CHAPTER-V

INTEGRAL REPRESENTATIONS FOR THE GENERALIZED HYPERGEOMETRIC FUNCTION $B_n^{(\alpha,\beta)}(x, y, w)$ AND MODIFIED KONHAUSER'S POLYNOMIAL $Z_n^{\alpha}(x; w, k)$

5.1 This chapter consists of two sections. In the section-I, an attempt has been made to derive different types of integral representations for the generalized hypergeometric function $B_n^{(\alpha,\beta)}(x, y, w)$ and in the section-II the similar integral representations for the modified Konhauser's polynomial $Z_n^{\alpha}(x; w, k)$. In all, we have obtained the following different types of integral representations, viz;

i) A contour integral representation

ii) Real integral representation

iii) Finite single integral representation

iv) Infinite single integral representation

v) Finite double integral representation

To obtain these representations, the following well known results have been used:

1) Maclaurin's theorem

If, $f(u) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, u^n}{n!}$, **---(5.1.1)**

then the coefficients $f^n(0),\ n = 0,1,2,....$ are given by means of the integrals

$$f^n(0) = \frac{n!}{2\pi i} \int^{(0+)} \frac{f(u)\, du}{u^{n+1}}, \quad n = 0,1,2,--- \qquad \textbf{---(5.1.2)}$$

2) $\int_0^{\pi/2} Cos^m\theta\ Cos(n\theta)\ d\theta = \frac{\Gamma(m+1)\ \pi}{2^{m+1}\ \Gamma(\frac{m+n+2}{2})\ \Gamma(\frac{m-n+2}{2})}$ **--- (5.1.3)**

3) $\int_0^1 t^{s-1} (\log t)^{x-1}\ dt = \frac{(-1)^{x-1}\,\Gamma(x)}{s^x}$, $\mathrm{Re}(x) > 1\ \ and\ \ s > 1.$ **--- (5.1.4)**

4) If Re(x) > 0, Re(y) > 0, Re(λ) > 0, then

$$\iint_A u^{x-1} v^{y-1}\ (1-u-v)^{\lambda-1}\ du\ dv = \frac{\Gamma(x)\Gamma(y)\ \Gamma(\lambda)}{\Gamma(x+y+\lambda)}, \qquad \textbf{--- (5.1.5)}$$

where the area A is bound by $u \ge 0,\ v \ge 0\ \ and\ \ u + v \le 1$.

5) If Re(m) > 0 and Re(n) > 0, then

$$\int_0^1 \int_0^1 \frac{(1-x)^{m-1}\ y^m\ (1-y)^{n-1}}{(1-xy)^{m+n-1}} dx\ dy = \beta(x,y)\,. \qquad \textbf{--- (5.1.6)}$$

SECTION-I

5.2 INTEGRAL REPRESENTATIONS FOR $B_n^{(\alpha,\beta)}(x,y,w)$

I. A CONTOUR INTEGRAL REPRESENTATION

From the generating relation (2.5.1), we have

$$\sum_{n=0}^{\infty} \frac{n!\, B_n^{(\alpha,\beta)}(x,y,w)}{(1+\alpha)_n\,(1+\beta)_n} t^n = e^t \; {}_1F_1\left[\begin{matrix} -\dfrac{y}{w}\;; \\ 1+\beta\;; \end{matrix}\; wt\right] {}_1F_1\left[\begin{matrix} \dfrac{x}{w}\;; \\ 1+\alpha\;; \end{matrix}\; -wt\right] \qquad \text{--- (5.2.1)}$$

Now, on taking

$$f(u) \;=\; e^u \; {}_1F_1\left[\begin{matrix} -\dfrac{y}{w}\;; \\ 1+\beta\;; \end{matrix}\; wu\right] {}_1F_1\left[\begin{matrix} \dfrac{x}{w}\;; \\ 1+\alpha\;; \end{matrix}\; -wu\right] \qquad \text{--- (5.2.2)}$$

and using the Maclaurin’s theorem

$$f(u) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, u^n}{n!}, \qquad \text{--- (5.2.3)}$$

We find the coefficients $f^n(0),\; n = 0,1,2,....$ are given by means of the integrals

$$f^n(0) = \frac{n!}{2\pi i} \int^{(0+)} \frac{f(u)\, du}{u^{n+1}}, \quad n = 0,1,2,.... \qquad \text{--- (5.2.4)}$$

Thus from (5.2.1), (5.2.2), (5.2.3) and (5.2.4), we arrive at the following theorem:

Theorem 1

If $e^{u}\ {}_1F_1\left[\begin{matrix}-\frac{y}{w}\ ; & wu\\ 1+\beta\,; & \end{matrix}\right]{}_1F_1\left[\begin{matrix}\frac{x}{w}\ ; & -wu\\ 1+\alpha\,; & \end{matrix}\right]=\sum_{n=0}^{\infty}\frac{n!\,B_n^{(\alpha,\beta)}(x,y,w)}{(1+\alpha)_n\,(1+\beta)_n}u^n$ then

$B_n^{(\alpha,\beta)}(x,y,w)$

$$=\frac{(1+\alpha)_n\,(1+\beta)_n}{n!\,(2\pi i)}\int^{0+}u^{-n-1}e^{u}\ {}_1F_1\left[\begin{matrix}-\frac{y}{w}\ ; & wu\\ 1+\beta\,; & \end{matrix}\right]{}_1F_1\left[\begin{matrix}\frac{x}{w}\ ; & -wu\\ 1+\alpha\,; & \end{matrix}\right]du\,, \qquad \textbf{--- (5.2.5)}$$

where the contour of integration encircles the origin of the u-plane in the positive direction.

II. THE REAL INTEGRAL REPRESENTATION

From (5.2.5), we get (on using the contour $u=e^{i\theta}$)

$$B_n^{(\alpha,\beta)}(x,y,w)\ =\ \frac{(1+\alpha)_n\,(1+\beta)_n}{n!\,(2\pi i)}\int_0^{2\pi}(e^{i\theta})^{-n-1}e^{e^{i\theta}}\ {}_1F_1\left[\begin{matrix}-\frac{y}{w}\ ; & we^{i\theta}\\ 1+\beta\,; & \end{matrix}\right]$$

$$\times\ {}_1F_1\left[\begin{matrix}\frac{x}{w}\ ; & -we^{i\theta}\\ 1+\alpha\,; & \end{matrix}\right](e^{i\theta}\,i\,d\theta)$$

$$=\frac{(1+\alpha)_n\,(1+\beta)_n}{n!\,(2\pi)}\sum_{r,s,m=0}^{\infty}\frac{\left(-\frac{y}{w}\right)_r\left(\frac{x}{w}\right)_s w^r\,(-w)^s}{(1+\beta)_r\,(1+\alpha)_s\,m!\,r!\,s!}\int_0^{2\pi}Cis(m-n+r+s)\theta\ \ d\theta$$

where $Cis\,\theta=(Cos\,\theta+i\,Sin\,\theta)$.

Consequently, we arrive at

$$B_n^{(\alpha,\beta)}(x,y,w)$$

$$=\frac{(1+\alpha)_n\,(1+\beta)_n}{n!\;\pi}\sum_{r,s,m=0}^{\infty}\frac{\left(-\frac{y}{w}\right)_r\left(\frac{x}{w}\right)_s w^r\,(-w)^s}{(1+\beta)_r\,(1+\alpha)_s\;m!\;r!\;s!}\int_0^{\pi} Cis(m-n+r+s)\theta\;d\theta$$

--- (5.2.6)

III. FINITE SINGLE INTEGRAL REPRESENTATION

We know that for Re(a) > 0 and Re(b-a) > 0,

$$\frac{(a)_j}{(b)_j}=\frac{\Gamma(b)}{\Gamma(a)\;\Gamma(b-a)}\int_0^1 t^{a+j-1}(1-t)^{b-a-1}\,dt \qquad \textbf{--- (5.2.7)}$$

Now, on considering

$$\frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2}\;F^{1:1;1}_{-:1;1}\begin{bmatrix} -n:-\frac{y}{w};\frac{x}{w} & ; & -w,w \\ -:1+\beta;1+\alpha; & & \end{bmatrix}$$

$$=\frac{(1+\alpha)_n\,(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\left(-\frac{y}{w}\right)_r\left(\frac{x}{w}\right)_s(-w)^r\,w^s\,(b)_r}{r!\;s!\;(1+\alpha)_s\,(1+\beta)_r\,(a)_r}$$

$$\times\frac{\Gamma(b)}{\Gamma(a)\;\Gamma(b-a)}\int_0^1 t^{a+r-1}(1-t)^{b-a-1}\,dt$$

$$=\frac{\Gamma(b)}{\Gamma(a)\;\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}$$

$$\times\left\{\frac{(1+\alpha)_n\,(1+\beta)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\left(-\frac{y}{w}\right)_r\left(\frac{x}{w}\right)_s(-wt)^r\,w^s\,(b)_r}{r!\;s!\;(1+\alpha)_s\,(1+\beta)_r\,(a)_r}\right\}dt$$

We thus arrive at the following

Theorem 2

If Re(a) > 0 and Re(b-a) > 0, then

$$B_n^{(\alpha,\beta)}(x,y,w) = \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)} \int_0^1 t^{a-1}(1-t)^{b-a-1}$$

$$\times \frac{(1+\alpha)_n (1+\beta)_n}{(n!)^2} \; F^{1:2;1}_{-:2;1}\left[\begin{matrix} -n : -\frac{y}{w}, b; \frac{x}{w} \quad ; \\ - : 1+\beta, a; 1+\alpha; \end{matrix} -wt, w\right] dt \qquad \textbf{--- (5.2.8)}$$

Theorem 3

If $\operatorname{Re}(a) > -\frac{1}{2}$ *and* $\operatorname{Re}(b) > -\frac{1}{2}$, *then*

$$B_n^{(\alpha,\beta)}(x,y,w) = \frac{\Gamma(a+b)(1+\alpha)_n (1+\beta)_n}{\Gamma(a)\Gamma(b)(n!)^2} \; 2\int_0^{\pi/2} (Sin\,t)^{2a-1}(Cos\,t)^{2b-1}$$

$$\times F^{1:3;1}_{-:4;1}\left[\begin{matrix} -n : -\frac{y}{w}, a+b; \frac{x}{w}; \\ - : 1+\beta, a, b; 1+\alpha; \end{matrix} -w\,Sin^2 t\,Cos^2 t, w\right] dt \qquad \textbf{--- (5.2.9)}$$

Proof:

From (2.3.2), we have

$$B_n^{(\alpha,\beta)}(x,y,w) = \frac{(1+\alpha)_n (1+\beta)_n}{(n!)^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} \left(-\frac{y}{w}\right)_r \left(\frac{x}{w}\right)_s (-w)^r w^s}{r!\, s!\, (1+\alpha)_s (1+\beta)_r}$$

$$\times \frac{(a+b)_r (a)_r (b)_r}{(b)_r (a)_r (a+b)_r}$$

$$= \frac{\Gamma(a+b)(1+\alpha)_n (1+\beta)_n}{\Gamma(a)\Gamma(b)\ (n!)^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(a+b)_r \left(-\frac{y}{w}\right)_r \left(\frac{x}{w}\right)_s (-w)^r w^s}{r!\ s!\ (1+\alpha)_s (a)_r (b)_r (1+\beta)_r}$$

$$\times 2 \int_0^{\pi/2} (Sin\,t)^{2(a+r)-1} (Cos\,t)^{2(b+r)-1}\, dt$$

$$= \frac{\Gamma(a+b)(1+\alpha)_n (1+\beta)_n}{\Gamma(a)\Gamma(b)\ (n!)^2}\ 2\int_0^{\pi/2} (Sin\,t)^{2a-1} (Cos\,t)^{2b-1}$$

$$\times F^{1:3;1}_{-:4;1}\left[\begin{matrix} -n:-\frac{y}{w}, a+b;\ \frac{x}{w}; \\ -:1+\beta, a, b; 1+\alpha; \end{matrix}\ -w\,Sin^2 t\,Cos^2 t, w\right] dt$$

IV. INFINITE SINGLE INTEGRAL REPRESENTATION

From (2.3.2), we have

$$B_n^{(\alpha,\beta)}(x,y,w) = \frac{(1+\alpha)_n (1+\beta)_n}{(n!)^2}\ F^{1:1;1}_{-:1;1}\left[\begin{matrix} -n:-\frac{y}{w}; \frac{x}{w} \quad ; \\ -:1+\beta; 1+\alpha; \end{matrix}\ -w, w\right]$$

$$= \frac{(1+\alpha)_n (1+\beta)_n}{(n!)^2\,\Gamma(a)} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} \left(-\frac{y}{w}\right)_r \left(\frac{x}{w}\right)_s (-w)^r w^s\,\Gamma(a+r)}{r!\ s!\ (1+\alpha)_s (1+\beta)_r\ (a)_r}$$

$$= \frac{(1+\alpha)_n (1+\beta)_n}{(n!)^2\,\Gamma(a)} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} \left(-\frac{y}{w}\right)_r \left(\frac{x}{w}\right)_s (-w)^r w^s}{r!\ s!\ (1+\alpha)_s (1+\beta)_r\ (a)_r} \int_{-\infty}^{\infty} e^{-t^2} t^{(2a+2r)-1} dt$$

on making use of integral formula $\Gamma(n) = \int_{-\infty}^{\infty} e^{-t^2} t^{2n-1}\, dt$

Hence, we get

Theorem 4

If $\text{Re}(a) > \frac{1}{2}$, then

$$B_n^{(\alpha,\beta)}(x,y,w) = \frac{(1+\alpha)_n\,(1+\beta)_n}{(n!)^2\;\Gamma(a)} \int_{-\infty}^{\infty} e^{-t^2} t^{2a-1}\, F^{1:1;1}_{-:2;1}\left[\begin{matrix} -n & : & -\frac{y}{w}, \frac{x}{w}; & -wt^2, w \\ - : 1+\beta, a; 1+\alpha; & & & \end{matrix}\right] dt$$

--- (5.2.10)

V. DOUBLE INTEGRAL REPRESENTATION

We know from Srivastava and Karlsson [201,p.275] that

If Re(a) > 0, Re(b) > 0, Re(λ) > 0, then

$$\iint u^{a-1} v^{b-1} (1-u-v)^{\lambda-1}\, du\, dv = \frac{\Gamma(a)\Gamma(b)\Gamma(\lambda)}{\Gamma(a+b+\lambda)} \;;\; u \ge 0, v \ge 0, u+v \le 1$$

Hence

$$B_n^{(\alpha,\beta)}(x,y,w) = \frac{(1+\alpha)_n\,(1+\beta)_n}{(n!)^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} \left(-\frac{y}{w}\right)_r \left(\frac{x}{w}\right)_s (-w)^r w^s}{r!\; s!\; (1+\alpha)_s\,(1+\beta)_r}$$

$$\times \frac{\Gamma(\lambda)\left(\frac{\lambda}{2}\right)_r \left(\frac{\lambda}{2}+\frac{1}{2}\right)_r \Gamma(a+r)\Gamma(b+r)\Gamma(\lambda-a-b)}{\Gamma(\lambda-a-b)\,\Gamma(a)\,\Gamma(b)\,(a)_r\,(b)_r\,\Gamma(\lambda+2r)}$$

$$= \frac{\Gamma(\lambda)\,(1+\alpha)_n\,(1+\beta)_n}{\Gamma(\lambda-a-b)\,\Gamma(a)\,\Gamma(b)\,(n!)^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} \left(\frac{\lambda}{2}\right)_r \left(\frac{\lambda}{2}+\frac{1}{2}\right)_r}{r!\; s!\; (a)_r\,(b)_r}$$

$$\times \frac{\left(-\frac{y}{w}\right)_r \left(\frac{x}{w}\right)_s (-w)^r w^s}{(1+\beta)_r\,(1+\alpha)_s} \iint u^{a+r-1} v^{b+r-1} (1-u-v)^{\lambda-a-b-1}\, du\, dv$$

Finally, we get

Theorem 5

If Re(a) > 0, Re(b) > 0, Re(λ) > 0, then

$$B_n^{(\alpha,\beta)}(x,y,w) = \frac{\Gamma(\lambda)(1+\alpha)_n(1+\beta)_n}{\Gamma(\lambda-a-b)\Gamma(a)\Gamma(b)(n!)^2}\iint u^{a-1}v^{b-1}(1-u-v)^{\lambda-a-b-1}$$

$$\times F^{1:3;1}_{-:3;1}\left[\begin{matrix}-n:\frac{\lambda}{2},\frac{\lambda}{2}+\frac{1}{2},-\frac{y}{w};\frac{x}{w} & ; \\ -:a,b,1+\beta\ ;1+\alpha & ;\end{matrix}-uvw,w\right]du\ dv \qquad \textbf{--- (5.2.11)}$$

5.3 INTEGRAL REPRESENTATIONS FOR $z_n^{\alpha}(x;w,k)$

I. A CONTOUR INTEGRAL REPRESENTATION

From the generating relation (2.5.18), we have

$$\sum_{n=0}^{\infty}\frac{z_n^{\alpha}(x;w,k)}{(1+\alpha)_{nk}}u^n = e^u\ {}_kF_k\left[\begin{matrix}\frac{x}{w},\frac{x}{w}-1,\ldots,\frac{x}{w}-k+1; \\ \Delta(k;1+\alpha)\qquad ;\end{matrix}-u(\frac{w}{k})^k\right] \qquad \textbf{--- (5.3.1)}$$

Now, on taking

$$f(u) = e^u\ {}_kF_k\left[\begin{matrix}\frac{x}{w},\frac{x}{w}-1,\ldots,\frac{x}{w}-k+1; \\ \Delta(k;1+\alpha)\qquad ;\end{matrix}-u(\frac{w}{k})^k\right] \qquad \textbf{--- (5.3.2)}$$

and using the Maclaurin’s theorem

$$f(u)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)\,u^n}{n!}, \qquad \textbf{--- (5.3.3)}$$

We find the coefficients $f^n(0),\ n=0,1,2,....$ are given by means of the integrals

$$f^n(0)=\frac{n!}{2\pi i}\int^{(0+)}\frac{f(u)\ du}{u^{n+1}},\quad n=0,1,2,.... \qquad \textbf{--- (5.3.4)}$$

Thus from (5.3.1), (5. 3.2), (5.3.3) and (5.3.4), we arrive at the following theorem:

Theorem 6

If $e^u\ {}_kF_k\left[\begin{matrix}\frac{x}{w},\frac{x}{w}-1,...,\frac{x}{w}-k+1; \\ \Delta(k;1+\alpha)\qquad ;\end{matrix}\ -u(\frac{w}{k})^k\right]=\sum_{n=0}^{\infty}\frac{z_n^\alpha(x;w,k)}{(1+\alpha)_{nk}}u^n$ then

$$z_n^\alpha(x;w,k)=\frac{(1+\alpha)_{nk}}{2\pi i}\int^{0+}u^{-n-1}e^u\ {}_kF_k\left[\begin{matrix}\frac{x}{w},\frac{x}{w}-1,...,\frac{x}{w}-k+1; \\ \Delta(k;1+\alpha)\qquad ;\end{matrix}\ -u(\frac{w}{k})^k\right]du,$$

--- (5.3.5)

where the contour of integration encircles the origin of the u-plane in the positive direction.

II. THE REAL INTEGRAL REPRESENTATION

From (5.1.5), we get (on using the contour $u=e^{i\theta}$)

$$z_n^\alpha(x;w,k)=\frac{(1+\alpha)_{nk}}{2\pi}\int_0^{2\pi}e^{-ni\theta}e^{e^{i\theta}}\ {}_kF_k\left[\begin{matrix}\frac{x}{w},\frac{x}{w}-1,...,\frac{x}{w}-k+1; \\ \Delta(k;1+\alpha)\qquad ;\end{matrix}\ -e^{i\theta}(\frac{w}{k})^k\right]d\theta$$

$$=\frac{(1+\alpha)_{nk}}{2\pi}\sum_{j,m=0}^{\infty}\frac{\left(\frac{x}{w}\right)_j\left(\frac{x}{w}-1\right)_j...\left(\frac{x}{w}-k+1\right)_j(-\frac{w^k}{k^k})^j}{\Delta(k;1+\alpha)\quad j!\quad m!}$$

$$\times\int_0^{2\pi}Cis(m+j-n)\ d\theta,$$

where $Cis\,\theta = (Cos\,\theta + i\,Sin\,\theta)$.

Consequently, we arrive at

$$z_n^{\alpha}(x;w,k) = \frac{(1+\alpha)_{nk}}{\pi} \sum_{j,m=0}^{\infty} \frac{\left(\frac{x}{w}\right)_j \left(\frac{x}{w}-1\right)_j \cdots \left(\frac{x}{w}-k+1\right)_j \left(-\frac{w^k}{k^k}\right)^j}{\Delta(k;1+\alpha) \quad j! \quad m!}$$

$$\times \int_0^{\pi} Cis(m+j-n)\; d\theta \qquad \textbf{--- (5.3.6)}$$

III. FINITE SINGLE INTEGRAL REPRESENTATION

We know that for Re(a) > 0 and Re(b-a) > 0,

$$\frac{(a)_j}{(b)_j} = \frac{\Gamma(b)}{\Gamma(a)\;\Gamma(b-a)} \int_0^1 t^{a+j-1}(1-t)^{b-a-1}\,dt \qquad \textbf{--- (5.3.7)}$$

Now, on considering

$$\frac{(1+\alpha)_{nk}}{n!}\; {}_{k+1}F_k\left[\begin{matrix} -n, \frac{x}{w}, \frac{x}{w}-1, \ldots, \frac{x}{w}-k+1; \\ \Delta(k;1+\alpha) \qquad ; \end{matrix} \left(\frac{w}{k}\right)^k\right]$$

$$= \frac{(1+\alpha)_{nk}}{n!} \sum_{j=0}^{n} \frac{(-n)_j \left(\frac{x}{w}\right)_j \left(\frac{x}{w}-1\right)_j \cdots \left(\frac{x}{w}-k+1\right)_j \left(\frac{w^k}{k^k}\right)^j}{\Delta_j(k;1+\alpha) \quad j!} \frac{(b)_j}{(a)_j}$$

$$\times \frac{\Gamma(b)}{\Gamma(a)\;\Gamma(b-a)} \int_0^1 t^{a+j-1}(1-t)^{b-a-1}\,dt$$

$$= \frac{(1+\alpha)_{nk}}{n!} \sum_{j=0}^{n} \frac{(-n)_j \left(\frac{x}{w}\right)_j \left(\frac{x}{w}-1\right)_j \ldots \left(\frac{x}{w}-k+1\right)_j \; (t\frac{w^k}{k^k})^j}{\Delta_j(k;1+\alpha) \quad j!} \frac{(b)_j}{(a)_j}$$

$$\times \frac{\Gamma(b)}{\Gamma(a)\;\Gamma(b-a)} \int_0^1 t^{a-1}(1-t)^{b-a-1}\, dt$$

We thus arrive at the following

Theorem 7

If Re(a) > 0 and Re(b-a) > 0, then

$$z_n^{\alpha}(x;w,k) = \frac{\Gamma(b)}{\Gamma(a)\;\Gamma(b-a)} \int_0^1 t^{a-1}(1-t)^{b-a-1}$$

$$\times \frac{(1+\alpha)_{nk}}{n!} \; {}_{k+2}F_{k+1}\left[\begin{matrix} -n, b, \frac{x}{w}, \frac{x}{w}-1, \ldots, \frac{x}{w}-k+1; \\ a, \Delta(k;1+\alpha) \qquad ; \end{matrix} \; t\frac{w^k}{k^k}\right] dt$$

--- (5.3.8)

Applications:

(i) By substituting $a=b$ in (5.3.8), we obtain

$$z_n^{\alpha}(x;w,k) = \int_0^1 t^{a-1}(1-t)^{-1}$$

$$\times \frac{(1+\alpha)_{nk}}{n!} \sum_{j=0}^{n} \frac{(-n)_j \left(\frac{x}{w}\right)_j \left(\frac{x}{w}-1\right)_j \ldots \left(\frac{x}{w}-k+1\right)_j}{\Delta_j(k;1+\alpha) \quad j!} \left(\frac{t\,w^k}{k^k}\right)^j dt$$

$$= \int_0^1 t^{a-1}(1-t)^{-1} Z_n^{\alpha}(x;wt,k)\; dt$$

--- (5.3.9)

(ii) By substituting $a=b$ and $w\to 0$ in (5.3.8), it leads to

$$z_n^{\alpha}(x;w,k)=\frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}$$

$$\times\frac{(1+\alpha)_{nk}}{n!}\;{}_{k+2}F_{k+1}\left[\begin{matrix}-n,b,\,\frac{x}{w},\frac{x}{w}-1,...,\frac{x}{w}-k+1; & t\,\frac{w^k}{k^k}\\ a,\Delta(k;1+\alpha) & ;\end{matrix}\right]dt$$

$$=\int_0^1 t^{a-1}(1-t)^{-1}Z_n^{\alpha}(xt,k)\;dt$$

--- (5.3.10)

(iii) By substituting $a=b$, $w\to 0$ and k = 1 in (5.3.8), it reduces to

$$z_n^{\alpha}(x;w,k)=\frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}$$

$$\times\frac{(1+\alpha)_{nk}}{n!}\;{}_{k+2}F_{k+1}\left[\begin{matrix}-n,b,\,\frac{x}{w},\frac{x}{w}-1,...,\frac{x}{w}-k+1; & t\,\frac{w^k}{k^k}\\ a,\Delta(k;1+\alpha) & ;\end{matrix}\right]dt$$

$$=\int_0^1 t^{a-1}(1-t)^{-1}L_n^{\alpha}(xt)\;dt$$

--- (5.3.11)

Theorem 8

If $\mathrm{Re}(a)>-\frac{1}{2}$ *and* $\mathrm{Re}(b)>-\frac{1}{2}$, then

$$z_n^{\alpha}(x;w,k)=\frac{2\,\Gamma(a+b)(1+\alpha)_{nk}}{n!\,\Gamma(a)\,\Gamma(b)}\,2\int_0^{\pi/2}(Sin\,t)^{2a-1}(Cos\,t)^{2b-1}$$

$$\times\left\{{}_{k+2}F_{k+2}\left[\begin{matrix}-n,a+b,\frac{x}{w},\frac{x}{w}-1,...,\frac{x}{w}-k+1; & \left(\frac{Sin^2t\;Cos^2t\,w^k}{k^k}\right)\\ a,b,\Delta(k;1+\alpha) & ;\end{matrix}\right]\right\}dt$$

--- (5.3.12)

Proof:

From (2.3.7), we have

$$\frac{(1+\alpha)_{nk}}{n!}\ {}_{k+1}F_k\left[\begin{matrix}-n,\frac{x}{w},\frac{x}{w}-1,...,\frac{x}{w}-k+1; & (\frac{w}{k})^k\\ \Delta(k;1+\alpha) & ;\end{matrix}\right]$$

$$=\frac{(1+\alpha)_{nk}}{n!}\sum_{j=0}^{n}\frac{(-n)_j\left(\frac{x}{w}\right)_j\left(\frac{x}{w}-1\right)_j...\left(\frac{x}{w}-k+1\right)_j(\frac{w^k}{k^k})^j(a+b)_j}{\Delta_j(k;1+\alpha)\quad j!\,(a)_j\ (b)_j}$$

$$\times\frac{\Gamma(a+b)\ \Gamma(a+j)\Gamma(b+j)}{\Gamma(a)\ \Gamma(b)\ \Gamma(a+b+j)}$$

$$=\frac{2\Gamma(a+b)(1+\alpha)_{nk}}{\Gamma(a)\ \Gamma(b)\ n!}\sum_{j=0}^{n}\frac{(-n)_j(a+b)_j\left(\frac{x}{w}\right)_j\left(\frac{x}{w}-1\right)_j...\left(\frac{x}{w}-k+1\right)_j(\frac{w^k}{k^k})^j}{(a)_j\ (b)_j\,\Delta_j(k;1+\alpha)\quad j!}$$

$$\times\int_0^{\pi/2}(Sin\,t)^{2(a+j)-1}(Cos\,t)^{2(b+j)-1}\ dt$$

$$=\frac{2\Gamma(a+b)(1+\alpha)_{nk}}{n!\,\Gamma(a)\ \Gamma(b)}2\int_0^{\pi/2}(Sin\,t)^{2a-1}(Cos\,t)^{2b-1}$$

$$\times\left\{{}_{k+2}F_{k+2}\left[\begin{matrix}-n,a+b,\frac{x}{w},\frac{x}{w}-1,...,\frac{x}{w}-k+1; & \left(\frac{Sin^2t\ Cos^2t\,w^k}{k^k}\right)\\ a,b,\Delta(k;1+\alpha) & ;\end{matrix}\right]\right\}\ dt$$

--- (5.3.13)

IV. INFINITE SINGLE INTEGRAL REPRESENTATION

Theorem 9

$$z_n^{\alpha}(x;w,k) = \frac{(1+\alpha)_{nk}}{n!\,\Gamma(a)} \int_{-\infty}^{\infty} e^{-t^2} t^{2a-1} {}_{k+1}F_k\left[\begin{array}{ll} -n, \frac{x}{w}, \frac{x}{w}-1, \ldots, \frac{x}{w}-k+1; & (t^2\frac{w}{k})^k \\ \Delta(k;1+\alpha), \quad a \quad ; & \end{array}\right] dt$$

--- (5.3.14)

Proof:

From (2.3.7), we have

$$z_n^{\alpha}(x;w,k) = \frac{(1+\alpha)_{nk}}{n!} \sum_{j=0}^{n} \frac{(-n)_j \left(\frac{x}{w}\right)_j \left(\frac{x}{w}-1\right)_j \ldots \left(\frac{x}{w}-k+1\right)_j (\frac{w^k}{k^k})^j}{\Delta_j(k;1+\alpha) \quad j!}$$

$$= \frac{(1+\alpha)_{nk}}{n!\,\Gamma(a)} \sum_{j=0}^{n} \frac{(-n)_j \left(\frac{x}{w}\right)_j \left(\frac{x}{w}-1\right)_j \ldots \left(\frac{x}{w}-k+1\right)_j (\frac{w^k}{k^k})^j}{\Delta_j(k;1+\alpha) \quad j!\,(a)_j} \int_{-\infty}^{\infty} e^{-t^2} t^{(2a+2j)-1} dt$$

on making use of integral formula $\Gamma(n) = \int_{-\infty}^{\infty} e^{-t^2} t^{2n-1}\, dt$

$$= \frac{(1+\alpha)_{nk}}{n!\,\Gamma(a)} \int_{-\infty}^{\infty} e^{-t^2} t^{2a-1} {}_{k+1}F_k\left[\begin{array}{ll} -n, \frac{x}{w}, \frac{x}{w}-1, \ldots, \frac{x}{w}-k+1; & (t^2\frac{w}{k})^k \\ \Delta(k;1+\alpha), \quad a \quad ; & \end{array}\right] dt$$

--- (5.3.15)

Applications:

(i) By taking $k = 1$ in (5.3.15), it leads to

$$J_n^{\alpha}(x,w) = \frac{(1+\alpha)_n}{n!\,\Gamma(a)} \int_{-\infty}^{\infty} e^{-t^2} t^{2a-1} \sum_{j=0}^{n} \frac{(-n)_j \left(\frac{x}{w}\right)_j \left(t^2 w\right)^j}{(1+\alpha)_j \quad j!\, a!}\, dt$$

$$= \frac{(1+\alpha)_n}{n!\,\Gamma(a)} \int_{-\infty}^{\infty} e^{-t^2} t^{2a-1} {}_2F_1\left[\begin{matrix} -n, \frac{x}{w} \;; \\ 1+\alpha, a; \end{matrix} \; t^2 w \right] dt \qquad \text{--- (5.3.16)}$$

(ii) By taking $k = 1$ and $w \to 0$ in (5.3.15), it leads to

$$L_n^{\alpha}(x) = \frac{(1+\alpha)_n}{n!\,\Gamma(a)} \int_{-\infty}^{\infty} e^{-t^2} t^{2a-1} \sum_{j=0}^{n} \frac{(-n)_j \left(t^2 x\right)^j}{(1+\alpha)_j \;\; j!\, a!} \, dt$$

$$= \frac{(1+\alpha)_n}{n!\,\Gamma(a)} \int_{-\infty}^{\infty} e^{-t^2} t^{2a-1} {}_1F_2\left[\begin{matrix} -n \;\; ; \\ 1+\alpha, a; \end{matrix} \; xt^2 \right] dt \qquad \text{--- (5.3.17)}$$

V. DOUBLE INTEGRAL REPRESENTATION

We know from Srivastava and Karlsson [201,p.275] that

If Re(a) > 0, Re(b) > 0, Re(λ) > 0, then

$$\iint u^{a-1} v^{b-1} (1-u-v)^{\lambda-1} \, du\, dv = \frac{\Gamma(a)\Gamma(b)\Gamma(\lambda)}{\Gamma(a+b+\lambda)} \;\; ; \; u \ge 0, v \ge 0, u+v \le 1$$

Hence

$$z_n^{\alpha}(x; w, k) = \frac{(1+\alpha)_{nk}}{n!} \sum_{j=0}^{n} \frac{(-n)_j \left(\frac{x}{w}\right)_j \left(\frac{x}{w}-1\right)_j \cdots \left(\frac{x}{w}-k+1\right)_j \left(\frac{w^k}{k^k}\right)^j}{\Delta_j(k;1+\alpha) \quad j!}$$

$$\times \frac{\Gamma(\lambda)\left(\frac{\lambda}{2}\right)_j \left(\frac{\lambda}{2}+\frac{1}{2}\right)_j \Gamma(a+j)\Gamma(b+j)\Gamma(\lambda-a-b)}{\Gamma(\lambda-a-b)\,\Gamma(a)\,\Gamma(b)\,(a)_j\,(b)_j\,\Gamma(\lambda+2j)}$$

$$= \frac{(1+\alpha)_{nk}\,\Gamma(\lambda)}{n!\,\Gamma(a)\,\Gamma(b)\,\Gamma(\lambda-a-b)} \sum_{j=0}^{n} \frac{(-n)_j \left(\frac{\lambda}{2}\right)_j \left(\frac{\lambda}{2}+\frac{1}{2}\right)_j}{\Delta_j(k;1+\alpha)}$$

$$\times \frac{\left(\frac{x}{w}\right)_j \left(\frac{x}{w}-1\right)_j \cdots \left(\frac{x}{w}-k+1\right)_j \; \left(\frac{w^k}{k^k}\right)^j}{j!\;\,(a)_j\,(b)_j} \iint u^{a+j-1}\, v^{b+j-1}\,(1-u-v)^{\lambda-a-b-1}\, du\, dv$$

Finally, we get

Theorem 10

If Re(a) > 0, Re(b) > 0, Re(λ) > 0, then

$$z_n^{\alpha}(x;w,k) = \frac{\Gamma(\lambda)\,(1+\alpha)_{nk}}{\Gamma(\lambda-a-b)\,\Gamma(a)\,\Gamma(b)\,n!} \iint u^{a-1}\, v^{b-1}\,(1-u-v)^{\lambda-a-b-1}$$

$$\times\, {}_{k+3}F_{k+2}\left[\begin{matrix} -n, \frac{\lambda}{2}, \frac{\lambda}{2}+\frac{1}{2}, \frac{x}{w}, \frac{x}{w}-1, \ldots, \frac{x}{w}-k+1; \\ a, b, \Delta(k;1+\alpha) \qquad ; \end{matrix} \; \frac{u\,v\,w^k}{k^k}\right] du\;\, dv \qquad \textbf{---(5.3.18)}$$