CHAPTER-III

RECURRENCE RELATIONS FOR THE GENERALIZED HYPERGEOMETRIC FUNCTION $B_n^{(\alpha,\beta)}(x,y,w)$ AND MODIFIED KONHAUSER'S POLYNOMIAL $Z_n^{\alpha}(x;w,k)$

3.1 The term recurrence relation was first used in systematic way by Sister Mary Celine Fasenmyer in a Michigan thesis in the year 1945. Later on, it became a necessity to deduce the different types of recurrence relations for each and every polynomial or function due to its various applications in the different fields of science and technology. These results are also important for discussing the group theoretic approach to the generating relations.

Inspired by the above argument, an attempt has been made to deduce the different types of recurrence relations in the present chapter. These relations are of the following types :

i) Pseudo recurrence relations

ii) Mixed recurrence relations

which are followed by their applications.

3.2 PSEUDO-RECURRENCE RELATIONS FOR $B_n^{(\alpha,\beta)}(x,y,w)$

In this section the following pseudo-recurrence relations are proved :

Theorem 1

$$\alpha\, B_n^{(\alpha,\beta)}(x,y,w) \;=(\alpha+n)\, B_n^{(\alpha-1,\beta)}(x,y,w)+\left(\frac{x}{w}-1\right)\left(\frac{\beta+n}{n}\right)B_{n-1}^{(\alpha+1,\beta)}(x,y,w) \quad \textbf{--- (3.2.1)}$$

Proof:

Consider $(\alpha+n)\dfrac{(\alpha)_n(1+\beta)_n}{(n!)^2}\displaystyle\sum_{r=0}^{n}\;\sum_{s=0}^{n-r}\;\frac{(-n)_{r+s}\left(-\frac{y}{w}\right)_s\left(\frac{x}{w}\right)_r(-w)^s\, w^r}{(\alpha)_r(1+\beta)_s\; r!\; s!}$

$$+\left(\frac{x}{w}-1\right)\left(\frac{\beta+n}{n}\right)\frac{(2+\alpha)_{n-1}(1+\beta)_{n-1}}{[(n-1)!]^2}$$

$$\times\sum_{r=0}^{n-1}\;\sum_{s=0}^{n-1-r}\;\frac{(-n+1)_{r+s}\left(-\frac{y}{w}\right)_s\left(\frac{x}{w}\right)_r(-w)^s\, w^r}{(2+\alpha)_r(1+\beta)_s\; r!\; s!}$$

$$=\frac{\alpha(1+\alpha)_n(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\;\sum_{s=0}^{n-r}\;\frac{(-n)_{r+s}\left(-\frac{y}{w}\right)_s\left(\frac{x}{w}\right)_r(-w)^s\, w^r}{(\alpha)_r(1+\beta)_s\; r!\; s!}$$

$$+\left(\frac{x}{w}-1\right)\frac{(1+\alpha)_n(1+\beta)_n\, n}{[n(n-1)!]^2}$$

$$\times\sum_{r=0}^{n-1}\;\sum_{s=0}^{n-1-r}\;\frac{(-1)^{r+s}(n-1)!\left(-\frac{y}{w}\right)_s\left(\frac{x}{w}\right)_r(-w)^s\, w^r(1+\alpha)}{[n-1-(r+s)]!\,(1+\alpha)_{r+1}(1+\beta)_s\; r!\; s!}$$

$$=\frac{\alpha(1+\alpha)_n(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\;\sum_{s=0}^{n-r}\;\frac{(-n)_{r+s}\left(-\frac{y}{w}\right)_s\left(\frac{x}{w}\right)_r(-w)^s\, w^r}{(\alpha)_r(1+\beta)_s\; r!\; s!}$$

$$+\left(\frac{x}{w}-1\right)\frac{(1+\alpha)_n(1+\beta)_n\, n}{(n!)^2}$$

$$\times\sum_{r=0}^{n-1}\sum_{s=0}^{n-1-r}\frac{(-1)^{r+s}(n-1)!\left(-\frac{y}{w}\right)_s\left(\frac{x}{w}\right)_r(-w)^s w^r}{[n-1-(r+s)]!\,(1+\alpha)_{r+1}(1+\beta)_s\ r!\,s!}$$

In the last double series write (r-1) for 'r' and it results in

$$=\frac{\alpha(1+\alpha)_n(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\left(-\frac{y}{w}\right)_s\left(\frac{x}{w}\right)_r(-w)^s w^r}{(\alpha)_r(1+\beta)_s\ r!\,s!}$$

$$+\left(\frac{x}{w}-1\right)\frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2}$$

$$\times\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-1)^{r-1+s}n!\left(-\frac{y}{w}\right)_s\left(\frac{x}{w}\right)_{r-1}(-w)^s w^{r-1}}{[n-(r+s)]!\,(1+\alpha)_r(1+\beta)_s\ (r-1)!\,s!}$$

$$=\frac{\alpha(1+\alpha)_n(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\left(-\frac{y}{w}\right)_s\left(\frac{x}{w}\right)_r(-w)^s w^r(\alpha+r)}{\alpha(1+\alpha)_r(1+\beta)_s\ r!\,s!}$$

$$+\frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2}$$

$$\times\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-1)(-1)^{r+s}n!\left(-\frac{y}{w}\right)_s\left(\frac{x}{w}-1\right)_r(-w)^s w^{r-1}(r)}{[n-(r+s)]!\,(1+\alpha)_r(1+\beta)_s\ r!\,s!}$$

$$=\frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\left(-\frac{y}{w}\right)_s\left(\frac{x}{w}\right)_r(-w)^s w^r(\alpha+r)}{(1+\alpha)_r(1+\beta)_s\ r!\,s!}$$

$$+\frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\left(-\frac{y}{w}\right)_s\left(\frac{x}{w}-1\right)_r(-w)^s w^{r-1}(-r)}{(1+\alpha)_r(1+\beta)_s\ r!\,s!}$$

$$= \frac{(1+\alpha)_n (1+\beta)_n}{(n!)^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} \left(-\frac{y}{w}\right)_s \left(\frac{x}{w}\right)_r (-w)^s \, w^r (\alpha + r)}{(1+\alpha)_r (1+\beta)_s \; r! \, s!}$$

$$+ \frac{(1+\alpha)_n (1+\beta)_n}{(n!)^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} \left(-\frac{y}{w}\right)_s \left(\frac{x}{w}\right)_r (-w)^s \, w^r (-r) \left(\frac{x}{w} - 1\right)}{(1+\alpha)_r (1+\beta)_s \; r! \, s! \; w\left(\frac{x}{w} + r - 1\right)}$$

By neglecting small quantity $\left(\frac{\frac{x}{w} - 1}{w(\frac{x}{w} + r - 1)} \right)$ and using (2.3.1), we obtain

$$= \frac{(1+\alpha)_n (1+\beta)_n}{(n!)^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} \left(-\frac{y}{w}\right)_s \left(\frac{x}{w}\right)_r (-w)^s \, w^r (\alpha + r - r)}{(1+\alpha)_r (1+\beta)_s \; r! \, s!}$$

$$= \alpha \, B_n^{(\alpha,\beta)}(x, y, w) \qquad \textbf{--- (3.2.2)}$$

Applications:

By taking the limit $w \to 0$ on both sides of (3.2.1), it leads to the recurrence relation of generalized Laguerre polynomial of two variables, which is obtained by Ragab, S.F.[159].

$$\alpha \, L_n^{(\alpha,\beta)}(x, y) = (\alpha + n) \, L_n^{(\alpha-1,\beta)}(x, y) + x \left(\frac{\beta + n}{n}\right) L_{n-1}^{(\alpha+1,\beta)}(x, y) \qquad \textbf{--- (3.2.3)}$$

Theorem 2

$$\beta \, B_n^{(\alpha,\beta)}(x, y, w) = (\beta + n) \, B_n^{(\alpha,\beta-1)}(x, y, w) + \left(\frac{y}{w} - 1\right)\left(\frac{\alpha + n}{n}\right) B_{n-1}^{(\alpha,\beta+1)}(x, y, w) \qquad \textbf{--- (3.2.4)}$$

Proof:

Consider $(\beta+n)\dfrac{(1+\alpha)_n(\beta)_n}{(n!)^2}\displaystyle\sum_{r=0}^{n}\sum_{s=0}^{n-r}\dfrac{(-n)_{r+s}\left(-\dfrac{y}{w}\right)_r\left(\dfrac{x}{w}\right)_s(-w)^r\,w^s}{(1+\alpha)_s(\beta)_r\;r!\,s!}$

$$+\left(\frac{y}{w}-1\right)\left(\frac{\alpha+n}{n}\right)\frac{(1+\alpha)_{n-1}(2+\beta)_{n-1}}{[(n-1)!]^2}$$

$$\times\sum_{r=0}^{n-1}\sum_{s=0}^{n-1-r}\frac{(-n+1)_{r+s}\left(-\dfrac{y}{w}\right)_r\left(\dfrac{x}{w}\right)_s(-w)^r\,w^s}{(1+\alpha)_r(2+\beta)_s\;r!\,s!}$$

$$=\frac{\beta(1+\alpha)_n(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\left(-\dfrac{y}{w}\right)_r\left(\dfrac{x}{w}\right)_s(-w)^r\,w^s}{(1+\alpha)_s(\beta)_r\;r!\,s!}$$

$$+\left(\frac{y}{w}-1\right)\frac{(1+\alpha)_n(1+\beta)_n\,n}{[n(n-1)!]^2(1+\beta)}$$

$$\times\sum_{r=0}^{n-1}\sum_{s=0}^{n-1-r}\frac{(-1)^{r+s}(n-1)!\left(-\dfrac{y}{w}\right)_r\left(\dfrac{x}{w}\right)_s(-w)^r\,w^s(1+\beta)}{[n-1-(r+s)]!\,(1+\alpha)_s(1+\beta)_{r+1}\;r!\,s!}$$

$$=\frac{\beta(1+\alpha)_n(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\left(-\dfrac{y}{w}\right)_r\left(\dfrac{x}{w}\right)_s(-w)^r\,w^s}{(1+\alpha)_s(\beta)_r\;r!\,s!}$$

$$+\left(\frac{y}{w}-1\right)\frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n-1}\sum_{s=0}^{n-1-r}\frac{(-1)^{r+s}\,n(n-1)!\left(-\dfrac{y}{w}\right)_r\left(\dfrac{x}{w}\right)_s(-w)^r\,w^s}{[n-1-(r+s)]!\,(1+\alpha)_s(1+\beta)_{r+1}\;r!\,s!}$$

In the last double series write (r-1) for ‘r’ and it results in

$$=\frac{\beta(1+\alpha)_n(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\left(-\dfrac{y}{w}\right)_r\left(\dfrac{x}{w}\right)_s(-w)^r\,w^s}{(1+\alpha)_s(\beta)_r\;r!\,s!}$$

$$+\left(\frac{y}{w}-1\right)\frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2}$$

$$\times\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-1)^{r-1+s}\,n!\left(-\frac{y}{w}\right)_{r-1}\left(\frac{x}{w}\right)_s(-w)^{r-1}\,w^s}{[n-(r+s)]!\,(1+\alpha)_s(1+\beta)_r\;(r-1)!\,s!}$$

$$=\frac{\beta(1+\alpha)_n(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\left(-\frac{y}{w}\right)_r\left(\frac{x}{w}\right)_s(-w)^r\,w^s(\beta+r)}{\beta(1+\alpha)_s(1+\beta)_r\;r!\,s!}$$

$$+\frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2}$$

$$\times\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-1)^{r+s}\,n!\left(-\frac{y}{w}\right)_{r-1}\left(\frac{x}{w}\right)_s(-w)^{r-1}\,w^s(-r)}{[n-(r+s)]!\,(1+\alpha)_s(1+\beta)_r\;r!\,s!}\left(\frac{y}{w}-1\right)$$

$$=\frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\left(-\frac{y}{w}\right)_r\left(\frac{x}{w}\right)_s(-w)^r\,w^s(\beta+r)}{(1+\alpha)_s(1+\beta)_r\;r!\,s!}$$

$$+\frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2}$$

$$\times\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\left(-\frac{y}{w}\right)_r\left(\frac{x}{w}\right)_s(-w)^r\,w^s(-r)\left(\frac{y}{w}-1\right)}{(1+\alpha)_s(1+\beta)_r\;r!\,s!\left(-\frac{y}{w}+r-1\right)(-w)}$$

By neglecting small quantity $\left(\dfrac{\frac{y}{w}-1}{(-w)(-\frac{y}{w}+r-1)}\right)$ and using (2.3.1), we obtain

$$= \frac{(1+\alpha)_n (1+\beta)_n}{(n!)^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} \left(-\frac{y}{w}\right)_r \left(\frac{x}{w}\right)_s (-w)^r \, w^s (\beta + r - r)}{(1+\alpha)_s (1+\beta)_r \; r! \, s!}$$

$$= \beta \, B_n^{(\alpha,\beta)}(x, y, w) \qquad \textbf{--- (3.2.5)}$$

Applications:

By taking the limit $w \to 0$ on both sides of (3.2.4), it leads to the recurrence relation of generalized Laguerre polynomial of two variables, which is obtained by Ragab, S.F.[159].

$$\beta \, L_n^{(\alpha,\beta)}(x, y) = (\beta + n) \, L_n^{(\alpha,\beta-1)}(x, y) + y \left(\frac{\alpha + n}{n}\right) L_{n-1}^{(\alpha,\beta+1)}(x, y) \qquad \textbf{--- (3.2.6)}$$

3.3 RECURRENCE RELATIONS FOR $Z_n^{\alpha}(x; w, k)$

In this section the following recurrence relations are to be proved :

Theorem 3

$$k \, x^{[kw]} \, Z_n^{\alpha+k}(x + w; w, k) = \alpha Z_{n+1}^{\alpha}(x; w, k) - (kn + \alpha + k) \, Z_{n+1}^{\alpha-1}(x; w, k) \qquad \textbf{--- (3.3.1)}$$

Proof:

Consider $\alpha Z_{n+1}^{\alpha}(x; w, k) - (kn + \alpha + k) \, Z_{n+1}^{\alpha-1}(x; w, k)$

$$= \alpha \frac{((n+1)k + \alpha)\Gamma[nk + \alpha + k]}{(n+1)!} \sum_{j=0}^{n+1} \frac{(-1)^j \, (n+1)_{C_j} {}^{[jw]} \left\{x^{[kw]}\right\}}{(kj + \alpha)\, \Gamma(kj + \alpha)}$$

$$- (kn + \alpha + k) \frac{\Gamma[kn + \alpha + k]}{(n+1)!} \sum_{j=0}^{n+1} \frac{(-1)^j \, (n+1)_{C_j} {}^{[jw]} \left\{x^{[kw]}\right\}}{\Gamma(kj + \alpha)}$$

$$=\frac{\Gamma[nk+(\alpha+k)+1]}{(n+1)!}\sum_{j=0}^{n+1}\frac{(-1)^j\,(n+1)_{C_j}\left[\frac{\alpha}{kj+\alpha}-1\right]^{[jw]}\left\{x^{[kw]}\right\}}{\Gamma(kj+\alpha)}$$

$$=\frac{\Gamma(kn+(\alpha+k)+1)}{(n+1)!}\sum_{j=0}^{n+1}\frac{(-1)^j\,(n+1)!(-kj)^{[jw]}\left\{x^{[kw]}\right\}}{j!\,(n+1-j)!\,(kj+\alpha)\,\Gamma(kj+\alpha)}$$

$$=k\,\frac{\Gamma(kn+(\alpha+k)+1)}{n!}\times\sum_{j=1}^{n+1}\frac{(-1)^{j-1}\,n!^{[jw]}\left\{x^{[kw]}\right\}}{(j-1)!\,(n-(j-1))!\,\Gamma(k(j-1)+(\alpha+k)+1)}$$

By using (1.5.17), it can be obtained

$$=k\,\frac{\Gamma(kn+(\alpha+k)+1)}{n!}\sum_{j=1}^{n+1}\frac{(-1)^{j-1}\,n!\left(\frac{x}{w}\right)_j\left(\frac{x}{w}-1\right)_j\cdots\left(\frac{x}{w}-k+1\right)_j\,w^{kj}}{(j-1)!\,(n-(j-1))!\,\Gamma(k(j-1)+(\alpha+k)+1)}$$

By writing (*j*+1) for *j*, we obtain

$$=k\,\frac{\Gamma(kn+(\alpha+k)+1)}{n!}$$

$$\times\sum_{j=0}^{n}\frac{(-1)^j\,n!\left(\frac{x}{w}\right)_{j+1}\left(\frac{x}{w}-1\right)_{j+1}\cdots\left(\frac{x}{w}-k+1\right)_{j+1}\,w^{kj}\,w^k}{j!\,(n-j)!\,\Gamma(kj+(\alpha+k)+1)}$$

$$=k\,w^k\,\frac{\Gamma(kn+(\alpha+k)+1)}{n!}\sum_{j=0}^{n}\frac{(-1)^j\,n_{C_j}\left(\frac{x}{w}\right)\left(\frac{x}{w}-1\right)\cdots\left(\frac{x}{w}-k+1\right)w^{kj}}{\Gamma(kj+(\alpha+k)+1)}$$

$$\times\left(\frac{x}{w}+1\right)_j\left(\frac{x}{w}+1-1\right)_j\cdots\left(\frac{x}{w}-k+1+1\right)_j$$

$$=k\left(\frac{x}{w}\right)\left(\frac{x}{w}-1\right)\cdots\left(\frac{x}{w}-k+1\right)w^k\,\frac{\Gamma(nk+(\alpha+k)+1)}{n!}$$

$$\times \sum_{j=0}^{n} \frac{(-1)^j \, n_{C_j} \left(\frac{x}{w}+1\right)_j \left(\frac{x}{w}\right)_j \cdots \left(\frac{x}{w}-k+1+1\right)_j \, w^{kj}}{\Gamma(kj+(\alpha+k)+1)}$$

$$= k \; x^{[kw]} \, Z_n^{\alpha+k}(x+w; w, k) \qquad \textbf{--- (3.3.2)}$$

Applications:

(i) By applying $w \to 0$ in (3.3.1), it leads to the recurrence relation on Konhauser's polynomial obtained by Srivastava, H.M. [199].

$$k \, x^k \, Z_n^{\alpha+k}(x; k) = \alpha Z_{n+1}^{\alpha}(x; k) - (kn+\alpha+k) \, Z_{n+1}^{\alpha-1}(x; k) \qquad \textbf{--- (3.3.3)}$$

(ii) By taking *k* = 1, in (33.1), it leads to a recurrence relation on Modified Jacobi polynomial

$$x \, j_n^{\alpha+1}(x+w, w) = \alpha J_{n+1}^{\alpha}(x, w) - (n+\alpha+1) \, J_{n+1}^{\alpha-1}(x, w) \qquad \textbf{--- (3.3.4)}$$

(iii) By taking $w \to 0$ and k = 1 in (3.3.1), it leads to a famous recurrence relation on Laguerrre polynomials.

$$x \, L_n^{\alpha+1}(x) = \alpha \, L_{n+1}^{\alpha}(x) - (n+\alpha+1) \, L_{n+1}^{\alpha-1}(x) \qquad \textbf{--- (3.3.5)}$$

Theorem 4

$$(n+1) \, Z_{n+1}^{\alpha}(x; w, k) = (1+\alpha) \, Z_n^{\alpha+1}(x; w, k) - \prod_{i=1}^{k-1} \left(\frac{x}{w} - i\right) w^k \; Z_n^{\alpha}(x+w; w, k)$$

$$- \prod_{i=1}^{k-1} \left(\frac{x}{w} - i\right) \; k \, w^k Z_{n-1}^{\alpha+1}(x+w; w, k) \qquad \textbf{--- (3.3.6)}$$

Proof:

From (2.5.23),

$$\sum_{n=0}^{\infty} Z_n^{\alpha}(x;w,k)\,t^n = \frac{1}{(1-t)^{1+\alpha}}\;{}_kF_0\left[\begin{array}{c} \frac{x}{w}, \frac{x}{w}-1, \ldots, \frac{x}{w}-k+1; \\ - \end{array} \;;\; -\frac{w^k t}{(1-t)^k}\right]$$

Differentiating the above with respect to't', we obtain

$$\sum_{n=0}^{\infty} n\,Z_n^{\alpha}(x;w,k)\,t^{n-1}$$

$$= \frac{(1+\alpha)}{(1-t)^{2+\alpha}}\;{}_kF_0\left[\begin{array}{c} \frac{x}{w}, \frac{x}{w}-1, \ldots, \frac{x}{w}-k+1; \\ - \end{array} \;;\; -\frac{w^k t}{(1-t)^k}\right]$$

$$+ \frac{1}{(1-t)^{1+\alpha}} \times \frac{d}{dt}\left\{\sum_{r=0}^{\infty} \frac{\left(\frac{x}{w}\right)_r \left(\frac{x}{w}-1\right)_r \ldots \left(\frac{x}{w}-k+1\right)_r \; w^{kr}\, t^r\,(-1)^r}{r!\;(1-t)^{kr}}\right\}$$

$$= \frac{(1+\alpha)}{(1-t)^{2+\alpha}}\;{}_kF_0\left[\begin{array}{c} \frac{x}{w}, \frac{x}{w}-1, \ldots, \frac{x}{w}-k+1; \\ - \end{array} \;;\; -\frac{w^k t}{(1-t)^k}\right]$$

$$+ \frac{1}{(1-t)^{1+\alpha}} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{w}\right)_r \left(\frac{x}{w}-1\right)_r \ldots \left(\frac{x}{w}-k+1\right)_r \, w^{kr}}{r!}$$

$$\times \left\{ r\,t^{r-1}(1-t)^{-kr} + t^r\,(-kr)(-1)(1-t)^{-kr-1} \right\}$$

$$= \frac{(1+\alpha)}{(1-t)^{2+\alpha}}\;{}_kF_0\left[\begin{array}{c} \frac{x}{w}, \frac{x}{w}-1, \ldots, \frac{x}{w}-k+1; \\ - \end{array} \;;\; -\frac{w^k t}{(1-t)^k}\right]$$

$$+\frac{1}{(1-t)^{1+\alpha}}\sum_{r=1}^{\infty}\frac{(-1)^r\left(\frac{x}{w}\right)_r\left(\frac{x}{w}-1\right)_r\ldots\left(\frac{x}{w}-k+1\right)_r w^{kr}t^{r-1}}{(r-1)!\;(1-t)^{kr}}$$

$$+\frac{1}{(1-t)^{2+\alpha}}\sum_{r=1}^{\infty}\frac{(-1)^r\left(\frac{x}{w}\right)_r\left(\frac{x}{w}-1\right)_r\ldots\left(\frac{x}{w}-k+1\right)_r w^{kr}k\,t^{r}}{(r-1)!\;(1-t)^{kr}}$$

Replacing (r+1) for ‘r’, we get

$$=\frac{(1+\alpha)}{(1-t)^{2+\alpha}}\,{}_kF_0\left[\begin{matrix}\frac{x}{w},\frac{x}{w}-1,\ldots,\frac{x}{w}-k+1; & -\frac{w^k t}{(1-t)^k}\\ - & ;\end{matrix}\right]$$

$$+\frac{1}{(1-t)^{1+\alpha}}\sum_{r=0}^{\infty}\frac{(-1)^{r+1}\left(\frac{x}{w}\right)_{r+1}\left(\frac{x}{w}-1\right)_{r+1}\ldots\left(\frac{x}{w}-k+1\right)_{r+1} w^{kr}t^{r}w^{k}}{r!\;(1-t)^{kr+k}}$$

$$+\frac{1}{(1-t)^{2+\alpha}}\sum_{r=0}^{\infty}\frac{(-1)^{r+1}\left(\frac{x}{w}\right)_{r+1}\left(\frac{x}{w}-1\right)_{r+1}\ldots\left(\frac{x}{w}-k+1\right)_{r+1} w^{kr}\,w^{k}k\,t^{r+1}}{r!\;(1-t)^{kr+k}}$$

$$=\frac{(1+\alpha)}{(1-t)^{2+\alpha}}\,{}_kF_0\left[\begin{matrix}\frac{x}{w},\frac{x}{w}-1,\ldots,\frac{x}{w}-k+1; & -\frac{w^k t}{(1-t)^k}\\ - & ;\end{matrix}\right]$$

$$-\frac{1}{(1-t)^{1+\alpha}}\sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{x}{w}\right)\left(\frac{x}{w}-1\right)\ldots\left(\frac{x}{w}-k+1\right)w^{kr}t^{r}w^{k}}{r!}$$

$$\times\frac{\left(\frac{x}{w}+1\right)_r\left(\frac{x}{w}-1+1\right)_r\ldots\left(\frac{x}{w}-k+1+1\right)_r w^{kr}t^{r}w^{k}}{(1-t)^{kr}\,(1-t)^{k}}$$

$$-\frac{1}{(1-t)^{2+\alpha}}\sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{x}{w}\right)\left(\frac{x}{w}-1\right)\ldots\left(\frac{x}{w}-k+1\right)}{r!}$$

$$\times\frac{\left(\frac{x}{w}+1\right)_r\left(\frac{x}{w}-1+1\right)_r\ldots\left(\frac{x}{w}-k+1+1\right)_r w^{kr}t^r w^k k}{(1-t)^{kr}\,(1-t)^k}$$

$$=\frac{(1+\alpha)}{(1-t)^{2+\alpha}}\,{}_kF_0\left[\begin{matrix}\frac{x}{w},\frac{x}{w}-1,\ldots,\frac{x}{w}-k+1; & -\frac{w^k t}{(1-t)^k}\\ - & ;\end{matrix}\right]$$

$$-\frac{\left(\frac{x}{w}\right)\left(\frac{x}{w}-1\right)\ldots\left(\frac{x}{w}-k+1\right)w^k}{(1-t)^k\,(1-t)^{1+\alpha}}$$

$$\times\sum_{r=0}^{\infty}\frac{\left(\frac{x}{w}+1\right)_r\left(\frac{x}{w}+1-1\right)_r\ldots\left(\frac{x}{w}-k+1+1\right)_r(-1)^r w^{kr}t^r}{r!(1-t)^{kr}}$$

$$-\frac{\left(\frac{x}{w}\right)\left(\frac{x}{w}-1\right)\ldots\left(\frac{x}{w}-k+1\right)w^k\,kt}{(1-t)^k\,(1-t)^{2+\alpha}}$$

$$\times\sum_{r=0}^{\infty}\frac{\left(\frac{x}{w}+1\right)_r\left(\frac{x}{w}+1-1\right)_r\ldots\left(\frac{x}{w}-k+1+1\right)_r(-1)^r w^{kr}t^r}{r!\,(1-t)^{kr}}$$

$$=\frac{(1+\alpha)}{(1-t)^{2+\alpha}}\,{}_kF_0\left[\begin{matrix}\frac{x}{w},\frac{x}{w}-1,\ldots,\frac{x}{w}-k+1; & -\frac{w^k t}{(1-t)^k}\\ - & ;\end{matrix}\right]$$

$$-\prod_{i=1}^{k-1}\left(\frac{x}{w}-i\right)\frac{w^k}{(1-t)^k}\sum_{n=0}^{\infty}Z_n^{\alpha}(x+w;w,k)\,t^n$$

$$-k\prod_{i=1}^{k-1}\left(\frac{x}{w}-i\right)\frac{w^k\,t}{(1-t)^k}\sum_{n=0}^{\infty}\;Z_n^{\alpha+1}(x+w;w,k)\;t^n$$

$$=(1+\alpha)\sum_{n=0}^{\infty}\;Z_n^{\alpha+1}(x;w,k)\;t^n-\prod_{i=1}^{k-1}\left(\frac{x}{w}-i\right)w^k\,(1-t)^{-k}\sum_{n=0}^{\infty}\;Z_n^{\alpha}(x+w;w,k)\;t^n$$

$$-k\prod_{i=1}^{k-1}\left(\frac{x}{w}-i\right)w^k\,(1-t)^{-k}\sum_{n=0}^{\infty}\;Z_n^{\alpha+1}(x+w;w,k)\;t^{n+1}$$

Expanding $(1-t)^{-k}$ and comparing the t^n coefficients on both sides, the following can be obtained

$$(n+1)\,Z_{n+1}^{\alpha}(x;w,k)=(1+\alpha)\,Z_n^{\alpha+1}(x;w,k)-\prod_{i=1}^{k-1}(\frac{x}{w}-i)\,w^k\;Z_n^{\alpha}(x+w;w,k)$$

$$-\prod_{i=1}^{k-1}(\frac{x}{w}-i)\;k\,w^k Z_{n-1}^{\alpha+1}(x+w;w,k)$$ **--- (3.3.7)**

Applications:

(i) By applying $w\to 0$ in (3.3.6), it leads to

$$(n+1)\;Z_{n+1}^{\alpha}(x;k)=(1+\alpha)Z_n^{\alpha+1}(x;k)-\prod_{i=1}^{k-1}x\,Z_n^{\alpha}(x;k)-\prod_{i=1}^{k-1}xk\,Z_{n-1}^{\alpha+1}(x;k)$$ **--- (3.3.8)**

(ii) By taking k = 1 in (3.3.6), it reduces to a recurrence relation on Modified Jacobi polynomial.

$$(n+1)\,J_{n+1}^{\alpha}(x,w)=(1+\alpha)J_n^{\alpha+1}(x,w)-\prod_{i=1}^{k-1}(x-iw)\,J_n^{\alpha}(x+w,w)$$

$$-\prod_{i=1}^{k-1}(x-iw)\,k\,J_{n-1}^{\alpha+1}(x+w,w)$$ **--- (3.3.9)**

(iii) By taking $w\to 0$ and k = 1 in (3.3.6), it leads to a recurrence relation on

Laguerre polynomial.

$$(n+1)\,L_{n+1}^{\alpha}(x)=(1+\alpha)\,L_{n}^{\alpha+1}(x)-x\,L_{n}^{\alpha}(x)-x\,L_{n-1}^{\alpha+1}(x) \qquad \textbf{--- (3.3.10)}$$