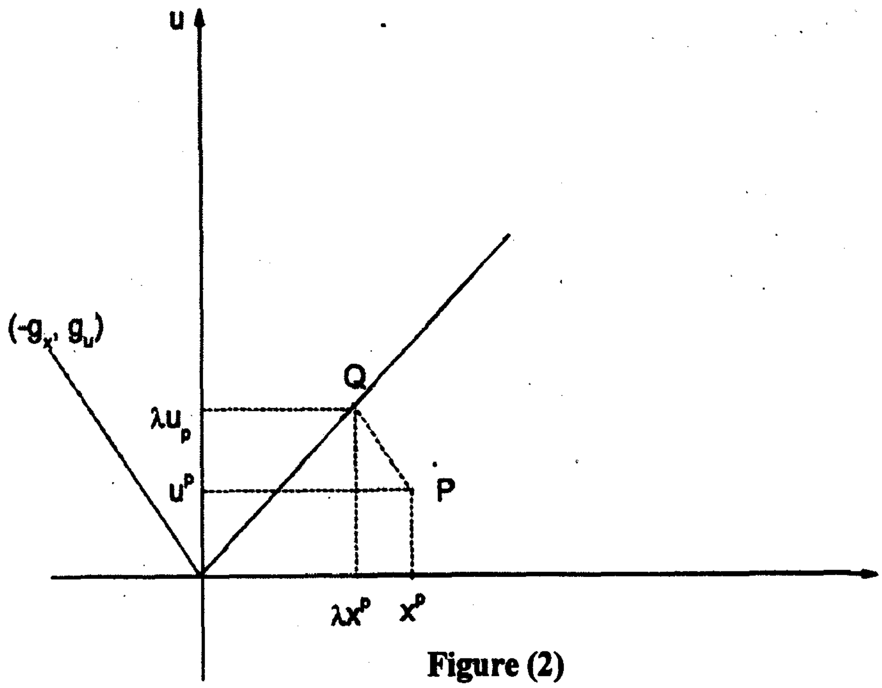

Figure (2)

In the above figure input is measured along horizontal axis and output along vertical axis. The straight line that possess through the origin is production frontier that admits constant returns to scale. The producer who operates at P is technically inefficient. With input x^P output u^P is produced. Input x^P is decreased and output u^P is increased simultaneously in the direction of the directional vector $(-g_x, g_u)$ until the production frontier is reached. The directional distance function reaches the point,

$$(x_0 - g_x \lambda,\ u_0 + g_u \lambda)$$

where $\lambda = \vec{D}(x_0, u_0; g_x, g_u)$

$\vec{D} = 0 \Rightarrow$ Directional productive efficiency

$\vec{D} > 0 \Rightarrow$ Directional Productive inefficiency

Under certain conditions we can compute radial efficiency measures out of the directional distance functions. Directional efficiency score is subjective to the choice of the directional vector $(-g_x, g_u)$.

2.5 Shephard's Duality Theorem:*

Input price space is dual to input quantity space. Both these spaces are connected by what is called the Shephard's lemma. This lemma also provides input demand equations and input shadow prices. Let Q (u, p) be the factor minimal cost function. Then Shephard's lemma is stated as follows:

$$\frac{\partial Q(u,p)}{\partial p_i} = x_i\ (u,p)$$

where p_i : unit price of i^{th} input

u : output vector

x_i (u,p) is input demand for i^{th} input

which is a function of output and inputs prices.

Given a parametric production function the Shephard's lemma helps to find the dual cost function which is also parametric.

2.6 Environmental Problems – Measurement of Efficiency

T. Subramanyam and C.S. Reddy** (2012), formulated a Data Envelopment Analysis model to measure risk efficiency in commercial banks. The study treated non-performing assets (NPA) as environmental variable. In this study overall cost efficiency was decomposed multiplicatively into price efficiency, cost based risk efficiency and Farrell's cost efficiency.

$$\underbrace{\frac{Q(x_0,p^{\min})}{px_0}}_{\text{Overall cost efficiency}} = \left[\underbrace{\frac{Q(u_0,p^{\min})}{Q(u_0,p_0)}}_{\text{Price efficiency}} \quad \underbrace{\frac{Q(u_0,p_0)}{Q(u_0,p,u_{b0})}}_{\text{Cost based risk efficiency}} \quad \underbrace{\frac{Q(u_0,p_0)}{px_0}}_{\text{Farrell's cost efficiency}}\right]$$

where, $Q(u_0,p^{\min}) = Min\ p^{\min}x$

* R.W. Shepard (1978), Op.cit.

** T.Subrahmanyam and C.S Reddy (2012), 'Cost Based Risk Efficiency', International Journal of Information, Business and Management, Vol.4, No.1, 90-104.

subject to $\sum_{j=1}^{n} \lambda_j \, x_{ij} \le x_i, \; i = 1,2,....m$

$$\sum_{j=1}^{n} \lambda_j \, u_{rj} \ge u_{r0}, \; r = 1,2,.....s.$$

where $p_i^{min} = \underset{j}{Min} \; p_{ij}$

$$Q(u_0, p_0, u_{b0}) = Min \; p_0 \, x$$

subject to $\sum_{j=1}^{n} \lambda_j \, x_{ij} \le x_i, \; i = 1,2,.....m$

$$\sum_{j=1}^{n} \lambda_j \, u_{rj} \ge u_{r0}, \; r = 1,2,.....s$$

$$\sum_{j=1}^{n} \lambda_j \, u_{bj} \ge u_{b0}$$

2.7 Cobb-Douglas Production Frontier*

To measure efficiency of production, the chief tool is production function. The earliest contribution to the theory of production was the Cobb-Douglas production frontier. This production function may be expressed as,

$$u = A \prod_{i=1}^{n} x_i^{\alpha_i} \qquad \text{---- (2.7.1)}$$

where u : output

x_i : i^{th} input

* Cobb, C.W., Douglas, P.H. (1928), 'A Theory of Production', American Economic Review, 10.

P.1 **Marginal product if i^th^ input:**

$$\frac{\partial u}{\partial x_i} = \frac{\alpha_i u}{x_i}, \quad i = 1,2,....n$$

P.2 Elasticity of output with respect to i^th^ input:

$$\frac{\partial \ln u}{\partial \ln x_i} = \alpha_i, \quad i = 1,2,....n$$

P.3 Elasticity of substitution :

$$\sigma_{ij} = \frac{d \ln\left(\frac{x_i}{x_j}\right)}{d \ln\left(\frac{\partial u/\partial x_j}{\partial u/\partial x_i}\right)} = 1 \ \forall \ (i,j) \qquad \text{----} \qquad (2.7.2)$$

P.4 The producer attains equilibrium if,

$$\left(\frac{\partial u/\partial x_j}{\partial u/\partial x_i}\right) = \frac{p_j}{p_i}$$

P.5 The equilibrium condition can be induced into σ_{ij} as follows:

$$\sigma_{ij} = \frac{d \ln\left(\frac{x_i}{x_j}\right)}{d \ln\left(\frac{p_j}{p_i}\right)}, \quad \forall \ (i,j)$$

P.6 The expansion path of the producer whose production technology is governed by a Cobb-Douglas frontier, is a straight line that passes through the origin.

P.7 For Cobb-Douglas production frontier, for the equilibrium of the producer we require,

$$\frac{\alpha_i u}{x_i} \Big/ \frac{\alpha_j u}{x_j} = \frac{p_i}{p_j}$$

$$\left(\frac{\alpha_i}{x_i}\right)\left(\frac{x_j}{\alpha_j}\right) = \frac{p_i}{p_j}$$

$$\frac{x_j}{x_i} = \left(\frac{\alpha_j}{\alpha_i}\right)\left(\frac{p_i}{p_j}\right)$$

$$x_j = a(p)\, x_i \qquad \text{---- (2.7.3)}$$

where $a(p) = \left(\frac{\alpha_j}{\alpha_i}\right)\left(\frac{p_i}{p_j}\right)$

In empirical analysis of economic data a very widely used production frontier is the Cobb-Douglas production frontier.

P.8 $u = f(x)$ is non-decreasing in inputs.

$$\frac{\partial u}{\partial x_i} = \frac{\alpha_i\, u}{x_i} > 0$$

P.9 It is concave in inputs.

$$\frac{\partial^2 u}{\partial x_i} = \alpha_i\,(\alpha_i - 1)\,\frac{u}{x^2}$$

since $0 < \alpha_i < 1, \ \forall\, i$

2.8 Constant Elasticity of Substitution Production Frontier*:

An improvement over the Cobb-Douglas production function is the Constant Elasticity of Substitution (CES) production function. As $\sigma \to 1$ the CES production function converges to the Cobb-Douglas production function.

$$u = A\left(\sum_{i=1}^{n} \eta_i\, x_i^{-\rho}\right)^{-v/\rho}$$

$$\eta_i \geq 0$$

$$\sum_{i=1}^{n} \eta_i = 1$$

$$\text{Elasticity of substitution}: \sigma = \frac{1}{1+\rho} \geq 0$$

Returns to scale parameter : ν

* Arrow et al. (1961), 'Capital-Labour Substitution and Economic Efficiency', Review of Economics and Statistics, 63.

A.1 Marginal product of i^{th} input:

$$\frac{\partial u}{\partial x_i} = \frac{\upsilon\, \eta_i\, x_i^{-\frac{1}{\sigma}}\, u}{\sum_{i=1}^{n} \eta_i\, x_i^{-\rho}}, \quad i = 1,2,\ldots n$$

A.2 The producer attains equilibrium when ever,

$$\left(\frac{\partial u/\partial x_i}{\partial u/\partial x_j}\right) = \frac{\upsilon\, \eta_i\, x_i^{-\frac{1}{\sigma}}\, u}{\sum^{n} \eta_i\, x_i^{-\rho}} \quad \frac{\sum_{i=1}^{n} \eta_i\, x_i^{-\rho}}{\upsilon\, \eta_j\, x_j^{-\frac{1}{\sigma}}\, u}$$

$$\frac{\eta_i}{\eta_j}\left(\frac{x_i}{x_j}\right)^{\frac{1}{\sigma}} = \frac{p_i}{p_j}$$

$$= \frac{\eta_i}{\eta_j}\left(\frac{x_i}{x_j}\right)^{\sigma} = \frac{p_i}{p_j} \qquad \text{---- (2.8.1)}$$

A.3 Expansion path:

$$\left(\frac{x_i}{x_j}\right)^{\frac{1}{\sigma}} = \left(\frac{\eta_j}{\eta_i}\right)\left(\frac{p_i}{p_j}\right) \qquad \text{---- (2.8.2)}$$

$$\frac{x_j}{x_i} = \left(\frac{\eta_j}{\eta_i}\,\frac{p_i}{p_j}\right)^{\sigma}$$

$$x_j = a(\eta, p, \sigma)\, x_i$$

where $a(\eta, p, \sigma) = \left(\frac{\eta_j}{\eta_i}\,\frac{p_i}{p_i}\right)^{\sigma}$

A.4 Elasticity of substitution:

$$\ln\left(\frac{\eta_i}{\eta_j}\right) + \frac{1}{\sigma}\ln\left(\frac{x_j}{x_i}\right) = \ln\left(\frac{p_i}{p_j}\right)$$

$$\frac{1}{\sigma}\, d\ln\left(\frac{x_j}{x_i}\right) = d\ln\left(\frac{p_i}{p_j}\right)$$

$$\frac{d\ln\left(\frac{x_j}{x_i}\right)}{d\ln\left(\frac{p_i}{p_j}\right)} = \sigma, \ \forall\ (i,j) \qquad \text{---- (2.8.3)}$$

A.5 Elasticity of output with respect to ith input x_i:

$$\frac{\partial \ln u}{\partial \ln x_i} = \frac{\partial u}{\partial x_i}\,\frac{x_i}{u} = \frac{\upsilon\, \eta_i\, x_i^{\frac{1}{\sigma}}\, u}{\sum_{i=1}^{n} \eta_i\, x_i^{-\rho}}\;\frac{x_i}{u}$$

$$= \frac{\upsilon\, \eta_i\, x_i^{\frac{1}{\sigma}+1}}{\sum_{i=1}^{n} \eta_i\, x_i^{-\rho}} \qquad \text{----} \qquad (2.8.4)$$

The CES production function does not identify complementarity of inputs. Two inputs are said to be substitutes, if

$$\sigma_{ij} > 0$$

They are said to be complements if

$$\sigma_{ij} < 0$$

2.9 Translog Production Frontier*:

A flexible production function is the translog production function, which is second order approximation of Taylor's series of an arbitrary production function. Let us consider an arbitrary production function with a scalar output and two inputs.

$$u = f(x_1, x_2)$$

Taylor's series expansion of the above production function may be expressed as,

$$f(x_1, x_2) = f(0,0) + \frac{x_1}{1!}\,\frac{\partial f}{\partial x_1} + \frac{x_2}{1!}\,\frac{\partial f}{\partial x_2}$$

$$+ \frac{x_1^2}{2!}\,\frac{\partial^2 f}{\partial x_1^2} + \frac{x_2^2}{2!}\,\frac{\partial^2 f}{\partial x_2^2} + 2\,\frac{x_1 \cdot x_2}{2!}\,\frac{\partial^2 f}{\partial x_1\, \partial x_2} + R$$

* Jorgenson, D.W. and Grilliches, Z. (1971), 'Divisia Index Numbers and Productivity Measurement', Review of Income and Wealth, 2, 227-29.

where R is the remainder of Taylor's series expansion. Expansion is about (0,0).

$$f(x_1,x_2) = f(0,0) + \frac{\partial f}{\partial x_1}(0,0)\, x_1 + \frac{\partial f}{\partial x_2}(0,0)\, x_2$$

$$+ \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2}(0,0)\, x_1^2 + \frac{1}{2}\frac{\partial^2 f}{\partial x_2^2}(0,0)\, x_2^2 + \frac{2x_1\, x_2}{2}\frac{\partial^2 f}{\partial x_1\, \partial x_2}(0,0)$$

The partial derivatives evaluated at (0,0) are unknowns. We take them as unkown parameters to be estimated.

$$f(x_1,x_2) = \alpha_0 + \alpha_1\, x_1 + \alpha_2\, x_2 + \frac{1}{2}\alpha_n\, (\ln x_1)^2 + \frac{1}{2}\alpha_{22} \ln\, (\ln x_2)^2$$

$$+ \alpha_{12} \ln x_1 \ln x_2$$

We have two partial derivatives.

$$\alpha_{12} = \frac{\partial^2 f}{\partial x_1\, x_2}(0,0), \quad \alpha_{21} = \frac{\partial f}{\partial x_2\, \partial x_1}(0,0)$$

Since both partial derivatives or equal, we have, $\alpha_{12} = \alpha_{21}$. This is called the property of symmetry.

The quadratic production function may be expressed as,

$$u = f(x_1,x_2) = \alpha_0 + \sum_{i=1}^{2}\alpha_i\, x_i + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\alpha_{ij}\, x_i\, x_j$$

If $\bar{u} = \ln u$

$\bar{x}_i = \ln x_i$, i = 1,2,....n

We get the translog production frontier as follows:

$$\ln u = \alpha_0 + \sum_{i=1}^{n}\alpha_i \ln x_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij} \ln x_i \ln x_j$$

STRUCTURAL PROPERTIES

P.1 If the production function is linear homogeneous it admits constant returns to scale.

$$f(\lambda x) = \lambda f(x)$$

$$\alpha_0 + \sum_{i=1}^{n} \alpha_i \ln (\lambda x_i) + \frac{1}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} \ln (\lambda x_i) \ln (\lambda x_j)$$

$$\alpha_0 + \sum_{i=1}^{n} \alpha_i \ln x_i + \frac{1}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} \ln x_i \ln x_j + \sum \alpha_i (\ln \lambda) +$$

$$\frac{1}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} \ln \lambda \ln x_i + \frac{1}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} \ln \lambda \ln x_j + \frac{1}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} (\ln \lambda)^2$$

Homogeneity requires that,

$$\sum \alpha_i = 1$$

$$\sum_i \alpha_{ij} = 0$$

$$\sum_j \alpha_{ij} = 0$$

$$\sum_i \sum_j \alpha_{ij} = 0$$

P.2 Symmetry requires that, $\alpha_{ij} = \alpha_{ji}$

P.3 The translog production function is non-decreasing in the components of x.

$$\frac{\partial \ln u}{\partial \ln x_i} > 0$$

i = 1,2,.....n

$$\frac{\partial \ln u}{\partial \ln x_i} = \alpha_i + \sum_{j=1}^{n} \alpha_{ij} \ln x_j > 0$$

P.4 It is a concave function in the components of x.

$$\frac{\partial^2 \ln u}{\partial \ln x_i^2} = \alpha_{ii} < 0$$

The translog production function may be estimated imposing the above restrictions on the parameters.

To estimate productive efficiency factor minimal cost function was also used widely. There is duality between the input space and input price space. The celebrated Shephard's lemma connects the input space with input price space. Minimization of production cost yields input demand equations.

2.9 INPUT DEMAND EQUATIONS – COBB-DOUGLAS PRODUCTION FUNCTION

A producer who minimizes cost of production attains equilibrium by equating the ratio of marginal products with the ratio of input prices.

$$\frac{\partial u}{\partial x_i} = \frac{\alpha_i u}{x_i}$$

$$\frac{\partial u}{\partial x_j} = \frac{\alpha_j u}{x_j}$$

$$\frac{\partial u}{\partial x_i} \Big/ \frac{\partial u}{\partial x_j} = \frac{\alpha_i u / x_i}{\alpha_j u / x_j} = \frac{p_i}{p_j}$$

$$\frac{\alpha_i}{\alpha_j} \frac{x_j}{x_i} = \frac{p_i}{p_j}$$

$$x_i = \frac{\alpha_i}{\alpha_j} \frac{p_j}{p_i} x_j \qquad (2.9.1)$$

$$x_i = (\alpha_i / \alpha_j) \, p_i^{-1} \, p_j x_j$$

All inputs governed by a Cobb-Douglas frontier are substitutes. As such x_i and x_j are substitutes.

$$\frac{\partial x_i}{\partial p_i} = \left(\frac{\alpha_i}{\alpha_j}\right) p_i^{-2} \cdot p_j \, x_j < 0 \qquad \text{---- } (2.9.2)$$

As price of i^{th} input decreases, demand for i^{th} input ir ses.

$$\frac{\partial x_i}{\partial p_j} = \left(\frac{\alpha_i}{\alpha_j}\right) p_i^{-1} x_j > 0 \qquad \text{---- (2.9.3)}$$

As price of j^{th} input decreases demand for i^{th} input increases since inputs x_i and x_j are substitutes.

2.10 Input Demand Equations – CES Production function

For a cost minimizing producer equilibrium requires that,

$$\left(\frac{\eta_i}{\eta_j}\right)\left(x_j\right) = \frac{p_i}{p_j} \qquad \text{-- (2.10.1)}$$

$$\left(\frac{\eta_i}{\eta_j}\right)\left(\frac{x_j}{x_i}\right)^{\sigma} = \frac{p_i}{p_j}$$

$$\frac{x_j}{} - \left(\frac{p_i}{p_j}\right)\left(\frac{\eta_i}{\eta_j}\right) \qquad \text{-- (2.10.2)}$$

$$x_j = B(\eta)\ p_i^{\sigma}\ p_j^{-\sigma}\ x_i$$

where $B(\eta) = \left(\frac{\eta_i}{\eta_j}\right)$ (2.10.3)

$\sigma \geq 0$

If $\sigma = 0$, x_i can not be substituted for x_j

$x_j = x_i$

If $\sigma > 0$, x_i and x_j are substitutable

$$\frac{\partial x_j}{\partial p_j} = -\sigma\ B(\eta)\ p_i^{\sigma}\ p_j^{-\sigma-1} < 0 \qquad (2.10.4)$$

$$\frac{\partial x_j}{\partial p_i} = \sigma\ B(\eta)\ p_i^{\sigma-1}\ p_j^{-\sigma} > 0 \qquad (2.10.5)$$

2.11 Input Demand equations – Translog Production function

$$\frac{\partial \ln u}{\partial \ln x_i} = \alpha_i + \sum_{i=1}^{n} \alpha_{ij} \ln x_j \tag{2.11.1}$$

$$\frac{\partial u}{\partial x_i}\frac{x_i}{u} = \alpha_i + \sum_{i=1}^{n} \alpha_{ij} \ln x_j$$

$$\frac{\partial u}{\partial x_i} = \left(\frac{u}{x_i}\right. \quad \alpha_i + \sum_{j=1}^{n} \alpha_{ij} \ln x_j$$

$$\frac{\partial u}{\partial x_j} = \frac{u}{x_j}\left[\alpha_j + \sum_{i=1}^{n} \alpha_{ij} \ln x_i\right]$$

$$\left.\frac{u}{x_i}\right) \Big/ \left(\frac{u}{x_i}\right. \quad \frac{\alpha_j + \sum_{i=1}^{n} \alpha_{ij} \ln x_j}{\alpha_j + \sum_{i=1}^{n} \alpha_{ij} \ln x_i} \quad \frac{p_i}{p_j}$$

$$x_j = x_i \; \frac{p_i}{p_j} \; \frac{\left(\alpha_i + \sum_{j=1}^{n} \alpha_{ij} \ln x_j\right.}{\left(\alpha_j + \sum_{i=1}^{n} \alpha_{ij} \ln x_i\right.} \tag{2.11.2}$$

For all the popular production functions above we have seen that, the demand for inputs is a function that depends upon input prices and output u. Thus, factor minimal cost function can be expressed as,

$$Q\ (u,p) = \sum_{i=1}^{n} p_i \; x_i \; (u,\ p) \quad \text{----} \quad (2.11.3)$$

where p_i is price of i^{th} input

x_i (u,p) is demand for i^{th} input that depends upon output u and input price vector p.

2.12 SHEPHARD'S LEMMA

$$\frac{\partial Q(u,p)}{\partial p_i} = x_i\ (u,p)$$

One important short coming of the production functions discussed above is that they cannot handle multiple outputs. In this context it is desirable to use cost functions (dual) instead of the production functions themselves.

2.13 FACTOR MINIMAL COST FUNCTION – COBB-DOUGLAS PRODUCTION FUNCTION

$$u = A \prod_{i=1}^{n} x_i^{\alpha_i}$$

There is self-duality between Cobb-Douglas production and its cost frontier.

$$Q\ (u,p) = B\, u^{\frac{1}{\theta}} \prod_{i=1}^{n} p_i^{\alpha_i/\theta}$$

where θ is returns to scale parameter.

If returns to scale are constant, we have, θ=1. Consequently, the factor minimal cost function reduces to,

$$Q\ (u,p) = B\, u \prod_{i=1}^{n} p_i^{\alpha_i}$$

$$\frac{Q(u,p)}{u} = B \prod_{i=1}^{n} p_i^{\alpha_i}$$

$$C_A = B \prod_{i=1}^{n} p_i^{\alpha_i}$$

where C_A is average cost.

The above average cost function is in Cobb-Douglas form. Therefore, there is self-duality between Cobb-Douglas production and cost function.

.14 Factor minimal cost function – Translog Production function

P.1 $$\ln Q(u,p) = \alpha_0 + \sum_{i=1}^{n} \alpha_i \ln p_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} (\ln p_i)(\ln p_j) \qquad \text{---- (2.14.1)}$$

$$\frac{\partial \ln Q}{\partial \ln p_i} = \alpha_i + \sum_{j=1}^{n} \alpha_{ij} \ln p_j \qquad i = 1,2,\ldots,m \qquad \text{---- (2.14.2)}$$

$$\frac{\frac{\partial Q}{\partial p_i} p_i}{Q} = \alpha_j + \sum_{j=1}^{n} \alpha_{ij} \ln p_j$$

Using Shephard's lemma, we have,

$$\frac{\partial Q}{\partial p_i} = x_i$$

$$\frac{p_i\, x_i}{Q} = \alpha_i + \sum_{j=1}^{n} \alpha_{ij} \ln p_j \qquad \text{---- (2.14.3)}$$

$$S_i = \alpha_i + \sum_{j=1}^{n} \alpha_{ij} \ln p_j \qquad \text{---- (2.14.4)}$$

i = 1,2,.....n

where S_i can be interpreted as share of i^{th} input in total cost of production.

But, all shares sum to unity.

$$\sum_{i=1}^{n} S_i = 1 \qquad \text{---- (2.14.5)}$$

One of the above share equations can be obtained from other shares due to linear dependence.

$$S_i = \alpha_i + \sum_{j=1}^{n-1} \alpha_{ij} \ln\left(\frac{p_j}{p_m}\right), \quad i = 1,2,\ldots m-1 \qquad \text{---- (2.14.6)}$$

(m-1) linearly independent equations.

P.2 The factor minimal cost function is always linear homogeneous in input prices. This imposes parametric restrictions, such as,

$$\sum_{i=1}^{m} \alpha_i = 1$$

$$\sum_{i=1}^{n} \alpha_{ij} = 0 \qquad \text{---- (2.14.7)}$$

$$\sum_{j=1}^{n} \alpha_{ij} = 0$$

P.3 The parameters should satisfy the property of symmetry.

$$\alpha_{ij} = \alpha_{ji} \qquad \text{---- (2.14.8)}$$

2.15 R.W.Shephard (1970) in his monumental work on theory of Cost and Production functions introduced the concept of input distance function whose inverse measures input technical efficiency. It is defined on input level sets L(u).

$$L(u) = \{x : x \textit{ produces } u\}$$

All x $\in$L(u) can produce output u, some of them are efficient and other are inefficient input vectors. These sets satisfy certain structural properties.

A.1 $L(o) = R_n^+$

Every non-negative input vector can produce null output vector.

A.2 $x \in L(u),\ \bar{x} \geq x \Rightarrow \bar{x} \in L(u)$

If x produces u any input vector larger than x can also produce u.

A.3 $\bar{u} \geq u \Rightarrow L(\bar{u}) \subseteq L(u)$

Input vectors which can produce $\bar{u}$ can also produce u.

A.4 L(u) is convex.

A.5 L(u) closed.

A.6 If $x \in L(u)$ *then* $\lambda x \in L(u)$ *where* $\lambda \geq 1$

The structural properties A.2 and A.3 are respectively called as strong disposability of inputs and outputs respectively. Property A.6 refers to weak disposability of inputs.

If production function is piece wise linear, L(u) is shown below:

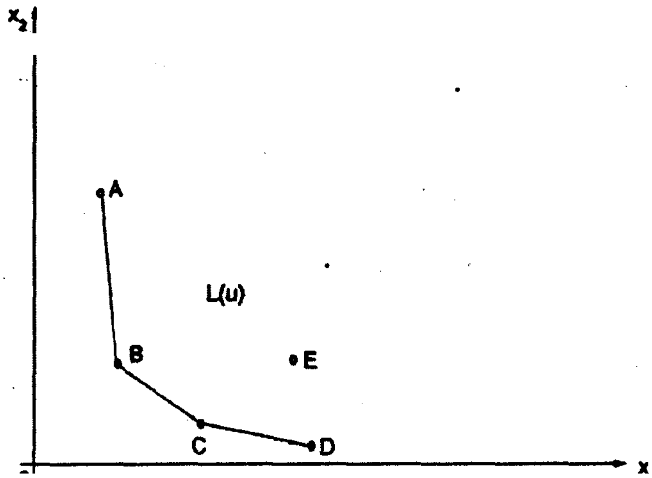

Figure (3)

In the figure above input level set is L(u). The isoquant of L(u) is determined by the points A, B, C and D and the linear segments joining them. A, B, C and D are efficient points and E is inefficient point.

SHEPHARD'S INPUT DISTANCE FUNCTION

$$D(u_0, x_0) = [Min \{\lambda : \lambda\ x_0 \in L(u_0)\}]^{-1} \geq 1$$

$[D(u_0,x_0)]^{-1}$ measures input technical efficiency.

Shephard also constructed output distance function, whose definition is based on output level sets $P(x)$.

$$P(x) = \{u : x \text{ produces } u\}$$

A.1 P(o) = ∅, No free lunch

A.2 $u \in P(x), \bar{u} \leq u \Rightarrow \bar{u} \in P(x)$

If x produces u, it can produce output vector less than x.

A.3 $x \geq \bar{x} \Rightarrow P(\bar{x}) \subseteq P(x)$

Larger inputs produce larger outputs.

A.4 P(x) is convex.

A.5 P(x) is closed.

If production function is piece wise linear, output level set is shown as below:

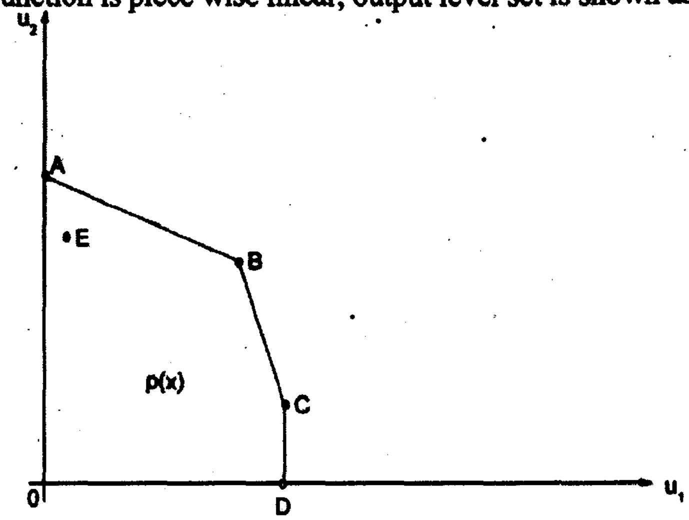

Figure (4)

The first output is measured along horizontal axis and the second output is measured along vertical axis. The points A, B, C and D and the line segments joining them determine output isoquant. The producer who operates at E is inefficient.

OUTPUT DISTANCE FUNCTION

$$D_0\ (x_0, u_0) = [Max\ \{\theta : \theta\ u_0 \in P(x_0)\}]^{-1}$$

The inverse of output distance function measures output technical efficiency.