CHAPTER-II

GENERALIZED HYPERGEOMETRIC FUNCTION $B_n^{(\alpha,\beta)}(x,y,w)$

AND MODIFIED KONHAUSER'S POLYNOMIAL $Z_n^{\alpha}(x;w,k)$

2.1 Several polynomials like the Legendre, the Laguerre, the Hermite, the Jacobi and many of their generalizations have been introduced and studied at great length from time to time in a number of different ways by various researchers. Recently, Satyanarayana,B[166] has introduced Generalized Hypergeometric functions $I_n^{\alpha}(x,w)$ and $H_n^{\alpha}(x,w)$ for a discrete variable x as

$$I_{n;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w)=\frac{1}{n!(x-\mu w)^{[\alpha w]}}\Delta_{x,w}^{n}[(x-\mu w)^{[(\alpha+n)w]}\ {}_{p+1}F_q((a_p),-\frac{x}{w}+\lambda;(b_q);w] \quad \text{--- (2.1.1)}$$

$$H_{n;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w)=\frac{1}{n!(x-\mu w)^{[(\alpha+n)w]}}\Delta_{x,w}^{n}[(x-\mu w)^{[\alpha w]}\ {}_{p+1}F_q((a_p),-\frac{x}{w}+\lambda;(b_q);w] \quad \text{--- (2.1.2)}$$

Ragab, S.F.[159] introduced a two variable Laguerre polynomial as

$$L_n^{(\alpha,\beta)}(x,y)=\frac{(1+\alpha)_n\,(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\ \sum_{s=0}^{n-r}\ \frac{(-n)_{r+s}\ (y)^r\ (x)^s}{r!\ s!\ (1+\alpha)_s\ (1+\beta)_r} \quad \text{--- (2.1.3)}$$

Parihar and Patel [154] introduced a Modified Jacobi Polynomial as

$$J_n^{\alpha}(x,w)=\frac{(1+\alpha)_n}{n!}\ {}_2F_1\left[-n,\frac{x}{w};1+\alpha;w\right] \quad \text{--- (2.1.4)}$$

and Konhauser [104] introduced an explicit expression for the polynomials $Z_n^{\alpha}(x;k)$

as

$$Z_n^{\alpha}(x;k) = \frac{\Gamma(kn+\alpha+1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj+\alpha+1)} \qquad \text{--- (2.1.5)}$$

The basic aim of our present chapter is to provide the generalizations of the polynomials given by Ragab, S.F. and Konhauser in two different directions, the two functions $B_n^{(\alpha,\beta)}(x,y,w)$ and $Z_n^{\alpha}(x;w,k)$ are defined and they are valid for the discrete values of the variable x at equal intervals. Due to the nature of the variable in these functions, it may be widely used in various fields such as statistics particularly in Markoff-Chain and other physical sciences.

The Laguerre polynomial of two variables defined by Ragab, S.F. [159] and the Laguerre polynomials can be deduced directly from $B_n^{(\alpha,\beta)}(x,y,w)$ by assigning specific values to the various parameters involved therein. Modified Jacobi polynomial set defined by Parihar and Patel [154] and Konhauser's polynomial set defined by Konhauser, J.D.E. [104] can be deduced directly from $Z_n^{\alpha}(x;w,k)$ by assigning specific values to the various parameters involved therein.

After defining these two functions, the main stress is given to obtain the set of conditions for their convergence. The hypergeometric representations have also been given.

After giving hypergeometric representations, the linear generating relations for these two polynomial sets has been given. An extended linear generating relation for $Z_n^{\alpha}(x;w,k)$ is also given.

At the end, an attempt has been made to deduce the summation series of

$B_n^{(\alpha,\beta)}(x,y,w)$.

All these results are followed by their applications to the Laguerre polynomial of two variables, Laguerre polynomials, Modified Jacobi polynomial and Konhauser's polynomial.

2.2 DEFINITIONS

In the present section, two sets of polynomials $B_n^{(\alpha,\beta)}(x,y,w)$ and $Z_n^{\alpha}(x;w,k)$ are defined by using a difference operator technique.

The first one is introduced and studied in a class of generalized hypergeometric function of two discrete variables which is defined as follows:

$$B_n^{(\alpha,\beta)}(x,y,w)=\frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{n!}\sum_{r=0}^{n}\frac{(-1)^r\, y^{[rw]}\, J_{n-r}^{(\alpha)}(x,w)}{r!\,\Gamma(n+\alpha-r+1)\,\Gamma(\mathrm{r}+\beta+1)} \qquad \textbf{---(2.2.1)}$$

where $J_n^{(\alpha)}(x,w)$ is modified Jacobi polynomial [154].

Later, the modified Konhauser Polynomial is defined by a

formula $$Z_n^{\alpha}(x;w,k)=\frac{\Gamma(kn+\alpha+1)}{n!}\sum_{j=0}^{n}\frac{(-1)^j\, n_{C_j}{}^{[jw]}\left\{x^{[kw]}\right\}}{\Gamma(kj+\alpha+1)} \qquad \textbf{---(2.2.2)}$$

The above functions are valid under the following set of conditions:

i) α, β are non-negative integers,

ii) *n* is any positive integer or zero,

iii) *k* =1,2,3,---

iv) *j* is a value in the set {0,1,2,3,---}

v) *w* is a small increment of *x* such that $|w|<1$, and

vi) *x*,*y* be any two discrete variables.

2.3 HYPERGEOMETRIC REPRESENTATIONS

2.3.1 The function $B_n^{(\alpha,\beta)}(x,y,w)$ defined in (2.2.1) can be expressed as a double sum by using (1.3.3) and observing the relations (1.5.1), (1.5.19) and (1.5.20).

$$B_n^{(\alpha,\beta)}(x,y,w)=\frac{(1+\alpha)_n\,(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\left(-\frac{y}{w}\right)_r\left(\frac{x}{w}\right)_s(-w)^r\,(w)^s}{r!\,s!\,(1+\alpha)_s\,(1+\beta)_r} \qquad \textbf{---(2.3.1)}$$

$$=\frac{(1+\alpha)_n\,(1+\beta)_n}{(n!)^2}\,F_{-:1;1}^{1:1;1}\left[\begin{matrix}-n:-\frac{y}{w},\ \frac{x}{w};-w,w\\ -:1+\beta;1+\alpha;\end{matrix}\right] \qquad \textbf{---(2.3.2)}$$

where $F_{q:s,v}^{p:r;u}$ is a double hypergeometric function[201].

Proof:

From the definition (2.2.1),

$$B_n^{(\alpha,\beta)}(x,y,w)=\frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{n!}\sum_{r=0}^{n}\frac{(-1)^r\,y^{[rw]}\,J_{n-r}^{(\alpha)}(x,w)}{r!\,\Gamma(n+\alpha-r+1)\,\Gamma(\mathrm{r}+\beta+1)}$$

$$=\frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{n!}\sum_{r=0}^{n}\frac{\left(-\frac{y}{w}\right)_r(w)^r\,J_{n-r}^{(\alpha)}(x,w)}{r!\,\Gamma(n+\alpha-r+1)\,\Gamma(\mathrm{r}+\beta+1)}$$

By using (2.1.4)

$$=\frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{n!}\sum_{r=0}^{n}\frac{\left(-\frac{y}{w}\right)_r w^r}{r!\,\Gamma(n+\alpha-r+1)\,\Gamma(\mathrm{r}+\beta+1)}$$

$$\times\frac{(1+\alpha)_{n-r}}{(n-r)!}\,{}_2F_1\left[-(n-r),\frac{x}{w};1+\alpha;w\right]$$

$$=\frac{(1+\alpha)_n\,(1+\beta)_n}{n!}\sum_{r=0}^{n}\frac{\left(-\frac{y}{w}\right)_r w^r\,(1+\alpha)_{n-r}}{r!\,(n-r)!\,(1+\beta)_r\,(1+\alpha)_{n-r}}\sum_{s=0}^{n-r}\frac{\left(-(n-r)\right)_s\left(\frac{x}{w}\right)_s w^s}{s!\,(1+\alpha)_s}$$

$$=\frac{(1+\alpha)_n\,(1+\beta)_n}{n!}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{\left(-\frac{y}{w}\right)_r\left(\frac{x}{w}\right)_s(-(n-r))_s\,w^r\,w^s}{s!\,r!\,(n-r)!\,(1+\beta)_r\,(1+\alpha)_s}$$

$$=\frac{(1+\alpha)_n\,(1+\beta)_n}{n!}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-1)^s\,(n-r)!\left(-\frac{y}{w}\right)_r\left(\frac{x}{w}\right)_s w^r\,w^s}{(n-r)!\,s!\,r!\,(n-r-s)!(1+\beta)_r\,(1+\alpha)_s}$$

$$=\frac{(1+\alpha)_n\,(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-1)^{r+s}\,n!\left(-\frac{y}{w}\right)_r\left(\frac{x}{w}\right)_s(-w)^r\,w^s}{s!\,r!\,(n-r-s)!\,(1+\beta)_r\,(1+\alpha)_s}$$

$$=\frac{(1+\alpha)_n\,(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\left(-\frac{y}{w}\right)_r\left(\frac{x}{w}\right)_s(-w)^r\,w^s}{s!\,r!\,(1+\beta)_r\,(1+\alpha)_s}$$

$$=\frac{(1+\alpha)_n\,(1+\beta)_n}{(n!)^2}\,F^{1:1;1}_{-:1;1}\left[\begin{matrix}-n:-\frac{y}{w},\ \frac{x}{w};\\ -:1+\beta;1+\alpha;\end{matrix}\,-w,w\right] \qquad \textbf{--- (2.3.3)}$$

Remark: One can easily see that the generalized hypergeometric function (2.3.3) will terminate due to the presence of the parameter -*y*/*w* if y is an integral multiple of *w* or $w \to 0$, otherwise a polynomial.

Applications:

(i) Taking the limit $w \to 0$ in (2.3.2), it leads to

$$L_n^{(\alpha,\beta)}(x,y)=\frac{(1+\alpha)_n\,(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\,(y)^r\,(x)^s}{r!\,s!\,(1+\alpha)_s\,(1+\beta)_r} \qquad \textbf{---(2.3.4)}$$

where $L_n^{(\alpha,\beta)}(x,y)$ is Laguerre polynomial of two variables defined by Ragab [159].

(ii) By writing $\beta = 0, y = 0$ and taking $w \to 0$ in (2.3.2), it reduces to

$$\underset{w \to 0}{Lt}\, B_n^{(\alpha,0)}(x, 0, w) = L_n^{\alpha}(x) \qquad \textbf{---(2.3.5)}$$

where $L_n^{\alpha}(x)$ is Laguerre polynomial, Rainville, E.D.[161].

(iii) By writing $\alpha = 0, x = 0$ and taking $w \to 0$ in (2.3.2), it reduces to

$$\underset{w \to 0}{Lt}\, B_n^{(0,\beta)}(0, y, w) = L_n^{\beta}(y) \qquad \textbf{---(2.3.6)}$$

where $L_n^{\beta}(y)$ is Laguerre polynomial, Rainville, E.D.[161].

2.3.2 The function $Z_n^{\alpha}(x; w, k)$ defined in (2.2.2) can be expressed as a hypergeometric form by using (1.5.17) and observing the relations (1.5.1) and (1.5.20).

$$Z_n^{\alpha}(x; w, k) = \frac{(1+\alpha)_{kn}}{n!}\, {}_{k+1}F_k \left[\begin{matrix} -n, \dfrac{x}{w}, \dfrac{x}{w} - 1, ---, \dfrac{x}{w} - k + 1; \left(\dfrac{w}{k}\right)^k \\ \Delta(k; 1+\alpha) \qquad ; \end{matrix} \right] \qquad \textbf{--- (2.3.7)}$$

Proof:

$$Z_n^{\alpha}(x; w, k) = \frac{\Gamma(kn+\alpha+1)}{n!} \sum_{j=0}^{n} \frac{(-1)^j\, n_{C_j}\, {}^{[jw]}\left\{x^{[kw]}\right\}}{\Gamma(kj+\alpha+1)}$$

$$= \frac{\Gamma(kn+\alpha+1)}{n!} \sum_{j=0}^{n} \frac{(-1)^j\, n! \left(\dfrac{x}{w}\right)_j \left(\dfrac{x}{w} - 1\right)_j --- \left(\dfrac{x}{w} - k + 1\right)_j w^{kj}}{j!\,(n-j)!\,\Gamma(1+\alpha)\,(1+\alpha)_{kj}}$$

$$= \frac{\Gamma(kn+\alpha+1)}{\Gamma(1+\alpha)\, n!} \sum_{j=0}^{n} \frac{(-n)_j\, n! \left(\dfrac{x}{w}\right)_j \left(\dfrac{x}{w} - 1\right)_j --- \left(\dfrac{x}{w} - k + 1\right)_j w^{kj}}{k^{kj} \left(\dfrac{\alpha+1}{k}\right)_j \left(\dfrac{\alpha+2}{k}\right)_j --- \left(\dfrac{\alpha+k}{k}\right)_j j!}$$

$$= \frac{(1+\alpha)_{kn}}{n!} \; {}_{k+1}F_k \left[\begin{matrix} -n, \frac{x}{w}, \frac{x}{w}-1, ---, \frac{x}{w}-k+1; \\ \Delta(k;1+\alpha) \qquad ; \end{matrix} \left(\frac{w}{k}\right)^k \right] \qquad \text{--- (2.3.8)}$$

Applications:

(i) when $w \to 0$ in (2.3.8), it leads to

$$\underset{w\to 0}{Lt}\, Z_n^{\alpha}(x; w, k) = \frac{(1+\alpha)_{nk}}{n!} \; {}_1F_k \left[\begin{matrix} -n \qquad ; \\ \Delta(k;1+\alpha); \end{matrix} \left(\frac{x}{k}\right)^k \right]$$

$$= Z_n^{\alpha}(x;k) \qquad \text{--- (2.3.9)}$$

where $Z_n^{\alpha}(x;k)$ is a Konhauser Polynomial [104]

(ii) If k = 1, (2.3.8) reduces to

$$Z_n^{\alpha}(x; w, 1) = \frac{(1+\alpha)_n}{n!} \; {}_2F_1 \left[\begin{matrix} -n, \frac{x}{w}; \\ 1+\alpha \;\; ; \end{matrix} w \right]$$

$$= J_n^{\alpha}(x, w) \qquad \text{--- (2.3.10)}$$

where $J_n^{\alpha}(x, w)$ is modified Jacobi Polynomial [154].

(iii) By substituting $w \to 0$ and k = 1 in (2.3.8), it leads to

$$\underset{w\to 0}{Lt}\, Z_n^{\alpha}(x; w, 1) = \frac{(1+\alpha)}{n!} \; {}_1F_1 \left[\begin{matrix} -n \quad ; \\ 1+\alpha \; ; \end{matrix} x \right]$$

$$= L_n^{\alpha}(x) \qquad \text{--- (2.3.11)}$$

where $L_n^{\alpha}(x)$ is a Laguerre Polynomial, Rainville, E.D.[161]

2.4 CONVERGENCE

(i) Convergence for $B_n^{(\alpha,\beta)}(x,y,w)$

On account of (2.3.1), it can be obtained

$$B_n^{(\alpha,\beta)}(x,y,w) = \frac{(1+\alpha)_n\,(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\frac{(-n)_r\left(-\frac{y}{w}\right)_r(-w)^r}{r!\,(1+\beta)_r}\;{}_2F_1\left[\begin{matrix}-n+r,\frac{x}{w}; & w\\ 1+\alpha\;\;; & \end{matrix}\right]$$

$$= \sum_{r=0}^{n}\frac{(-n)_r\left(-\frac{y}{w}\right)_r(-w)^r}{r!\,(1+\beta)_r}A_{n,r}$$

where $A_{n,r} = \frac{(1+\alpha)_n\,(1+\beta)_n}{(n!)^2}\;{}_2F_1\left[\begin{matrix}-n+r,\frac{x}{w}; & w\\ 1+\alpha\;\;; & \end{matrix}\right]$ **--- (2.4.1)**

Here $A_{n,r}$ is a polynomial and hence convergent.

Now, $$\lim_{r\to\infty}\left|\frac{(-n)_{r+1}\left(-\frac{y}{w}\right)_{r+1}(-w)^{r+1}}{(r+1)!\,(1+\beta)_{r+1}}\cdot\frac{r!\,(1+\beta)_r\;A_{n,r+1}}{(-n)_r\left(-\frac{y}{w}\right)_r(-w)^r\,A_{n,r}}\right|$$

$$= \lim_{r\to\infty}\left|\frac{(-n+r)\left(-\frac{y}{w}+r\right)w}{(r+1)\,(1+\beta+r)}\right|\lim_{r\to\infty}\left|\frac{A_{n,r+1}}{A_{n,r}}\right| = |w\,s|$$ **--- (2.4.2)**

where $s = \lim_{r\to\infty}\left|\frac{A_{n,r+1}}{A_{n,r}}\right|$

Thus, it can be concluded that the given function is convergent if $|w\,s|>1$ i.e., $|w|>\frac{1}{|s|}$

and divergent if $|w\,s|<1$ i.e., $|w|<\frac{1}{|s|}$.

Moreover, if $|w\,s|=1$, then it can be proceeded as follows:

Let $\delta=\frac{1}{2}\mathrm{Re}(\frac{y}{w}+1+\beta+n)>0$ and by comparing the terms of the

series $B_n^{(\alpha,\beta)}(x,y,w)=\sum_{r=0}^{n}\frac{(-n)_r\left(-\frac{y}{w}\right)_r(-w)^r}{r!\,(1+\beta)_r}A_{n,r}$

$$=A_{n,0}+\sum_{r=1}^{n}\frac{(-n)_r\left(-\frac{y}{w}\right)_r(-w)^r}{r!\,(1+\beta)_r}A_{n,r} \qquad \textbf{--- (2.4.3)}$$

with the corresponding terms of the series $\sum_{r=1}^{\infty}\frac{1}{r^{1+\delta}}$, which is known to be

convergent, the following can be obtained

$$\lim_{r\to\infty}\left|\frac{r^{1+\delta}(-n)_r\left(-\frac{y}{w}\right)_r}{r!\,(1+\beta)_r\,s^r}A_{n,r}\right|$$

$$=\lim_{r\to\infty}\left|\frac{(-n)_r}{(r-1)!\,r^{-n}}\cdot\frac{\left(-\frac{y}{w}\right)_r}{(r-1)!\,r^{-y/w}}\cdot\frac{(r-1)!\,r^{1+\beta}}{(1+\beta)_r}\cdot\frac{(r-1)!\,r^{1+\delta}}{s^r\,r!\,r^{1+\beta+\frac{y}{w}+n}}\right|\lim_{r\to\infty}|A_{n,r}|$$

$$= \frac{\rho}{\Gamma(-n)} \cdot \frac{1}{\Gamma\left(-\frac{y}{w}\right)} \cdot \frac{\Gamma(1+\beta)}{1} \cdot \lim_{r\to\infty} \left| \frac{1}{s^r \ r^{1+\beta+\frac{y}{w}+n-\delta}} \right|$$

where $\lim_{r\to\infty} \left| A_{n,r} \right| = \rho$, is finite

$$= 0 \qquad \textbf{--- (2.4.4)}$$

Thus, $\operatorname{Re}(\frac{y}{w}+1+\beta+n-\delta) = 2\delta-\delta=\delta>0$ and 's' is finite. Hence, the function

$B_n^{(\alpha,\beta)}(x,y,w)$ is absolutely convergent on $|w\,s|=1$ when $\operatorname{Re}(\frac{y}{w}+1+\beta+n)>0$.

(ii) Convergence for $Z_n^{\alpha}(x;w,k)$

The convergence of $Z_n^{\alpha}(x;w,k)$ also can be proved in the same manner since it is also a terminating series.

2.5 GENERATING RELATIONS

Generating relations always play an important role in the study of polynomials or functions. The term generating relation has been further classified as linear generating relation, bilinear generating relation and bilateral generating relation. The complete definitions of these terms have already been discussed in Chapter-I. In this section linear generating relations for the functions $B_n^{(\alpha,\beta)}(x,y,w)$ and $Z_n^{\alpha}(x;w,k)$ have been derived.

2.5.1 Linear Generating Relation for $B_n^{(\alpha,\beta)}(x,y,w)$

Suppose x is not an integral multiple of w or $w \to 0$.

Theorem 1

$$\sum_{n=0}^{\infty} \frac{n!\ B_n^{(\alpha,\beta)}(x,y,w)\,t^n}{(1+\alpha)_n\ (1+\beta)_n} = e^t\ {}_1F_1\left(-\frac{y}{w};1+\beta;wt\right)\ {}_1F_1\left(\frac{x}{w};1+\alpha;-wt\right) \qquad \textbf{---(2.5.1)}$$

Proof :

Consider the series

$$\sum_{n=0}^{\infty} \frac{n!\ B_n^{(\alpha,\beta)}(x,y,w)\,t^n}{(1+\alpha)_n\ (1+\beta)_n}$$

$$= \sum_{n=0}^{\infty} \frac{n!\ \Gamma(n+\alpha+1)\ \Gamma(n+\beta+1)}{n!(1+\alpha)_n\ (1+\beta)_n} \sum_{r=0}^{n} \frac{(-1)^r\ y^{[r\,w]}\ J_{n-r}^{\alpha}(x,w)\,t^n}{r!\Gamma(n+\alpha-r+1)\ \Gamma(\mathrm{r}+\beta+1)}$$

Changing $(n+r)$ for 'n' in the last summation,

$$= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha+1)\ \Gamma(n+\beta+1)}{(1+\alpha)_n\ (1+\beta)_n} \sum_{r=0}^{\infty} \frac{(w)^r \left(-\frac{y}{w}\right)_r J_n^{\alpha}(x,w)\,t^{n+r}}{r!\ \Gamma(n+\alpha+1)\ \Gamma(r+\beta+1)}$$

$$= \sum_{n=0}^{\infty} \frac{J_n^{\alpha}(x,w)\,t^n\ \Gamma(n+\beta+1)}{(1+\alpha)_n\ \Gamma(n+\beta+1)} \sum_{r=0}^{\infty} \frac{w^r \left(-\frac{y}{w}\right)_r t^r\ \Gamma(\beta+1)}{r!\Gamma(\beta+r+1)}$$

$$= \sum_{n=0}^{\infty} \frac{J_n^{\alpha}(x,w)\,t^n}{(1+\alpha)_n} \sum_{r=0}^{\infty} \frac{\left(-\frac{y}{w}\right)_r w^r\ t^r}{r!\ (1+\beta)_r}$$

$$= e^t\ {}_1F_1\left(-\frac{y}{w};1+\beta;wt\right)\ {}_1F_1\left(\frac{x}{w};1+\alpha;-wt\right) \qquad \textbf{--- (2.5.2)}$$

Applications:

(i) Taking the limit $w \to 0$ in (2.5.2), it leads to

$$\sum_{n=0}^{\infty} \frac{n!L_n^{(\alpha,\beta)}(x,y)\,t^n}{(1+\alpha)_n\,(1+\beta)_n} = e^t \,{}_0F_1(-;1+\alpha;-xt)\,{}_0F_1(-;1+\beta;-yt) \qquad \textbf{--- (2.5.3)}$$

where $L_n^{(\alpha,\beta)}(x,y)$ is Laguerre polynomial of two variables, please see Chatterjea, S.K. [46].

(ii) Applying the limit $w \to 0$ and $y = 0$ in (2.5.2), it reduces to

$$\frac{n!L_n^{(\alpha,\beta)}(x,0)}{(1+\beta)_n} = L_n^{(\alpha)}(x) \qquad \textbf{--- (2.5.4)}$$

For the above result, please see Chatterjea, S.K. [46].

(iii) Applying the limit $w \to 0$ and $x = 0$ in (2.5.2), it reduces to

$$\frac{n!L_n^{(\alpha,\beta)}(0,y)}{(1+\alpha)_n} = L_n^{(\beta)}(y) \qquad \textbf{--- (2.5.5)}$$

For the above result, please see Chatterjea, S.K. [46].

(iv) The applications (ii) and (iii), in particular, yields to the interesting results

$$L_n^{(\alpha,0)}(x,0) = L_n^{(\alpha)}(x) \qquad \textbf{--- (2.5.6)}$$

$$L_n^{(\alpha,0)}(x,0) = L_n^{(\alpha)}(x) \qquad \textbf{--- (2.5.7)}$$

For the above result, please see Chatterjea, S.K. [46].

2.5.2 Linear Generating Relation for $Z_n^{\alpha}(x;w,k)$

Suppose that either x is not an integral multiple of w or $w \to 0$.

Theorem 2

$$\sum_{n=0}^{\infty} \frac{Z_n^{\alpha}(x;w,k)}{(1+\alpha)_{nk}} w^n \left(\frac{t}{w}\right)_n$$

$$= (1-w)^{-t/w} \; {}_{k+1}F_k\left[\begin{matrix} \frac{t}{w}, \frac{x}{w}, \frac{x}{w}-1, ---, \frac{x}{w}-k+1; \\ \Delta(k;1+\alpha) \end{matrix} ; \frac{-w}{1-w}\left(\frac{w}{k}\right)^k\right] \qquad \textbf{--- (2.5.8)}$$

Proof:

Consider $\sum_{n=0}^{\infty} \frac{Z_n^{\alpha}(x;w,k)}{(1+\alpha)_{nk}} w^n \left(\frac{t}{w}\right)_n$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{t}{w}\right)_n w^n \, \Gamma(kn+\alpha+1)}{(1+\alpha)_{nk} \quad n!} \sum_{j=0}^{n} \frac{(-1)^j \, n_{C_j} \; {}^{[jw]}\left\{x^{[kw]}\right\}}{\Gamma(kj+\alpha+1)}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(-1)^j \, n!\left(\frac{t}{w}\right)_n w^n \; \Gamma(1+\alpha) \; {}^{[jw]}\left\{x^{[kw]}\right\}}{n! \; j! \; (n-j)! \; \Gamma(kn+\alpha+1)}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(-1)^j \left(\frac{t}{w}\right)_n w^n \; {}^{[jw]}\left\{x^{[kw]}\right\}}{j! \; (n-j)! \; (1+\alpha)_{kj}}$$

Replacing '*n*' by '*n+j*', it results in

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{t}{w}\right)_{n+j} w^{n+j} \; {}^{[jw]}\left\{x^{[kw]}\right\}}{j! \; (n)! \; (1+\alpha)_{kj}}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{t}{w}\right)_j w^j \; {}^{[jw]}\left\{x^{[kw]}\right\}}{j! \; (1+\alpha)_{kj}} \sum_{n=0}^{\infty} \frac{\left(\frac{t}{w}+j\right)_n w^n}{n!}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{t}{w}\right)_j w^j \; {}^{[jw]}\left\{x^{[kw]}\right\}}{j! \; (1+\alpha)_{kj}} (1-w)^{-\frac{t}{w}-j}$$

By using (1.5.17), the following can be obtained

$$=(1-w)^{-\frac{t}{w}}\sum_{j=0}^{\infty}\frac{\left(\frac{t}{w}\right)_j\left(\frac{x}{w}\right)_j\left(\frac{x}{w}-1\right)_j---\left(\frac{x}{w}-k+1\right)_j(-w)^j\,w^{kj}}{j!\;(1+\alpha)_{kj}\,(1-w)^j}$$

$$=(1-w)^{-\frac{t}{w}}\;{}_{k+1}F_k\left[\begin{matrix}\frac{t}{w},\frac{x}{w},\left(\frac{x}{w}-1\right),---\left(\frac{x}{w}-k+1\right); & -\frac{w}{(1-w)}\left(\frac{w}{k}\right)^k\\ \Delta(k;1+\alpha) \qquad ; & \end{matrix}\right] \quad \text{--- (2.5.9)}$$

Applications:

(i) When $w\to 0$ in (2.5.9), it leads to

$$\sum_{n=0}^{\infty}\frac{Z_n^{\alpha}(x;k)\,t^n}{(1+\alpha)_{nk}}=e^t\;{}_0F_k\left[-;\frac{\alpha+1}{k},\frac{\alpha+2}{k},---,\frac{\alpha+k}{k};-\frac{x^k}{k^k}t\right] \quad \text{--- (2.5.10)}$$

which is the generating function obtained by Srivastava, H.M. [199], where $Z_n^{\alpha}(x;k)$ is the Konhauser's polynomial.

(ii) Substituting k = 1 in (2.5.9), it reduces to

$$\sum_{n=0}^{\infty}\frac{J_n^{\alpha}(x,w)\left(\frac{t}{w}\right)_n w^n}{(1+\alpha)_n}=(1-w)^{-\frac{t}{w}}\;{}_2F_1\left[\begin{matrix}\frac{t}{w},\frac{x}{w}; & -\frac{w^2}{1-w}\\ 1+\alpha\;; & \end{matrix}\right] \quad \text{--- (2.5.11)}$$

which is the generating function given by Parihar and Patel [154], where $J_n^{\alpha}(x,w)$ is the Modified Jacobi polynomial.

(iii) Applying $w\to 0$ and k = 1 in (2.5.9), it leads to the known generating function on Laguerre Polynomials [161].

$$\sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)\, t^n}{(1+\alpha)_n} = e^t\ {}_0F_1\left[-;1+\alpha;-xt\right] \qquad \text{--- (2.5.12)}$$

Theorem 3

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n Z_n^{\alpha}(x;w,k)\, t^n}{(1+\alpha)_{nk}}$$

$$= (1-t)^{-\lambda}\ {}_{k+1}F_k\left[\begin{matrix}\lambda, \frac{x}{w}, \frac{x}{w}-1, ---, \frac{x}{w}-k+1; \\ \Delta(k;1+\alpha) \qquad ;\end{matrix} \frac{t}{t-1}\left(\frac{w}{k}\right)^k\right] \qquad \text{--- (2.5.13)}$$

Proof:

Consider $\sum_{n=0}^{\infty} \frac{(\lambda)_n Z_n^{\alpha}(x;w,k)\, t^n}{(1+\alpha)_{nk}}$

$$= \sum_{n=0}^{\infty} \frac{(\lambda)_n t^n\, \Gamma(\alpha+1)\Gamma(kn+\alpha+1)}{\Gamma(nk+\alpha+1)\quad n!} \sum_{j=0}^{n} \frac{(-1)^j\, n!\ {}^{[jw]}\left\{x^{[kw]}\right\}}{j!\ (n-j)!\ \Gamma(kj+\alpha+1)}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(-1)^j (\lambda)_n\, t^n\ {}^{[jw]}\left\{x^{[kw]}\right\} \Gamma(\alpha+1)}{j!\ (n-j)!\ \Gamma(kj+\alpha+1)}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(-1)^j (\lambda)_n\, t^n\ {}^{[jw]}\left\{x^{[kw]}\right\}}{j!\ (n-j)!\ (1+\alpha)_{kj}}$$

Replacing '*n*' by '*n*+*j*', it results in

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j (\lambda)_{n+j}\, t^{n+j}\ {}^{[jw]}\left\{x^{[kw]}\right\}}{j!\ (n)!\ (1+\alpha)_{kj}}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j (\lambda)_j\, t^j\ {}^{[jw]}\left\{x^{[kw]}\right\}}{j!\quad (1+\alpha)_{kj}} \sum_{n=0}^{\infty} \frac{(\lambda+j)_n\, t^n}{n!}$$

$$=\sum_{j=0}^{\infty} \frac{(-1)^j (\lambda)_j \; t^j \; {}^{[jw]}\left\{x^{[kw]}\right\}}{j! \quad (1+\alpha)_{kj}} (1-t)^{-\lambda-j}$$

By using (1.5.17), the following can be obtained

$$=(1-t)^{-\lambda} \sum_{j=0}^{\infty} \frac{(-1)^j (\lambda)_j \left(\frac{x}{w}\right)_j \left(\frac{x}{w}-1\right)_j --- \left(\frac{x}{w}-k+1\right)_j t^j \, w^{kj}}{j! \quad (1+\alpha)_{kj} \; (1-t)^j}$$

$$=(1-t)^{-\lambda} \sum_{j=0}^{\infty} \frac{(-1)^j (\lambda)_j \left(\frac{x}{w}\right)_j \left(\frac{x}{w}-1\right)_j --- \left(\frac{x}{w}-k+1\right)_j -t^j w^{kj}}{j! \; k^{kj} \left(\frac{\alpha+1}{k}\right)_j \left(\frac{\alpha+2}{k}\right)_j --- \left(\frac{\alpha+k}{k}\right)_j (1-t)^j}$$

$$= (1-t)^{-\lambda} \; {}_{k+1}F_k \left[\begin{matrix} \lambda, \frac{x}{w}, \frac{x}{w}-1, ---, \frac{x}{w}-k+1; \\ \Delta(k;1+\alpha) \qquad ; \end{matrix} \; \frac{t}{t-1}\left(\frac{w}{k}\right)^k \right]$$

--- (2.5.14)

Applications:

(i) When $w \to 0$ in (2.5.14), it leads to

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n Z_n^{\alpha}(x;k) t^n}{(1+\alpha)_{nk}} = (1-t)^{-\lambda} \; {}_1F_k \left[\lambda; \frac{\alpha+1}{k}, \frac{\alpha+2}{k}, ---, \frac{\alpha+k}{k}; \frac{x^k}{(\mathrm{t}-1)k^k} t \right]$$

--- (2.5.15)

which is the generating function obtained by Srivastava, H.M. [199], where $Z_n^{\alpha}(x;k)$ is the Konhauser's polynomial.

(ii) Writing k = 1 in (2.5.14), it reduces to

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n J_n^{\alpha}(x,w)\left(\frac{t}{w}\right)_n w^n}{(1+\alpha)_n} = (1-t)^{-\lambda} {}_2F_1\left[\begin{matrix}\lambda, \frac{x}{w}; & -\frac{wt}{1-t} \\ 1+\alpha\ ; & \end{matrix}\right]$$ **--- (2.5.16)**

which is the generating function given by Parihar and Patel [154], where $J_n^{\alpha}(x,w)$ is the Modified Jacobi polynomial.

(iii) Applying $w \to 0$ and k = 1 in (2.5.14), it leads to the known generating function on Laguerre Polynomials [205].

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n L_n^{\alpha}(x)\ t^n}{(1+\alpha)_n} = (1-t)^{-\lambda} {}_1F_1\left[\lambda; 1+\alpha; \frac{xt}{t-1}\right]$$ **--- (2.5.17)**

Theorem 4

$$\sum_{n=0}^{\infty} \frac{Z_n^{\alpha}(x;w,k)t^n}{(1+\alpha)_{nk}} = \mathrm{e}^t\ {}_kF_k\left[\begin{matrix}\frac{x}{w}, \frac{x}{w}-1, ---, \frac{x}{w}-k+1; & -t\left(\frac{w}{k}\right)^k \\ \Delta(k;1+\alpha) \qquad ; & \end{matrix}\right]$$ **--- (2.5.18)**

Proof :

Consider $\sum_{n=0}^{\infty} \frac{Z_n^{\alpha}(x;w,k)t^n}{(1+\alpha)_{nk}}$

$$= \sum_{n=0}^{\infty} \frac{t^n\, \Gamma(\alpha+1)\ \Gamma(kn+\alpha+1)}{\Gamma(kn+\alpha+1)\quad n!} \sum_{j=0}^{n} \frac{(-1)^j\ n!\ {}^{[jw]}\left\{x^{[kw]}\right\}}{j!(n-j)!\ \Gamma(kj+\alpha+1)}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{t^n\,(-1)^j\, \Gamma(\alpha+1)\ {}^{[jw]}\left\{x^{[kw]}\right\}}{j!(n-j)!\Gamma(kj+\alpha+1)}$$

Replacing 'n' by '$n+j$', it results in

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^{n+j} (-1)^{j} \; {}^{[jw]}\left\{x^{[kw]}\right\}}{j!\, n!\, (1+\alpha)_{kj}}$$

$$= e^{t} \sum_{j=0}^{\infty} \frac{(-1)^{j} \left(\frac{x}{w}\right)_{j} \left(\frac{x}{w}-1\right)_{j} ---\left(\frac{x}{w}-k+1\right)_{j} w^{kj}\, t^{j}}{j!\, k^{kj} \left(\frac{\alpha+1}{k}\right)_{j} \left(\frac{\alpha+2}{k}\right)_{j} ---\left(\frac{\alpha+k}{k}\right)_{j}}$$

$$= e^{t} \; {}_{k}F_{k}\left[\begin{matrix} \frac{x}{w}, \frac{x}{w}-1, ---\frac{x}{w}-k+1; \\ \Delta(k;\alpha+1) \qquad ; \end{matrix} -t\frac{w^{k}}{k^{k}}\right] \qquad \textbf{--- (2.5.19)}$$

Applications:

(i) When $w \to 0$ in (2.5.19), it leads to

$$\sum_{n=0}^{\infty} \frac{Z_n^{\alpha}(x;k)\, t^{n}}{(1+\alpha)_{nk}} = e^{t} \; {}_{0}F_{k}\left[-; \frac{\alpha+1}{k}, \frac{\alpha+2}{k}, ---, \frac{\alpha+k}{k}; \frac{-x^{k}}{k^{k}} t\right] \qquad \textbf{--- (2.5.20)}$$

which is the generating function obtained by Srivastava, H.M. [199], where $Z_n^{\alpha}(x;k)$ is the Konhauser's polynomial.

(ii) By putting k = 1 in (2.5.19), it reduces to

$$\sum_{n=0}^{\infty} \frac{J_n^{\alpha}(x,w)\; t^{n}}{(1+\alpha)_{n}} = \mathrm{e}^{t} \; {}_{1}F_{1}\left[\begin{matrix} \frac{x}{w} \;\; ; \\ 1+\alpha \; ; \end{matrix} -wt\right] \qquad \textbf{--- (2.5.21)}$$

where $J_n^{\alpha}(x,w)$ is the Modified Jacobi polynomial.

(iii) By taking $w \to 0$ and k = 1 in (2.5.19), it leads to the famous generating function on Laguerre Polynomials [161].

$$\sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)\ t^n}{(1+\alpha)_n} = e^t\ {}_0F_1\left[-;1+\alpha;-xt\right]$$ **--- (2.5.22)**

Theorem 5

When $\lambda = 1+\alpha$ in Theorem (3), it reduces to

$$\sum_{n=0}^{\infty} Z_n^{\alpha}(x;w,k)\,t^n = \frac{1}{(1-t)^{1+\alpha}}\ {}_kF_0\left[\begin{matrix}\frac{x}{w},\frac{x}{w}-1,---,\frac{x}{w}-k+1; & -\frac{w^k t}{(1-t)^k}\\ - & ;\end{matrix}\right]$$ **--- (2.5.23)**

This can also be written as

$$\sum_{n=0}^{\infty} Z_n^{\alpha}(x;w,k)\,t^n = \frac{1}{(1-t)^{1+\alpha}}\left[1-\frac{w^k t}{(1-t)^k}\right]$$ **---(2.5.24)**

Proof:

Consider $\sum_{n=0}^{\infty} Z_n^{\alpha}(x;w,k)\,t^n$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{\Gamma(kn+\alpha+1)\ (-1)^j\ n!^{\ [jw]}\left\{x^{[kw]}\right\}\Gamma(\alpha+1)}{n!\ \ \Gamma(\alpha+1)\ j!\ \ (n-j)!\Gamma(kj+\alpha+1)}\,t^n$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{(\alpha+1)_{kn}\ \ (-1)^{j\ \ [jw]}\left\{x^{[kw]}\right\}}{j!\ \ (n-j)!\ (\alpha+1)_{kj}}\,t^n$$

Replacing ‘*n*’ by ‘*n+j*’, it results in

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \frac{(\alpha+1)_{k(n+j)}\ \ (-1)^{j\ \ [jw]}\left\{x^{[kw]}\right\}}{j!\ \ n!\ (\alpha+1)_{kj}}\,t^{n+j}$$

$$=\sum_{j=0}^{\infty}\frac{(-1)^j \; {}^{[jw]}\left\{x^{[kw]}\right\} t^j}{j!}\sum_{n=0}^{\infty}\frac{(\alpha+1+kj)_{nk}\, t^n}{n!}$$

$$=\sum_{j=0}^{\infty}\frac{(-1)^j \; {}^{[jw]}\left\{x^{[kw]}\right\} t^j}{j!}(1-t)^{-1-\alpha-kj}$$

By using (1.5.17), the following can be obtained

$$=(1-t)^{-1-\alpha}\sum_{j=0}^{\infty}\frac{\left(\frac{x}{w}\right)_j\left(\frac{x}{w}-1\right)_j---\left(\frac{x}{w}-k+1\right)_j\; w^{kj}(-1)^j\, t^j}{j!\;(1-t)^{kj}}$$

$$=(1-t)^{-1-\alpha}\;{}_kF_0\left[\begin{matrix}\frac{x}{w},\frac{x}{w}-1,---,\frac{x}{w}-k+1; & -\frac{w^k}{(1-t)^k}t\\ - & ;\end{matrix}\right] \qquad \textbf{--- (2.5.25)}$$

Applications:

(i) By putting k = 1 in (2.5.24), it reduces to (See Gen.fn. (2.4) of Parihar and Patel, V.1 (1979), 41-46, J.Indian Acad.Maths.)

$$\sum_{n=0}^{\infty} J_n^{\alpha}(x,w)\; t^n=\frac{1}{(1-t)^{1+\alpha}}[1-\frac{wt}{1-t}]^{-x/w} \qquad \textbf{--- (2.5.26)}$$

which is the generating function given by Parihar and Patel [154], where $J_n^{\alpha}(x,w)$ is the Modified Jacobi polynomial.

(ii) By taking $w\to 0$ and k = 1 in (2.5.24), it leads to the famous generating function on Laguerre Polynomials [161]

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x)\; t^n=\frac{1}{(1-t)^{1+\alpha}}\,\mathrm{e}^{-xt/(1-t)} \qquad \textbf{--- (2.5.27)}$$

Theorem 6

$$\sum_{n=0}^{\infty} \frac{Z_n^{\alpha}(x;w,k)}{\Gamma(kn+\alpha+1)} t^n = \frac{e^t}{\Gamma(\alpha+1)} {}_kF_k \left[\begin{matrix} \frac{x}{w}, \frac{x}{w}-1, ---, \frac{x}{w}-k+1; \\ \Delta(k;\alpha+1) \quad ; \end{matrix} -\frac{w^k t}{k^k} \right] \qquad \text{--- (2.5.28)}$$

Proof:

Consider $\sum_{n=0}^{\infty} \frac{Z_n^{\alpha}(x;w,k)}{\Gamma(kn+\alpha+1)} t^n$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(kn+\alpha+1)\, t^n}{\Gamma(kn+\alpha+1)\, n!} \sum_{j=0}^{n} \frac{(-1)^j\, n_{C_j}\ {}^{[jw]}\left\{x^{[kw]}\right\}}{\Gamma(kj+\alpha+1)}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(-1)^j\ {}^{[jw]}\left\{x^{[kw]}\right\}\, t^n}{j!\,(n-j)!\,\Gamma(kj+\alpha+1)}$$

Replacing 'n' by '$n+j$', it results in

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j\ {}^{[jw]}\left\{x^{[kw]}\right\}\, t^{n+j}}{j!\, n!\, \Gamma(kj+\alpha+1)}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j\ {}^{[jw]}\left\{x^{[kw]}\right\}\, t^j}{j!\ \Gamma(\alpha+1)\ (\alpha+1)_{kj}} \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

$$= \frac{e^t}{\Gamma(\alpha+1)} \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{x}{w}\right)_j \left(\frac{x}{w}-1\right)_j --- \left(\frac{x}{w}-k+1\right)_j w^{kj}\ t^j}{j!\, k^{kj} \left(\frac{\alpha+1}{k}\right)_j \left(\frac{\alpha+2}{k}\right)_j --- \left(\frac{\alpha+k}{k}\right)_j}$$

$$= \frac{e^t}{\Gamma(\alpha+1)} {}_kF_k \left[\begin{matrix} \frac{x}{w}, \frac{x}{w}-1, ---, \frac{x}{w}-k+1; \\ \Delta(k;\alpha+1) \quad ; \end{matrix} -t\frac{w^k}{k^k} \right]$$

--- (2.5.29)

Applications:

(i) When $w \to 0$ in (2.5.28), it leads to the known generating function on Konhauser's polynomial.

$$\sum_{n=0}^{\infty} \frac{Z_n^{\alpha}(x;k)t^n}{(1+\alpha)_{nk}} = \frac{e^t}{\Gamma(\alpha+1)}\, {}_0F_k\left[-;\frac{\alpha+1}{k},\frac{\alpha+2}{k},---,\frac{\alpha+k}{k};\frac{-x^k}{k^k}t\right]$$

--- (2.5.30)

(ii) By writing k = 1 in (2.5.28), it reduces to the generating function on Modified Jacobi polynomial.

$$\sum_{n=0}^{\infty} \frac{J_n^{\alpha}(x,w)\ t^n}{\Gamma(1+\alpha+n)} = \frac{\mathrm{e}^t}{\Gamma(\alpha+1)}\, {}_1F_1\left[\begin{matrix}\frac{x}{w}\ ; \\ 1+\alpha\ ;\end{matrix} -wt\right]$$

--- (2.5.31)

(iii) By applying $w \to 0$ and k = 1 in (2.5.28), it leads to the famous generating function on Laguerre Polynomials [161].

$$\sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)\ t^n}{(1+\alpha)_n} = \mathrm{e}^t\ {}_0F_1\left[-;1+\alpha;-xt\right]$$

--- (2.5.32)

Theorem 7

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n Z_n^{\alpha}(x;w,k)}{\Gamma(kn+\alpha+1)}t^n = \frac{(1-t)^{-\lambda}}{\Gamma(\alpha+1)}\, {}_{k+1}F_k\left[\begin{matrix}\lambda,\frac{x}{w},\frac{x}{w}-1,---,\frac{x}{w}-k+1; \\ \Delta(k;\alpha+1)\qquad ;\end{matrix} \frac{w^k t}{k^k(t-1)}\right]$$

--- (2.5.33)

Proof:

Consider $\sum_{n=0}^{\infty} \frac{(\lambda)_n Z_n^{\alpha}(x;w,k)}{\Gamma(kn+\alpha+1)}t^n$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(\lambda)_n (-1)^j \, {}^{[jw]}\left\{x^{[kw]}\right\} t^n}{j!\,(n-j)!\,\Gamma(kj+\alpha+1)}$$

Replacing '*n*' by '*n*+*j*', it results in

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\lambda)_{n+j} (-1)^j \, {}^{[jw]}\left\{x^{[kw]}\right\} t^{n+j}}{j!\, n!\, \Gamma(kj+\alpha+1)}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j (\lambda)_j \, {}^{[jw]}\left\{x^{[kw]}\right\} t^j}{j!\ \Gamma(\alpha+1)\ (\alpha+1)_{kj}} \sum_{n=0}^{\infty} \frac{(\lambda+j)_n t^n}{n!}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j (\lambda)_j \, {}^{[jw]}\left\{x^{[kw]}\right\} t^j}{j!\ \Gamma(\alpha+1)\ (\alpha+1)_{kj}} (1-t)^{-\lambda-j}$$

$$= \frac{(1-t)^{-\lambda}}{\Gamma(\alpha+1)} \sum_{j=0}^{\infty} \frac{(-1)^j (\lambda)_j \left(\frac{x}{w}\right)_j \left(\frac{x}{w}-1\right)_j \cdots \left(\frac{x}{w}-k+1\right)_j w^{kj}\ t^j}{j!\, k^{kj} \left(\frac{\alpha+1}{k}\right)_j \left(\frac{\alpha+2}{k}\right)_j \cdots \left(\frac{\alpha+k}{k}\right)_j (1-t)^j}$$

$$= \frac{(1-t)^{-\lambda}}{\Gamma(\alpha+1)} {}_{k+1}F_k \left[\begin{matrix} \lambda, \frac{x}{w}, \frac{x}{w}-1, \ldots, \frac{x}{w}-k+1; \\ \Delta(k;\alpha+1) \quad ; \end{matrix} \ -\frac{w^k t}{k^k (1-t)} \right] \qquad \text{--- (2.5.34)}$$

Applications:

(i) When $w \to 0$ in (2.5.34), it leads to the known generating function on

Konhauser's polynomial.

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n Z_n^{\alpha}(x;k) t^n}{\Gamma(nk+\alpha+1)} = \frac{(1-t)^{-\lambda}}{\Gamma(\alpha+1)} {}_1F_k \left[\begin{matrix} \lambda \quad ; \\ \frac{\alpha+1}{k}, \frac{\alpha+2}{k}, \ldots, \frac{\alpha+k}{k}; \end{matrix} \ \frac{x^k t}{k^k (t-1)} \right] \qquad \text{--- (2.5.35)}$$

(ii) By putting k = 1 in (2.5.34), it reduces to generating function on Modified

Jacobi polynomial by Parihar and Patel [154].

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n J_n^{\alpha}(x,w)\, t^n}{(1+\alpha)_n} = (1-t)^{-\lambda}\; {}_2F_1\left[\begin{matrix}\lambda, \frac{x}{w}; & -\frac{wt}{1-t} \\ 1+\alpha\;; & \end{matrix}\right] \qquad \text{--- (2.5.36)}$$

(iii) By taking $w \to 0$ and k = 1 in (2.5.34), it leads to the famous generating function on Laguerre Polynomials [205].

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n L_n^{\alpha}(x)\, t^n}{(1+\alpha)_n} = (1-t)^{-\lambda}\; {}_1F_1\left[\lambda; 1+\alpha;\; \frac{xt}{t-1}\right] \qquad \text{--- (2.5.37)}$$

2.6 EXTENDED LINEAR GENERATING RELATION

In this article, a class of extended linear generating relation for $Z_n^{\alpha}(x;w,k)$ is derived.

Theorem 8

$$\sum_{n=0}^{\infty} \binom{m+n}{n} \frac{Z_{m+n}^{\alpha}(x;w,k)t^n}{(1+\alpha)_{k(m+n)}}$$

$$= \sum_{n=m}^{\infty} \binom{n}{m} \frac{t^{n-m}\,(-x^k)^{n\,[nw]}\{x^{[kw]}\}}{n!}\; {}_1F_1[n+1; n-m+1; t] \qquad \text{--- (2.6.1)}$$

Proof:

Considering the double series

$$\sum_{m=0}^{\infty} z^m \sum_{n=0}^{\infty} \binom{m+n}{n} \frac{Z_{m+n}^{\alpha}(x;w,k)t^n}{(1+\alpha)_{k(m+n)}}$$

$$=\sum_{m=0}^{\infty} z^m \sum_{n=0}^{\infty} \binom{m+n}{m} \frac{Z_{m+n}^{\alpha}(x;w,k)t^n}{(1+\alpha)_{k(m+n)}}$$

Replacing '*n*' by '(*n*-*m*)', it results in

$$=\sum_{n=0}^{\infty} \frac{Z_{m+n}^{\alpha}(x;w,k)}{(1+\alpha)_{kn}} \sum_{m=0}^{\infty} \binom{n}{m} t^{n-m}\ z^m$$

$$=\sum_{n=0}^{\infty} \frac{Z_{n}^{\alpha}(x;w,k)}{(1+\alpha)_{kn}}\ (z+t)^n$$

Using (2.5.18), the following can be obtained

$$= e^{z+t}\ {}_kF_k\left[\begin{matrix} \frac{x}{w}, \frac{x}{w}-1, \ldots, \frac{x}{w}-k+1; \\ \Delta(k;1+\alpha) \qquad ; \end{matrix}\ -(z+t)\frac{w^k}{k^k}\right]$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{x}{w}\right)_n \left(\frac{x}{w}-1\right)_n \ldots \left(\frac{x}{w}-k+1\right)_n (-1)^n\ (z+t)^n\, w^{nk}}{(1+\alpha)_{nk}\ n!} e^{z+t}$$

$$= \sum_{n=0}^{\infty}\ \sum_{\nu=0}^{\infty} \frac{\left(\frac{x}{w}\right)_n \left(\frac{x}{w}-1\right)_n \ldots \left(\frac{x}{w}-k+1\right)_n (-1)^n\ (z+t)^n\, w^{nk}\,(z+t)^{\nu}}{(1+\alpha)_{nk}\ n!\,\nu!}$$

$$= \sum_{n=0}^{\infty}\ \sum_{\nu=0}^{\infty} \frac{(-1)^n \left(\frac{x}{w}\right)_n \left(\frac{x}{w}-1\right)_n \ldots \left(\frac{x}{w}-k+1\right)_n (z+t)^{n+\nu}\, w^{nk}}{(1+\alpha)_{nk}\ n!\ \nu!}$$

$$= \sum_{n,\nu=0}^{\infty} \frac{(-1)^n \left(\frac{x}{w}\right)_n \left(\frac{x}{w}-1\right)_n \ldots \left(\frac{x}{w}-k+1\right)_n w^{nk}}{(1+\alpha)_{nk}\ n!\ \nu!} \sum_{m=0}^{n+\nu} \binom{n+\nu}{m} z^m t^{n+\nu-m}$$

$$= \sum_{m=0}^{\infty} z^m \sum_{n+\nu\ge m} \frac{t^{n-m}(-1)^n \left(\frac{x}{w}\right)_n \left(\frac{x}{w}-1\right)_n \ldots \left(\frac{x}{w}-k+1\right)_n n!(n+1)_{\nu}\, w^{nk} t^{\nu}}{(n-m)!\,(n-m+1)_{\nu}\,(1+\alpha)_{nk}\ m!n!\ \nu!}$$

$$= \sum_{m=0}^{\infty} z^m \sum_{n=0}^{\infty} \frac{n_{C_m} t^{n-m} (-1)^n \left(\frac{x}{w}\right)_n \left(\frac{x}{w}-1\right)_n \ldots \left(\frac{x}{w}-k+1\right)_n w^{nk}}{(1+\alpha)_{nk} \; n!}$$

$$\times \sum_{v=0}^{\infty} \frac{(n+1)_v \; t^v}{(n-m+1)_v \; v!}$$

$$= \sum_{m=0}^{\infty} z^m \sum_{n=0}^{\infty} \frac{n_{C_m} t^{n-m} (-1)^n \left(\frac{x}{w}\right)_n \left(\frac{x}{w}-1\right)_n \ldots \left(\frac{x}{w}-k+1\right)_n w^{nk}}{(1+\alpha)_{nk} \; n!}$$

$$\times {}_1F_1\left[\begin{matrix} n+1 \; ; \\ n-m+1; \end{matrix} t\right]$$

On equating the coefficient of Z^m, it results in

$$\sum_{n=0}^{\infty} \binom{m+n}{n} \frac{Z_{m+n}^{\alpha}(x;w,k)t^n}{(1+\alpha)_{k(m+n)}}$$

$$= \sum_{n=m}^{\infty} \frac{n_{C_m} t^{n-m} (-1)^n \left(\frac{x}{w}\right)_n \left(\frac{x}{w}-1\right)_n \ldots \left(\frac{x}{w}-k+1\right)_n w^{nk}}{(1+\alpha)_{nk} \; n!} \, {}_1F_1\left[\begin{matrix} n+1 \; ; \\ n-m+1; \end{matrix} t\right]$$

$$= \sum_{n=m}^{\infty} \frac{n_{C_m} t^{n-m} (-1)^n \; {}^{[nw]}\left\{x^{[kw]}\right\}}{(1+\alpha)_{nk} \; n!} \, {}_1F_1\left[\begin{matrix} n+1 \; ; \\ n-m+1; \end{matrix} t\right] \qquad \text{--- (2.6.2)}$$

Applications:

(i) If m = 0, the result (2.6.1) becomes,

$$\sum_{n=0}^{\infty} \frac{Z_n^{\alpha}(x;w,k)t^n}{(1+\alpha)_{kn}} = \sum_{n=0}^{\infty} \frac{n_{C_0} (-t)^n \; {}^{[nw]}\left\{x^{[kw]}\right\}}{(1+\alpha)_{nk} \; n!} e^t$$

$$= e^t \sum_{n=0}^{\infty} \frac{\left(\frac{x}{w}\right)_n \left(\frac{x}{w}-1\right)_n \cdots \left(\frac{x}{w}-k+1\right)_n \left(-t\frac{w^k}{k^k}\right)^n}{\Delta_n(k;1+\alpha)\; n!}$$

$$= e^t \, {}_kF_k\left(\begin{matrix} \frac{x}{w}, \frac{x}{w}-1, \ldots, \frac{x}{w}-k+1; \\ \Delta(k;1+\alpha) \qquad ; \end{matrix} \; -t\frac{w^k}{k^k}\right) \qquad \textbf{--- (2.6.3)}$$

which is the known generating function (2.5.18).

(ii) If $w \to 0$, the result (2.6.1) leads to

$$\sum_{n=0}^{\infty} \binom{m+n}{n} \frac{Z^{\alpha}_{m+n}(x;k)t^n}{(1+\alpha)_{k(m+n)}} = \sum_{n=m}^{\infty} \frac{n_{C_m}\, t^{n-m}\,(-x^k)^n}{(1+\alpha)_{nk}\; n!} \, {}_1F_1\left[\begin{matrix} n+1 \;\; ; \\ n-m+1; \end{matrix} \, t\right] \qquad \textbf{--- (2.6.4)}$$

which is the known result obtained by Srivastava, H.M. [199].

(iii) By taking k = 1 and $w \to 0$, the result (2.6.1) leads to

$$\sum_{n=0}^{\infty} \binom{m+n}{n} \frac{L^{\alpha}_{m+n}(x)\,t^n}{(1+\alpha)_{(m+n)}}$$

$$= \sum_{n=m}^{\infty} \frac{n_{C_m}\, t^{n-m}\,(-1)^n\; x^n}{(1+\alpha)_n\; n!} \, {}_1F_1\left[\begin{matrix} n+1 \;\; ; \\ n-m+1; \end{matrix} \, t\right]$$

$$= \sum_{n=m}^{\infty} \sum_{j=0}^{\infty} \frac{n!\,t^{n-m}\,(-x)^n\,(n+1)_j\; t^j}{m!\,(n-m)!(1+\alpha)_n\; n!\,(n-m+1)_j\;\; j!}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(n+1)_j(-x)^n\;\; t^j}{(1+\alpha)_n\; n!\,(1)_j\;\; j!} \qquad \textbf{--- (2.6.5)}$$

2.7 SERIES EXPANSIONS

In this article, a large number of series expansions involving generalized hypergeometric function $B_n^{(\alpha,\beta)}(x,y,w)$ are derived. These new theorems involve not only generalized hypergeometric function but they also involve the Laguerre polynomial of two and one variable.

Theorem 9

$$\sum_{t=0}^{m} \frac{{}^{m}C_t\,(-n)_t\,[(n-t)!]^2\,B_{n-t}^{(\alpha+t,\beta)}(x+tw,y,w)\left(\frac{x}{w}\right)_t w^t}{(1+\alpha)_t\,(1+\alpha-m)_t\,(1+\alpha+t)_{n-t}\,(1+\beta)_{n-t}}$$

$$=\frac{(n!)^2}{(1+\alpha-m)_n\,(1+\beta)_n}\,B_n^{(\alpha-m,\beta)}(x,y,w) \qquad \text{--- (2.7.1)}$$

Proof:

Consider $\displaystyle\sum_{t=0}^{m} \frac{{}^{m}C_t\,(-n)_t\,[(n-t)!]^2\,B_{n-t}^{(\alpha+t,\beta)}(x+tw,y,w)\left(\frac{x}{w}\right)_t w^t}{(1+\alpha)_t\,(1+\alpha-m)_t\,(1+\alpha+t)_{n-t}\,(1+\beta)_{n-t}}$

$$=\sum_{t=0}^{m} \frac{{}^{m}C_t\,(-n)_t\,[(n-t)!]^2\left(\frac{x}{w}\right)_t w^t\,(1+\alpha+t)_{n-t}\,(1+\beta)_{n-t}}{(1+\alpha)_t\,(1+\alpha-m)_t\,(1+\alpha+t)_{n-t}\,(1+\beta)_{n-t}\,[(n-t)!]^2}$$

$$\times\sum_{r=0}^{n-t}\sum_{s=0}^{n-t-r}\frac{(-n+t)_{r+s}\left(-\frac{y}{w}\right)_s\left(\frac{x}{w}+t\right)_r(-w)^s w^r}{r!\,s!\,(1+\alpha+t)_r\,(1+\beta)_s}$$

$$=\sum_{t=0}^{m} \frac{{}^{m}C_t\,(-n)_t\left(\frac{x}{w}\right)_t w^t}{(1+\alpha)_t\,(1+\alpha-m)_t}\sum_{r=0}^{n-t}\sum_{s=0}^{n-t-r}\frac{(-n+t)_{r+s}\left(-\frac{y}{w}\right)_s\left(\frac{x}{w}+t\right)_r(-w)^s w^r}{r!\,s!\,(1+\alpha+t)_r\,(1+\beta)_s}$$

$$=\sum_{t=0}^{m} \frac{{}^{m}C_t\,(-n)_t\,w^t}{(1+\alpha)_t\,(1+\alpha-m)_t}\sum_{r=0}^{n-t}\sum_{s=0}^{n-t-r}\frac{(-n+t)_{r+s}\left(-\frac{y}{w}\right)_s\left(\frac{x}{w}\right)_{r+t}(-w)^s w^r}{r!\,s!\,(1+\alpha+t)_r\,(1+\beta)_s}$$

Here put $r = p\text{-}t$ and change the order of summation so that first summation becomes the last so getting

$$= \sum_{t=0}^{m} \frac{m_{C_t} (-n)_t}{(1+\alpha)_t (1+\alpha-m)_t} \sum_{p=0}^{p+s} \sum_{s=0}^{\leq n} \frac{(-n+t)_{p-t+s} \left(-\frac{y}{w}\right)_s \left(\frac{x}{w}\right)_p (-w)^s w^p}{(p-t)!\, s!\, (1+\alpha+t)_{p-t} (1+\beta)_s}$$

$$= \sum_{t=0}^{m} \frac{m_{C_t} (-1)^t\, n!\, \Gamma(1+\alpha)}{(n-t)!\, \Gamma(1+\alpha+t) (1+\alpha-m)_t}$$

$$\times \sum_{p=0}^{p+s} \sum_{s=0}^{\leq n} \frac{(-1)^{p-t+s} (n-t)!\, \Gamma(1+\alpha+t)}{(n-t-p+t-s)!\ \Gamma(1+\alpha+t+p-t)}$$

$$\times \frac{\left(-\frac{y}{w}\right)_s \left(\frac{x}{w}\right)_p (-w)^s w^p}{(p-t)!\, s!\ (1+\beta)_s}$$

$$= \sum_{t=0}^{m} \frac{m_{C_t} (-1)^t n!}{(1+\alpha-m)_t} \sum_{p=0}^{p+s} \sum_{s=0}^{n} \frac{(-1)^{p+s} (-1)^t\ \Gamma(1+\alpha) \left(-\frac{y}{w}\right)_s \left(\frac{x}{w}\right)_p (-w)^s w^p}{(n-p-s)!\ \Gamma(1+\alpha+p)(p-t)!\, s!\ (1+\beta)_s}$$

$$= \sum_{p=0}^{p+s} \sum_{s=0}^{n} \frac{(-n; p+s) \left(-\frac{y}{w}\right)_s \left(\frac{x}{w}\right)_p (-w)^s w^p}{p!\, s!\, (1+\alpha)_p (1+\beta)_s} \sum_{t=0}^{m} \frac{(-m)_t\ (-p)_t}{t!\, (1+\alpha-m)_t}$$

$$= \sum_{p=0}^{p+s} \sum_{s=0}^{n} \frac{(-n; p+s) \left(-\frac{y}{w}\right)_s \left(\frac{x}{w}\right)_p (-w)^s w^p}{p!\, s!\, (1+\alpha)_p (1+\beta)_s}\ {}_2F_1\left(-p, -m; 1+\alpha-m; 1\right)$$

Using (1.6.11), then it results in

$$= \sum_{p=0}^{p+s} \sum_{s=0}^{n} \frac{(-n;\, p+s)\left(-\frac{y}{w}\right)_s \left(\frac{x}{w}\right)_p (-w)^s w^p\, \Gamma(1+\alpha)}{p!\, s!\, (1+\beta)_s\, \Gamma(1+\alpha+p)}$$

$$\times \frac{\Gamma(1+\alpha+p)\, \Gamma(1+\alpha-m)}{\Gamma(1+\alpha-m+p)\, \Gamma(1+\alpha)}$$

$$= \sum_{p=0}^{p+s} \sum_{s=0}^{n} \frac{(-n;\, p+s)\left(-\frac{y}{w}\right)_s \left(\frac{x}{w}\right)_p (-w)^s w^p}{p!\, s!\, (1+\beta)_s\, (1+\alpha-m)_p}$$

$$= \frac{(n!)^2 (1+\alpha-m)_n\, (1+\beta)_n}{(1+\alpha-m)_n\, (1+\beta)_n\, (n!)^2} \sum_{p=0}^{p+s} \sum_{s=0}^{\leq n} \frac{(-n)_{p+s}\left(-\frac{y}{w}\right)_s \left(\frac{x}{w}\right)_p (-w)^s w^p}{p!\, s!\, (1+\beta)_s\, (1+\alpha-m)_p}$$

$$= \frac{(n!)^2}{(1+\alpha-m)_n (1+\beta)_n} B_n^{(\alpha-m,\beta)}(x,y,w)$$

--- (2.7.2)

Applications:

(i) Applying $w \to 0$ to (2.7.1), it leads to

$$\sum_{t=0}^{m} \frac{{}^{m}C_t\, (-n)_t\, [(n-t)!]^2\, x^t\, L_{n-t}^{(\alpha+t,\beta)}(x,y)}{(1+\alpha)_t\, (1+\alpha-m)_t\, (1+\alpha+t)_{n-t}\, (1+\beta)_{n-t}} = \frac{(n!)^2\, L_n^{(\alpha+m,\beta)}(x,y)}{(1+\alpha-m)_n\, (1+\beta)_n}$$

--- (2.7.3)

which is the known result obtained by Ragab, S.F. [159].

Theorem 10

$$\sum_{t=0}^{m} \frac{{}^{m}C_t\, (-n)_t\, [(n-t)!]^2\, B_{n-t}^{(\alpha,\beta+t)}(x, y-tw, w)\left(-\frac{y}{w}\right)_t (-w)^t}{(1+\beta)_t\, (1+\beta-m)_t\, (1+\beta+t)_{n-t}\, (1+\alpha)_{n-t}}$$

$$=\frac{(n!)^2}{(1+\beta-m)_n\,(1+\alpha)_n}\,B_n^{(\alpha,\beta-m)}(x,y,w) \qquad \textbf{--- (2.7.4)}$$

Proof:

Consider $\sum_{t=0}^{m}\ \frac{m_{C_t}\,(-n)_t\,[(n-t)!]^2\left(-\frac{y}{w}\right)_t(-w)^t(1+\beta+t)_{n-t}\,(1+\alpha)_{n-t}}{(1+\beta)_t\,(1+\beta-m)_t\,(1+\beta+t)_{n-t}\,(1+\alpha)_{n-t}\,[(n-t)!]^2}$

$$\times\sum_{r=0}^{n-t}\ \sum_{s=0}^{n-t-r}\ \frac{(-n+t)_{r+s}\left(\frac{x}{w}\right)_s\left(-\frac{y}{w}+t\right)_r(-w)^r\,w^s}{r!\,s!\,(1+\beta+t)_r\,(1+\alpha)_s}$$

$$=\sum_{t=0}^{m}\ \frac{m_{C_t}\,(-n)_t\left(-\frac{y}{w}\right)_t(-w)^t}{(1+\beta)_t\,(1+\beta-m)_t}\sum_{r=0}^{n-t}\ \sum_{s=0}^{n-t-r}\ \frac{(-n+t)_{r+s}\left(\frac{x}{w}\right)_s\left(-\frac{y}{w}+t\right)_r(-w)^r\,w^s}{r!\,s!\,(1+\beta+t)_r\,(1+\alpha)_s}$$

$$=\sum_{t=0}^{m}\ \frac{m_{C_t}\,(-n)_t\,(-w)^t}{(1+\beta)_t\,(1+\beta-m)_t}\sum_{r=0}^{n-t}\ \sum_{s=0}^{n-t-r}\ \frac{(-n+t)_{r+s}\left(\frac{x}{w}\right)_s\left(-\frac{y}{w}\right)_{r+t}(-w)^r\,w^s}{r!\,s!\,(1+\beta+t)_r\,(1+\alpha)_s}$$

Here put $r = p\text{-}t$ and change the order of summation so that first summation becomes the last so getting

$$=\sum_{t=0}^{m}\ \frac{m_{C_t}\,(-n)_t\,(-w)^t}{(1+\beta)_t\,(1+\beta-m)_t}\sum_{p=0}^{p+s}\ \sum_{s=0}^{\le\,n}\ \frac{(-n+t)_{p-t+s}\left(\frac{x}{w}\right)_s\left(-\frac{y}{w}\right)_p(w)^s\,(-w)^{p-t}}{(p-t)!\,s!\,(1+\alpha+t)_{p-t}\,(1+\beta)_s}$$

$$=\sum_{t=0}^{m}\ \frac{m_{C_t}\,(-1)^t\,n!\,\Gamma(1+\beta)}{(n-t)!\,\Gamma(1+\beta+t)\,(1+\beta-m)_t}$$

$$\times\sum_{p=0}^{p+s}\ \sum_{s=0}^{\le\,n}\ \frac{(-1)^{p-t+s}(n-t)!\,\Gamma(1+\beta+t)\left(-\frac{y}{w}\right)_p\left(\frac{x}{w}\right)_s(-w)^p\,w^s}{(n-t-p+t-s)!\ \Gamma(1+\beta+t+p-t)\,(1+\alpha)_s\,(p-t)!\,s!}$$

$$=\sum_{t=0}^{m}\frac{m_{C_t}(-1)^t n!}{(1+\beta-m)_t}\sum_{p=0}^{p+s}\sum_{s=0}^{n}\frac{(-1)^{p+s}(-1)^{-t}\,\Gamma(1+\beta)\left(-\frac{y}{w}\right)_p\left(\frac{x}{w}\right)_s(-w)^p w^s}{(n-p-s)!\,\Gamma(1+\beta+p)(p-t)!\,s!\,(1+\alpha)_s}$$

$$=\sum_{p=0}^{p+s}\sum_{s=0}^{n}\frac{(-n;p+s)\left(-\frac{y}{w}\right)_p\left(\frac{x}{w}\right)_s(-w)^p w^s}{p!\,s!\,(1+\alpha)_s(1+\beta)_p}\sum_{t=0}^{m}\frac{(-m)_t\,(-p)_t}{t!\,(1+\beta-m)_t}$$

$$=\sum_{p=0}^{p+s}\sum_{s=0}^{n}\frac{(-n;p+s)\left(-\frac{y}{w}\right)_p\left(\frac{x}{w}\right)_s(-w)^p w^s}{p!\,s!\,(1+\alpha)_s(1+\beta)_p}\,{}_2F_1\left(-p,-m;1+\beta-m;1\right)$$

Using (1.6.11), the following can be obtained

$$=\sum_{p=0}^{p+s}\sum_{s=0}^{n}\frac{(-n;p+s)\left(-\frac{y}{w}\right)_p\left(\frac{x}{w}\right)_s(-w)^p w^s\,\Gamma(1+\beta)}{p!\,s!\,(1+\alpha)_s\,\Gamma(1+\beta+p)}$$

$$\times\frac{\Gamma(1+\beta+p)\,\Gamma(1+\beta-m)}{\Gamma(1+\beta-m+p)\,\Gamma(1+\beta)}$$

$$=\sum_{p=0}^{p+s}\sum_{s=0}^{n}\frac{(-n;p+s)\left(-\frac{y}{w}\right)_p\left(\frac{x}{w}\right)_s(-w)^p w^s}{p!\,s!\,(1+\alpha)_s\,(1+\beta-m)_p}$$

$$=\frac{(n!)^2(1+\beta-m)_n\,(1+\alpha)_n}{(1+\beta-m)_n\,(1+\alpha)_n\,(n!)^2}\sum_{p=0}^{p+s}\sum_{s=0}^{\leq n}\frac{(-n)_{p+s}\left(-\frac{y}{w}\right)_p\left(\frac{x}{w}\right)_s(-w)^p w^s}{p!\,s!\,(1+\alpha)_s\,(1+\beta-m)_p}$$

$$=\frac{(n!)^2}{(1+\beta-m)_n(1+\alpha)_n}B_n^{(\alpha,\beta-m)}(x,y,w) \qquad \textbf{--- (2.7.5)}$$

Applications:

(i) By taking $w\to 0$ in (2.7.4), it leads to

$$\sum_{t=0}^{m} \frac{m_{C_t}\,(-n)_t\,[(n-t)!]^2\, y^t\, L_{n-t}^{(\alpha,\beta+t)}(x,y)}{(1+\beta)_t\,(1+\beta-m)_t\,(1+\beta+t)_{n-t}\,(1+\alpha)_{n-t}} = \frac{(n!)^2\, L_n^{(\alpha,\beta-m)}(x,y)}{(1+\beta-m)_n\,(1+\alpha)_n} \quad \text{--- (2.7.6)}$$

which is the known result obtained by Ragab, S.F. [159].

Theorem 11

$$\sum_{t=0}^{n} \frac{(-\beta-n)_t \left(\frac{z}{w}\right)_t B_{n-t}^{(\alpha+t,\beta)}(x,y,w)\,(-w)^t}{(-n)_t\; t!} = B_n^{(\alpha,\beta)}(x+z,y,w) \quad \text{--- (2.7.7)}$$

Proof:

Consider $\displaystyle\sum_{t=0}^{n} \frac{(-\beta-n)_t \left(\frac{z}{w}\right)_t (-w)^t (1+\alpha+t)_{n-t} (1+\beta)_{n-t}}{(-n)_t\; t!\,[(n-t)!]^2}$

$$\times \sum_{r=0}^{n-t} \sum_{s=0}^{n-t-r} \frac{(-n+t)_{r+s} \left(-\frac{y}{w}\right)_r \left(\frac{x}{w}\right)_s (-w)^r w^s}{r!\, s!\,(1+\alpha+t)_s (1+\beta)_r}$$

$$= \sum_{t=0}^{n} \frac{(1+\beta)_n \left(\frac{z}{w}\right)_t (w)^t (1+\alpha+t)_{n-t} \left(-n\right)_t}{t!\,(n!)^2}$$

$$\times \sum_{r=0}^{n-t} \sum_{s=0}^{n-t-r} \frac{(-n+t)_{r+s} \left(-\frac{y}{w}\right)_r \left(\frac{x}{w}\right)_s (-w)^r w^s}{r!\, s!\,(1+\alpha+t)_s\; (1+\beta)_r}$$

$$= \sum_{t=0}^{n} \frac{(1+\beta)_n \left(\frac{z}{w}\right)_t (w)^t (1+\alpha+t)_{n-t}}{t!\,(n!)^2}$$

$$\times \sum_{r=0}^{n-t} \sum_{s=0}^{n-t-r} \frac{(-n)_{t+r+s} \left(-\frac{y}{w}\right)_r \left(\frac{x}{w}\right)_s (-w)^r w^s}{r!\, s!\,(1+\alpha+t)_s\; (1+\beta)_r}$$

Changing '*p-t*' for '*s*', it results in

$$=\sum_{t=0}^{n} \frac{(1+\beta)_n \left(\frac{z}{w}\right)_t (w)^t (1+\alpha+t)_{n-t}}{t!\,(n!)^2}$$

$$\times\sum_{r=0}^{n-t} \sum_{p=0}^{n-r} \frac{(-n)_{p+r} \left(-\frac{y}{w}\right)_r \left(\frac{x}{w}\right)_{p-t} (-w)^r w^{p-t}}{r!\,(p-t)!\,(1+\alpha+t)_{p-t}\,(1+\beta)_r}$$

$$=\sum_{t=0}^{n} \frac{(1+\beta)_n \left(\frac{z}{w}\right)_t (1+\alpha)_n}{t!\,(n!)^2} \sum_{r=0}^{n-t} \sum_{p=0}^{n-r} \frac{(-n)_{p+r} \left(-\frac{y}{w}\right)_r \left(\frac{x}{w}\right)_{p-t} (-w)^r w^{p}}{r!\,(p-t)!\,(1+\alpha)_{p}\,(1+\beta)_r}$$

Changing the order of summation so that the first summation becomes the last and the following is obtained,

$$=\frac{(1+\alpha)_n (1+\beta)_n}{(n!)^2} \sum_{r=0}^{n} \sum_{p=0}^{n-r} \frac{(-n)_{p+r} \left(-\frac{y}{w}\right)_r \left(\frac{x}{w}\right)_{p-t} (-w)^r w^{p}}{r!\,p!\,(1+\alpha)_{p}\,(1+\beta)_r} \sum_{t=0}^{p} \frac{p!\left(\frac{z}{w}\right)_t}{(p-t)!\,t!}$$

$$=\frac{(1+\alpha)_n (1+\beta)_n}{(n!)^2} \sum_{r=0}^{n} \sum_{p=0}^{n-r} \frac{(-n)_{p+r} \left(-\frac{y}{w}\right)_r (-w)^r w^{p} \left(\frac{x+z}{w}\right)_p}{r!\,p!\,(1+\alpha)_{p}\,(1+\beta)_r}$$

$$=B_n^{(\alpha,\beta)}(x+z,y,w)$$

--- (2.7.8)

Applications:

(i) By taking $w \to 0$ in (2.7.7), it leads to

$$\sum_{t=0}^{m} \frac{(-\beta-n)_t (-z)^t L_{n-t}^{(\alpha+t,\beta)}(x,y)}{t!\,(-n)_t} = L_n^{(\alpha,\beta)}(x+z,y)$$

--- (2.7.9)

which is the known result obtained by Ragab, S.F. [159].

Theorem 12

$$\sum_{t=0}^{m} \frac{(-\alpha-n)_t \left(\frac{z}{w}\right)_t B_{n-t}^{(\alpha,\beta+t)}(x,y,w)(w)^t}{(-n)_t \; t!} = B_n^{(\alpha,\beta)}(x,y-z,w) \qquad \textbf{--- (2.7.10)}$$

Proof:

Consider $\displaystyle\sum_{t=0}^{n} \frac{(-\alpha-n)_t \left(\frac{z}{w}\right)_t \; w^t (1+\alpha)_{n-t} (1+\beta+t)_{n-t}}{(-n)_t \; t! \, [(n-t)!]^2}$

$$\times \sum_{r=0}^{n-t} \sum_{s=0}^{n-t-r} \frac{(-n+t)_{r+s} \left(-\frac{y}{w}\right)_r \left(\frac{x}{w}\right)_s (-w)^r w^s}{r! \, s! \, (1+\alpha)_s (1+\beta+t)_r}$$

$$= \sum_{t=0}^{n} \frac{(-\alpha-n)_t \left(\frac{z}{w}\right)_t \; w^t \; (-1)^t (1+\alpha)_n (1+\beta)_n \left[(-n)_t\right]^2}{(-n)_t \; t! \, [(n-t)!]^2 \, (-\alpha-n)_t \; (n!)^2 \, (1+\beta)_t}$$

$$\times \sum_{r=0}^{n-t} \sum_{s=0}^{n-t-r} \frac{(-n+t)_{r+s} \left(-\frac{y}{w}\right)_r \left(\frac{x}{w}\right)_s (-w)^r w^s}{r! \, s! \, (1+\alpha)_s (1+\beta+t)_r}$$

$$= \frac{(1+\alpha)_n (1+\beta)_n}{(n!)^2} \sum_{t=0}^{n} \sum_{r=0}^{n-t} \sum_{s=0}^{n-t-r} \frac{(-n)_{t+r+s} \left(-\frac{y}{w}\right)_r \left(\frac{x}{w}\right)_s \left(\frac{z}{w}\right)_t w^t (-w)^r w^s (-1)^t}{r! \, s! \, t! \;\; (1+\alpha)_s \; (1+\beta)_{r+t}}$$

Changing '*p*-*t*' for '*r*', it results in

$$= \frac{(1+\alpha)_n (1+\beta)_n}{(n!)^2} \sum_{t=0}^{n} \sum_{p=t}^{n} \sum_{s=0}^{n-p} \frac{(-n)_{p+s} \left(-\frac{y}{w}\right)_{p-t} \left(\frac{x}{w}\right)_s \left(\frac{z}{w}\right)_t w^t (-w)^{p-t} w^s (-1)^t}{(p-t)! \, s! \, t! \;\; (1+\alpha)_s \; (1+\beta)_p}$$

Changing the order of summation, so that the first summation becomes the last and it can be obtained as

$$=\frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2}\sum_{p=0}^{n}\sum_{s=0}^{n-p}\frac{(-n)_{p+s}\left(\frac{x}{w}\right)_s(-w)^p w^s}{p!\,s!\quad(1+\alpha)_s\,(1+\beta)_p}\sum_{t=0}^{p}\frac{\left(-\frac{y}{w}\right)_{p-t}\left(\frac{z}{w}\right)_t p!}{(p-t)!\,t!}$$

$$=\frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2}\sum_{p=0}^{n}\sum_{s=0}^{n-p}\frac{(-n)_{p+s}\left(-\left(\frac{y-z}{w}\right)\right)_p\left(\frac{x}{w}\right)_s(-w)^p w^s}{p!\,s!\quad(1+\alpha)_s\,(1+\beta)_p}$$

$$=B_n^{(\alpha,\beta)}(x,y-z,w) \qquad \textbf{--- (2.7.11)}$$

Applications:

(i) By taking $w\to 0$ in (2.7.10), it leads to

$$\sum_{t=0}^{m}\frac{(-\alpha-n)_t\,(z)^t\,L_{n-t}^{(\alpha,\beta+t)}(x,y)}{t!\,(-n)_t}=L_n^{(\alpha,\beta)}(x,y-z) \qquad \textbf{--- (2.7.12)}$$

which is the known result obtained by Ragab, S.F. [159].