CHAPTER-IV

BILINEAR AND BILATERAL GENERATING RELATIONS OF GENERALIZED HYPERGEOMETRIC FUNCTION $B_n^{(\alpha,\beta)}(x,y,w)$

AND

MODIFIED KONHAUSER'S POLYNOMIAL $Z_n^{\alpha}(x;w,k)$

4.1 One of the classifications of generating relation is bilinear generating relations and the other is bilateral generating relations. These two play an important role in the theory of special functions of any polynomial or function. The complete definitions of these have already been discussed in Chapter-I.

Chapter-IV mainly deals with various types of bilinear and bilateral generating relations. It consists of two sections corresponding to the two different functions $B_n^{(\alpha,\beta)}(x,y,w)$ and $Z_n^{\alpha}(x;w,k)$.

In Section-I, we have derived bilateral generating relations for the function $B_n^{(\alpha,\beta)}(x,y,w)$ with the generalized hypergeometric function $I_{n;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w)$, Gottlieb polynomial set introduced by Gottlieb, M.J. $\phi_n(x;\lambda)$ and the Generalized Sylvester polynomial $f_n(x;a)$, whereas the Section-II deals with the bilinear generating relation for the function $Z_n^{\alpha}(x;w,k)$ and later we derived bilateral generating relations for the function $Z_n^{\alpha}(x;w,k)$ with the generalized hypergeometric function $I_{n;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w)$, Gottlieb polynomial set introduced by Gottlieb, M.J. $\phi_n(x;\lambda)$ and the Generalized

Sylvester polynomial $f_n(x;a)$. An attempt has been made to deduce the applications of every result to the known classical polynomials.

SECTION-I

4.2 BILATERAL GENERATING RELATIONS FOR $B_n^{(\alpha,\beta)}(x,y,w)$

In this article, we obtain the bilateral generating relation for a class of generalized hypergeometric function $B_n^{(\alpha,\beta)}(x,y,w)$ with generalized hypergeometric function $I_{n;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w)$, Gottlieb polynomial set introduced by Gottlieb, M.J. $\phi_n(x;\lambda)$ and the Generalized Sylvester polynomial $f_n(x;a)$.

THEOREM 1

$$\sum_{n=0}^{\infty}\frac{(\rho)_n(n!)^2}{\left[(1+\alpha)_n\right]^2(1+\beta)_n}B_n^{(\alpha,\beta)}{}_{(x,y,v)}\,I_{n;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w)t^n$$

$$=(1-t)^{-\rho}F_{q+2:0;0;0;1;1}^{p+2:0;0;1;2;1}\left[\begin{matrix}[(a_p):1,1,1,0,1],[\rho:1,1,0,1,1],[\frac{x}{w}-\mu+1:1,1,0,0,1]: \\ [(b_q):1,1,1,0,1],[1+\alpha:1,1,0,0,1],\quad[\rho:0,1,0,1,1,]\quad:\end{matrix}\right.$$

$$\left.\begin{matrix}-;-;-\frac{x}{w}+\lambda;\rho,-\frac{y}{v};\frac{x}{v}; \\ -;-;\quad-\quad;1+\beta;1+\alpha;\end{matrix}\quad\frac{-wt}{1-t},-w,w,\frac{vt}{1-t},wv\right] \qquad \textbf{--- (4.2.1)}$$

Proof:

Consider $\sum_{n=0}^{\infty} \frac{(\rho)_n (n!)^2}{[(1+\alpha)_n]^2 (1+\beta)_n} \frac{(1+\alpha)_n (1+\beta)_n}{(n!)^2}$

$$\times F^{1:1;1}_{-:1;1}\left[\begin{matrix} -n : \frac{-y}{v}; \frac{x}{v}; \\ \\ - : 1+\alpha; 1+\beta; \end{matrix} \; -v, v\right] \times I^{\alpha;\mu;(a_p)}_{n;\lambda;(b_q)}(x,w) t^n$$

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{n-r} \sum_{r=0}^{n} \frac{(\rho)_n (-n+r)_s (-n)_r \left(-\frac{y}{v}\right)_r \left(\frac{x}{v}\right)_s (-v)^r v^s}{(1+\alpha)_n (1+\alpha)_s (1+\beta)_r \; r!\, s!} I^{\alpha;\mu;(a_p)}_{n;\lambda;(b_q)}(x,w) t^n$$

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{n-r} \sum_{r=0}^{n} \frac{(\rho)_n (-n+r)_s (-1)^r n! \left(-\frac{y}{v}\right)_r \left(\frac{x}{v}\right)_s (-v)^r v^s}{(1+\alpha)_n (1+\alpha)_s (1+\beta)_r (n-r)!\, r!\, s!} I^{\alpha;\mu;(a_p)}_{n;\lambda;(b_q)}(x,w) t^n$$

Replacing '*n*' by '*n*+*r*', it results in

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{n} \sum_{r=0}^{\infty} \frac{(\rho)_{n+r} (-n)_s (n+r)! \left(-\frac{y}{v}\right)_r \left(\frac{x}{v}\right)_s (v)^r v^s}{(1+\alpha)_{n+r} (1+\alpha)_s (1+\beta)_r (n)!\, r!\, s!} I^{\alpha;\mu;(a_p)}_{n+r;\lambda;(b_q)}(x,w) t^{n+r}$$

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{n} \sum_{r=0}^{\infty} \frac{(\rho+r)_n (\rho)_r \binom{n+r}{r} (-n)_s \left(-\frac{y}{v}\right)_r \left(\frac{x}{v}\right)_s}{(1+\alpha+r)_n (1+\alpha)_r (1+\alpha)_s (1+\beta)_r}$$

$$\times \frac{(vt)^r v^s}{s!} I^{\alpha;\mu;(a_p)}_{n+r;\lambda;(b_q)}(x,w) t^n$$

$$= \sum_{s=0}^{n} \frac{(-n)_s \left(\frac{x}{v}\right)_s v^s}{(1+\alpha)_s s!} \sum_{r=0}^{\infty} \frac{\left(-\frac{y}{v}\right)_r (vt)^r (\rho)_r}{(1+\alpha)_r (1+\beta)_r}$$

$$\times \sum_{n=0}^{\infty} \binom{n+r}{r} \frac{(\rho+r)_n}{(1+\alpha+r)_n} I_{n+r;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w) t^n$$

By using (1.3.31), it results in

$$= \sum_{s=0}^{n} \frac{(-n)_s \left(\frac{x}{v}\right)_s v^s}{(1+\alpha)_s s!} \sum_{r=0}^{\infty} \frac{\left(-\frac{y}{v}\right)_r (vt)^r (\rho)_r}{(1+\alpha)_r (1+\beta)_r}$$

$$\times (1-t)^{-\rho-r} \binom{\alpha+r}{r} F^{(3)} \left[\begin{array}{l} (a_p) :: \frac{x}{w} - \mu + 1; -; -; \rho + r; -r; -\frac{x}{w} + \lambda; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{-wt}{1-t}, w, w \\ (b_q) :: \quad 1+\alpha \;\; ; -; -; \; - \; ; - \; ; \quad - \quad ; \end{array} \right]$$

$$= \sum_{s=0}^{n} \frac{(-n)_s \left(\frac{x}{v}\right)_s v^s}{(1+\alpha)_s s!} \sum_{r=0}^{\infty} \frac{\left(-\frac{y}{v}\right)_r \left(\frac{vt}{1-t}\right)^r (\rho)_r}{(1+\alpha)_r (1+\beta)_r} (1-t)^{-\rho} \frac{(\alpha+r)!}{\alpha! r!}$$

$$\times \sum_{m,k=0}^{\infty} \sum_{n=0}^{r} \frac{(a_p)_{m+n+k} \left(\frac{x}{w} - \mu + 1\right)_{m+n} (\rho+r)_m (-1)^n r!}{(b_q)_{m+n+k} (1+\alpha)_{m+n} (r-n)!}$$

$$\times \frac{\left(-\frac{x}{w} + \lambda\right)_k \left(\frac{-wt}{1-t}\right)^m w^n w^k}{m! n! k!}$$

Replacing ‘r’ by ‘$r+n$’ in the last summation, we get

$$=\sum_{s=0}^{n}\frac{(-n)_s\left(\frac{x}{v}\right)_s v^s}{(1+\alpha)_s\, s!}\sum_{r=0}^{\infty}\frac{\left(-\frac{y}{v}\right)_r\left(\frac{vt}{1-t}\right)^r(\rho)_r}{(1+\alpha)_r(1+\beta)_r}(1-t)^{-\rho}\frac{(\alpha+r)!}{\alpha!}$$

$$\times\sum_{m,k,n=0}^{\infty}\frac{(a_p)_{m+n+k}\left(\frac{x}{w}-\mu+1\right)_{m+n}(\rho+r+n)_m}{(b_q)_{m+n+k}(1+\alpha)_{m+n}(r)!}$$

$$\times\sum_{m,k,n=0}^{\infty}\frac{\left(-\frac{x}{w}+\lambda\right)_k\left(\frac{-wt}{1-t}\right)^m(-w)^n w^k}{m!\,n!\,k!}$$

$$=\sum_{s=0}^{n}\frac{(-n)_s\left(\frac{x}{v}\right)_s v^s}{(1+\alpha)_s\, s!}(1-t)^{-\rho}\sum_{m,k,n,r=0}^{\infty}\frac{(a_p)_{m+n+k}\left(\frac{x}{w}-\mu+1\right)_{m+n}}{(b_q)_{m+n+k}(1+\alpha)_{m+n}}$$

$$\times\frac{(\rho)_r(\rho)_{m+n+r}\left(-\frac{x}{w}+\lambda\right)_k\left(-\frac{y}{v}\right)_r\left(\frac{-wt}{1-t}\right)^m(-w)^n w^k\left(\frac{vt}{1-t}\right)^r}{(b_q)_{m+n+k}(1+\alpha)_{m+n}(1+\beta)_r(\rho)_{n+r}\; m!n!k!\;(r)!}$$

$$=(1-t)^{-\rho}\sum_{s=0}^{\infty}\frac{\left(\frac{x}{v}\right)_s v^s(-1)^s}{(1+\alpha)_s\, s!}\sum_{m,k,n,r=0}^{\infty}\frac{(a_p)_{m+n+k+s}\left(\frac{x}{w}-\mu+1\right)_{m+n+s}}{(b_q)_{m+n+k+s}(1+\alpha)_{m+n+s}}$$

$$\times\frac{(\rho)_r(\rho)_{m+n+r+s}\left(-\frac{x}{w}+\lambda\right)_k\left(-\frac{y}{v}\right)_r\left(\frac{-wt}{1-t}\right)^m(-w)^{n+s} w^k\left(\frac{vt}{1-t}\right)^r}{(1+\beta)_r(\rho)_{n+r+s}\; m!\; n!\; k!\, r!}$$

Replacing 'n' by '$n+s$', it results in

$$=(1-t)^{-\rho}\sum_{m,k,n,r,s=0}^{\infty}\frac{(a_p)_{m+n+k+s}(\rho)_{m+n+r+s}\left(\frac{x}{w}-\mu+1\right)_{m+n+s}}{(b_q)_{m+n+k+s}(1+\alpha)_{m+n+s}}$$

$$\times \frac{\left(-\frac{x}{w}+\lambda\right)_k \left(-\frac{y}{v}\right)_r (\rho)_r \left(\frac{x}{v}\right)_s \left(\frac{-wt}{1-t}\right)^m (-w)^n w^k \left(\frac{vt}{1-t}\right)^r (wv)^s}{(\rho)_{n+r+s}(1+\alpha)_s(1+\beta)_r\, m!\, n!\, k!\ r!\, s!}$$

$$= (1-t)^{-\rho} F^{p+2:0;0;1;2;1}_{q+2:0;0;0;1;1}\left[\begin{matrix} [(a_p):1,1,1,0,1],[\rho:1,1,0,1,1],[\frac{x}{w}-\mu+1:1,1,0,0,1]: \\ [(b_q):1,1,1,0,1],[1+\alpha:1,1,0,0,1], \quad [\rho:0,1,0,1,1,] \quad : \end{matrix}\right.$$

$$\left.\begin{matrix} -;-;-\frac{x}{w}+\lambda;\rho,-\frac{y}{v};\frac{x}{v}; \\ -;-; \quad - \quad ;1+\beta;1+\alpha; \end{matrix} \quad \frac{-wt}{1-t},-w,w,\frac{vt}{1-t},wv\right] \qquad \textbf{--- (4.2.2)}$$

Applications:

(i) By setting $p = q$, $a_j = b_j$, j = 1,2,…, p and taking μ=1 and λ=0 in (4.2.1), we

get $\sum_{n=0}^{\infty} \frac{(\rho)_n (n!)^2}{\left[(1+\alpha)_n\right]^2 (1+\beta)_n} B_n^{(\alpha,\beta)}(x,y,v)\, J_n^{\alpha}(x,w) t^n$

$$= (1-t)^{-\rho} F^{2:0;0;2;1}_{2:0;0;1;1}\left[\begin{matrix} [\rho:1,1,1,1],[\frac{x}{w}:1,1,0,1] \quad : \\ [1+\alpha:1,1,0,1],[\rho:0,1,1,1]: \end{matrix}\right.$$

$$\left.\begin{matrix} -;-;\rho,-\frac{y}{v};\frac{x}{v}; \\ -;-;1+\beta;1+\alpha; \end{matrix} \quad \frac{-wt}{1-t},-w,w,\frac{vt}{1-t},wv\right] \qquad \textbf{--- (4.2.3)}$$

(ii) By setting $p = q$, $a_j = b_j$, j = 1,2,…, p and taking μ=1, λ=0 and w→0 in (4.2.1),

we obtain

$$\sum_{n=0}^{\infty}\frac{(\rho)_n(n!)^2}{\left[(1+\alpha)_n\right]^2(1+\beta)_n}B_n^{(\alpha,\beta)}{}_{(x,y,v)}\,L_n^{\alpha}(x)t^n$$

$$=(1-t)^{-\rho}F^{1:0;0;2;1}_{2:0;0;1;1}\left[\begin{array}{l}[\rho:1,1,1,1]:\\ [1+\alpha:1,1,0,1],[\rho:0,1,1,1,]:\end{array}\begin{array}{l}-;-;\rho,-\frac{y}{v};\frac{x}{v};\\ -;-;1+\beta;1+\alpha;\end{array}\frac{-xt}{1-t},-x,\frac{vt}{1-t},xv\right] \qquad \textbf{--- (4.2.4)}$$

(iii) Byn taking $v\to 0$ in (4.2.1), we obtain

$$\sum_{n=0}^{\infty}\frac{(\rho)_n(n!)^2}{\left[(1+\alpha)_n\right]^2(1+\beta)_n}L_n^{(\alpha,\beta)}{}_{(x,y)}\,I^{\alpha;\mu;(a_p)}_{n;\lambda;(b_q)}(x,w)t^n$$

$$=(1-t)^{-\rho}F^{p+2:0;0;1;1;0}_{q+2:0;0;0;1;1}\left[\begin{array}{l}[(a_p):1,1,1,0,1],[\rho:1,1,0,1,1],[\frac{x}{w}-\mu+1:1,1,0,0,1]:\\ [(b_q):1,1,1,0,1][1+\alpha:1,1,0,0,1],[\rho:0,1,0,1,1,]:\end{array}\begin{array}{l}-;-;-\frac{x}{w}+\lambda;\rho;-;\\ -;-;-;1+\beta;1+\alpha;\end{array}\frac{-wt}{1-t},-w,w,-\frac{yt}{1-t},wx\right] \qquad \textbf{--- (4.2.5)}$$

(iv) By setting $p = q$, $a_j = b_j$, j = 1,2,…, p and taking μ=1, λ=0 and $v \to 0$ in (4.2.1), we obtain

$$\sum_{n=0}^{\infty} \frac{(\rho)_n (n!)^2}{\left[(1+\alpha)_n\right]^2 (1+\beta)_n} L_n^{(\alpha,\beta)}(x,y)\, J_n^{\alpha}(x,w) t^n$$

$$= (1-t)^{-\rho} F^{2:0;0;1;0}_{2:0;0;1;1}\left[\begin{matrix} [\rho:1,1,1,1],[\frac{x}{w}:1,1,0,1] : \\ [1+\alpha:1,1,0,1],[\rho:0,1,1,1]: \end{matrix}\right.$$

$$\left.\begin{matrix} -;-; \quad \rho; \quad - ; \\ \\ -;-;1+\beta;1+\alpha; \end{matrix} \; \frac{-wt}{1-t}, -w, \frac{-yt}{1-t}, xw \right] \quad \textbf{--- (4.2.6)}$$

(v) By setting $p = q$, $a_j = b_j$, j = 1,2,---,p and taking μ=1, λ=0, $v \to 0$ and $w \to 0$ in (4.2.1), we obtain

$$\sum_{n=0}^{\infty} \frac{(\rho)_n (n!)^2}{\left[(1+\alpha)_n\right]^2 (1+\beta)_n} L_n^{(\alpha,\beta)}(x,y)\, L_n^{\alpha}(x) t^n$$

$$= (1-t)^{-\rho} F^{1:0;0;1;0}_{2:0;0;1;1}\left[\begin{matrix} [\rho:1,1,1,1] : \\ [1+\alpha:1,1,0,1],[\rho:0,1,1,1]: \end{matrix}\right.$$

$$\left.\begin{matrix} -;-; \quad \rho; \quad - ; \\ \\ -;-;1+\beta;1+\alpha; \end{matrix} \; \frac{-xt}{1-t}, -x, \frac{-yt}{1-t}, x^2 \right] \quad \textbf{--- (4.2.7)}$$

(vi) By setting $p = q$, $a_j = b_j$, j = 1,2,---,p and taking μ=1, λ=0, $v \to 0$, $w \to 0$, $\beta = 0$ and $y = 0$ in (4.2.1), we obtain

$$\sum_{n=0}^{\infty} \frac{(\rho)_n (n!)^2}{\left[(1+\alpha)_n\right]^2} L_n^{\alpha}(x) L_n^{\alpha}(x) t^n$$

$$= (1-t)^{-\rho} F_{2:0;0;1}^{1:0;0;0}\left[\begin{matrix} [\rho:1,1,1] & :-;-; \quad - ; \\ [1+\alpha:1,1,1],[\rho:0,1,1] & :-;-;1+\alpha; \end{matrix} \frac{-xt}{1-t}, -x, x^2\right] \qquad \text{--- (4.2.8)}$$

Theorem 2

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(1+\alpha)_n (1+\beta)_n} B_n^{(\alpha,\beta)}{}_{(x,y,v)}\, \phi_n(x;\lambda)\, t^n$$

$$= (1-t)^{x} (1-te^{-\lambda})^{-x-1} F_{2:0;0;1}^{3:0;0;1}\left[\begin{matrix} [-\frac{y}{v}:1,1,0],[1:1,1,1],\ [-x:1,0,1]:-;-;x/v; \\ [1+\beta:1,1,1], \quad - \quad , \quad [1:1,0,1]\ :-;-;1+\alpha; \end{matrix}\right.$$

$$\left. \frac{v(e^{\lambda}-1)t}{(e^{\lambda}-t)(1-t)}, \frac{vt}{e^{\lambda}-t}, -\frac{v^2(e^{\lambda}-1)t}{(e^{\lambda}-t)(1-t)}\right] \qquad \text{--- (4.2.9)}$$

Proof:

Consider $\displaystyle \sum_{n=0}^{\infty}\sum_{s=0}^{n-r}\sum_{r=0}^{n} \frac{(-n)_{r+s}\left(-\frac{y}{v}\right)_r \left(\frac{x}{v}\right)_s}{(1+\alpha)_s (1+\beta)_r} \frac{(-v)^r}{r!} \frac{v^s}{s!} \phi_n(x;\lambda) t^n$

$$= \sum_{n=0}^{\infty}\sum_{s=0}^{n-r}\sum_{r=0}^{n} \frac{(-n+r)_s (-n)_r \left(-\frac{y}{v}\right)_r \left(\frac{x}{v}\right)_s}{(1+\alpha)_s (1+\beta)_r} \frac{(-v)^r}{r!} \frac{v^s}{s!} \phi_n(x;\lambda) t^n$$

$$= \sum_{n=0}^{\infty}\sum_{s=0}^{n-r}\sum_{r=0}^{n} \frac{(-n+r)_s (-1)^r n! \left(-\frac{y}{v}\right)_r \left(\frac{x}{v}\right)_s}{(1+\alpha)_s (1+\beta)_r (n-r)!} \frac{(-v)^r}{r!} \frac{v^s}{s!} \phi_n(x;\lambda)\, t^n$$

Replacing '*n*' by '*n*+*r*', it results in

$$=\sum_{n=0}^{\infty}\sum_{s=0}^{n}\sum_{r=0}^{\infty}\frac{(-n)_s(n+r)!\left(-\frac{y}{v}\right)_r\left(\frac{x}{v}\right)_s}{(1+\alpha)_s(1+\beta)_r(n)!}\frac{(v)^r}{r!}\frac{v^s}{s!}\phi_{n+r}(x;\lambda)t^{n+r}$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{n}\frac{(-n)_s\left(\frac{x}{v}\right)_s v^s\left(-\frac{y}{v}\right)_r(vt)^r}{(1+\alpha)_s(1+\beta)_r\,s!}\sum_{n=0}^{\infty}\binom{n+r}{r}\phi_{n+r}(x;\lambda)t^n$$

By using (1.3.15), we obtain

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{n}\frac{(-n)_s\left(\frac{x}{v}\right)_s v^s\left(-\frac{y}{v}\right)_r(vt)^r}{(1+\alpha)_s(1+\beta)_r\,s!}$$

$$\times(1-t)^{x-r}(1-t\,e^{-\lambda})^{-x-1}\phi_r(x;\log_e\frac{e^{\lambda}-t}{1-t})$$

$$=\sum_{s=0}^{n}\frac{(-n)_s\left(\frac{x}{v}\right)_s v^s\left(-\frac{y}{v}\right)_r(\frac{vt}{1-t})^r}{(1+\alpha)_s(1+\beta)_r\,s!}(1-t)^x(1-t\,e^{-\lambda})^{-x-1}\phi_r(x;\log_e\frac{e^{\lambda}-t}{1-t})$$

By using (1.3.16), we obtain

$$=(1-t)^x(1-te^{-\lambda})^{-x-1}\sum_{r=0}^{\infty}\sum_{s=0}^{n}\frac{(-n)_s\left(\frac{x}{v}\right)_s v^s\left(-\frac{y}{v}\right)_r\left(\frac{vt}{1-t}\right)^r}{(1+\alpha)_s(1+\beta)_r\,s!}$$

$$\times e^{-r\log_e(\frac{e^{\lambda}-t}{1-t})}\,{}_2F_1(-r,-x;1;1-e^{\log_e(\frac{e^{\lambda}-t}{1-t})})$$

$$=(1-t)^x(1-te^{-\lambda})^{-x-1}\sum_{r=0}^{\infty}\sum_{s=0}^{n}\frac{\left(-\frac{y}{v}\right)_r\left(\frac{vt}{e^{\lambda}-t}\right)^r(-n)_s\left(\frac{x}{v}\right)_s v^s}{(1+\alpha)_s(1+\beta)_r\,s!}$$

$$\times \sum_{n=0}^{r} \frac{(-r)_n (-x)_n}{(1)_n} \frac{[1-(\frac{e^{\lambda}-t}{1-t})]^n}{n!}$$

$$=(1-t)^x (1-te^{-\lambda})^{-x-1} \sum_{r=0}^{\infty} \sum_{s=0}^{n} \sum_{n=0}^{r} \frac{\left(-\frac{y}{v}\right)_r \left(\frac{vt}{e^{\lambda}-t}\right)^r (-n)_s \left(\frac{x}{v}\right)_s v^s}{(1+\alpha)_s (1+\beta)_r s!}$$

$$\times \frac{(-1)^n r!(-x)_n}{(1)_n (r-n)!} \frac{[1-(\frac{e^{\lambda}-t}{1-t})]^n}{n!}$$

Replacing 'r' by '$r+n$' in the third summation, we get

$$=(1-t)^x (1-te^{-\lambda})^{-x-1} \sum_{r=0}^{\infty} \sum_{s=0}^{n} \sum_{n=0}^{\infty} \frac{(-n)_s \left(\frac{x}{v}\right)_s v^s \left(-\frac{y}{v}\right)_{r+n} \left(\frac{vt}{e^{\lambda}-t}\right)^{r+n}}{(1+\alpha)_s (1+\beta)_{r+n} s! (r)!}$$

$$\times \frac{(n+r)!(-x)_n}{(1)_n} \frac{[(\frac{e^{\lambda}-t}{1-t})-1]^n}{n!}$$

$$=(1-t)^x (1-te^{-\lambda})^{-x-1} \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{\left(-\frac{y}{v}\right)_{r+n} \left(\frac{vt}{e^{\lambda}-t}\right)^{r+n} (n+r)!}{(1+\beta)_{r+n} (r)! \ (1)_n}$$

$$\times \frac{(-x)_n (\frac{e^{\lambda}-1}{1-t})^n}{n!} \sum_{s=0}^{n} \frac{(-1)^s n! \left(\frac{x}{v}\right)_s v^s}{(n-s)!(1+\alpha)_s s!}$$

Replacing 'n' by '$n+s$' in the third summation, we get

$$=(1-t)^{x}(1-te^{-\lambda})^{-x-1}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\left(-\frac{y}{v}\right)_{r+n+s}\left(\frac{vt}{e^{\lambda}-t}\right)^{r}}{(1+\beta)_{r+n+s}\,(1+\alpha)_{s}}$$

$$\times\frac{\left(\frac{(e^{\lambda}-1)vt}{(e^{\lambda}-t)(1-t)}\right)^{n+s}(-x)_{n+s}\left(\frac{x}{v}\right)_{s}(-v)^{s}(n+r+s)!}{r!\;s!\,(1)_{n+s}\quad n!}$$

$$=(1-t)^{x}(1-te^{-\lambda})^{-x-1}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\left(-\frac{y}{v}\right)_{n+r+s}(1)_{n+r+s}(-x)_{n+s}\left(\frac{x}{v}\right)_{s}}{(1+\beta)_{n+r+s}\,(1)_{n+s}\,(1+\alpha)_{s}}$$

$$\times\frac{\left(\frac{v(e^{\lambda}-1)t}{(e^{\lambda}-t)(1-t)}\right)^{n}\left(\frac{vt}{e^{\lambda}-t}\right)^{r}\left(-\frac{(e^{\lambda}-1)v^{2}t}{(e^{\lambda}-t)(1-t)}\right)^{s}}{n!\;\;r!\;\;s!}$$

$$=(1-t)^{x}(1-te^{-\lambda})^{-x-1}F^{3:0;0;1}_{2:0;0;1}\left[\begin{matrix}[-\frac{y}{v}:1,1,0],[1:1,1,1],\,[-x:1,0,1]\;: \\ [1+\beta:1,1,1],\quad - \quad ,\quad [1:1,0,1]\;:\end{matrix}\right.$$

$$\left.\begin{matrix}-;-;x/v; \\ -;-;1+\alpha;\end{matrix}\;t_1,t_2,t_3\right] \qquad \textbf{--- (4.2.10)}$$

where $t_1=\frac{v(e^{\lambda}-1)t}{(e^{\lambda}-t)(1-t)},\quad t_2=\frac{vt}{e^{\lambda}-t},\quad t_3=-\frac{v^{2}(e^{\lambda}-1)t}{(e^{\lambda}-t)(1-t)}$

Applications:

(i) Taking the limit $v\to 0$ in (4.2.9), we obtain a bilateral generating relation of Laguerre polynomial with the Gottlieb polynomial.

$$\sum_{n=0}^{\infty}\frac{(n!)^{2}}{(1+\alpha)_{n}(1+\beta)_{n}}L_{n}^{(\alpha,\beta)}(x,y)\,\phi_{n}(x;\lambda)\,t^{n}$$

$$=(1-t)^{x}\,(1-te^{-\lambda})^{-x-1}F^{2:0;0;0}_{2:0;0;1}\left[\begin{matrix}[1:1,1,1],\,[-x:1,0,1]:-;-;\quad - \;;\\ [1+\beta:1,1,1],[1:1,0,1]:-;-;1+\alpha;\end{matrix}\,t_1,t_2,t_3\right]$$

--- (4.2.11)

$$where\quad t_1=\frac{y(e^{\lambda}-1)t}{(e^{\lambda}-t)(1-t)},\quad t_2=\frac{-yt}{e^{\lambda}-t},\quad t_3=\frac{xy(e^{\lambda}-1)t}{(e^{\lambda}-t)(1-t)}$$

Theorem 3

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(1+\alpha)_n(1+\beta)_n}B_n^{(\alpha,\beta)}(x,y,v)\,f_n(x;a)t^n$$

$$=(1-t)^{-x}e^{axt}F^{2:0;0;1}_{1:0;0;1}\left[\begin{matrix}[\frac{-y}{v}:1,1,1],[x:1,0,1]:-;-;\frac{x}{v}\quad ; \\ \left(\frac{vt}{1-t}\right),vtax,\left(\frac{-v^2t}{1-t}\right)\\ [1+\beta:1,1,1],-\quad :-;-;1+\alpha;\end{matrix}\right]$$

--- (4.2.12)

Proof:

$$\text{Consider}\ \sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha)_n(1+\beta)_n}{(1+\alpha)_n(1+\beta)_n(n!)^2}F^{1:1;1}_{-:1;1}\left[\begin{matrix}-n:\frac{-y}{v};\frac{x}{v};\\ \qquad -v,v\\ -:1+\beta;1+\alpha;\end{matrix}\right]f_n(x;a)t^n$$

$$=\sum_{n=0}^{\infty}F^{1:1;1}_{-:1;1}\left[\begin{matrix}-n:\frac{-y}{v};\frac{x}{v};\\ \qquad -v,v\\ -:1+\beta;1+\alpha;\end{matrix}\right]f_n(x;a)t^n$$

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{n-r} \sum_{r=0}^{n} \frac{(-n+r)_s (-n)_r \left(-\frac{y}{v}\right)_r \left(\frac{x}{v}\right)_s (-v)^r v^s}{(1+\alpha)_s (1+\beta)_r \ r! \ s!} f_n(x;a) t^n$$

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{n-r} \sum_{r=0}^{n} \frac{(-n+r)_s (-1)^r n! \left(-\frac{y}{v}\right)_r \left(\frac{x}{v}\right)_s (-v)^r v^s}{(1+\alpha)_s (1+\beta)_r (n-r)! \ r! \ s!} f_n(x;a) t^n$$

Replacing '*n*' by '*n*+*r*', it results in

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{n} \sum_{r=0}^{\infty} \frac{(-n)_s (n+r)! \left(-\frac{y}{v}\right)_r \left(\frac{x}{v}\right)_s (v)^r v^s}{n!(1+\alpha)_s (1+\beta)_r \ r! \, s!} f_{n+r}(x;a) t^{n+r}$$

$$= \sum_{s=0}^{n} \sum_{r=0}^{\infty} \frac{(-n)_s \left(-\frac{y}{v}\right)_r \left(\frac{x}{v}\right)_s (vt)^r v^s}{(1+\alpha)_s (1+\beta)_r \ s!} \sum_{n=0}^{\infty} \binom{n+r}{r} f_{n+r}(x;a) t^n$$

By using (1.3.21), we obtain

$$= \sum_{s=0}^{n} \sum_{r=0}^{\infty} \frac{(-n)_s \left(-\frac{y}{v}\right)_r \left(\frac{x}{v}\right)_s}{(1+\alpha)_s (1+\beta)_r} \frac{(vt)^r v^s}{s!} (1-t)^{-x-r} e^{axt} f_r(x;a(1-t))$$

$$= \sum_{s=0}^{n} \sum_{r=0}^{\infty} \frac{(-n)_s \left(-\frac{y}{v}\right)_r \left(\frac{x}{v}\right)_s}{(1+\alpha)_s (1+\beta)_r} \frac{\left(\frac{vt}{1-t}\right)^r v^s}{s!} (1-t)^{-x} e^{axt} f_r(x;a(1-t))$$

By using (1.3.22), we obtain

$$= (1-t)^{-x} e^{axt} \sum_{s=0}^{n} \sum_{r=0}^{\infty} \frac{(-n)_s \left(-\frac{y}{v}\right)_r \left(\frac{x}{v}\right)_s}{(1+\alpha)_s (1+\beta)_r} \frac{\left(\frac{vt}{1-t}\right)^r v^s}{s!}$$

$$\times \frac{(a(1-t)x)^r}{r!} {}_2F_0\left[-r, x; --; \frac{-1}{ax(1-t)}\right]$$

$$= (1-t)^{-x} e^{axt} \sum_{r=0}^{\infty} \frac{\left(-\frac{y}{v}\right)_r (vt)^r}{(1+\beta)_r} \sum_{s=0}^{n} \frac{(-n)_s \left(\frac{x}{v}\right)_s}{(1+\alpha)_s} \frac{v^s}{s!}$$

$$\times \frac{(ax)^r}{r!} \sum_{n=0}^{r} \frac{(-1)^n r!}{(r-n)!} (x)_n \frac{\left(\frac{-1}{a(1-t)x}\right)^n}{n!}$$

$$= (1-t)^{-x} e^{axt} \sum_{r=0}^{\infty} \frac{\left(-\frac{y}{v}\right)_r (vt)^r}{(1+\beta)_r} \sum_{s=0}^{n} \frac{(-n)_s \left(\frac{x}{v}\right)_s v^s}{(1+\alpha)_s s!}$$

$$\times (ax)^r \sum_{n=0}^{r} \frac{(x)_n \left(\frac{1}{a(1-t)x}\right)^n}{(r-n)!\, n!}$$

Replacing '*r*' by '*r*+*n*', it results in

$$= (1-t)^{-x} e^{axt} \sum_{r=0}^{\infty} \frac{\left(-\frac{y}{v}\right)_{r+n} (vtax)^{r+n}}{(1+\beta)_{r+n}} \sum_{s=0}^{n} \frac{(-n)_s \left(\frac{x}{v}\right)_s v^s}{(1+\alpha)_s s!}$$

$$\times \sum_{n=0}^{\infty} \frac{(x)_n \left(\frac{1}{a(1-t)x}\right)^n}{(r)!\, n!}$$

$$= (1-t)^{-x} e^{axt} \sum_{r=0}^{\infty} \frac{\left(-\frac{y}{v}\right)_{r+n} (vtax)^{r+n}}{(1+\beta)_{r+n}} \sum_{s=0}^{n} \frac{(-1)^s n! \left(\frac{x}{v}\right)_s v^s}{(n-s)!(1+\alpha)_s s!}$$

$$\times \sum_{n=0}^{\infty} \frac{(x)_n \left(\frac{1}{a(1-t)x}\right)^n}{(r)!\, n!}$$

Replacing 'n' by '$n+s$', we get

$$= (1-t)^{-x} e^{axt} \sum_{r=0}^{\infty} \frac{\left(-\frac{y}{v}\right)_{r+n+s} (vtax)^{r+n+s}}{(1+\beta)_{r+n+s}} \sum_{s=0}^{\infty} \frac{\left(\frac{x}{v}\right)_s (-v)^s}{(n)!(1+\alpha)_s\, s!}$$

$$\times \sum_{n=0}^{\infty} \frac{(x)_{n+s} \left(\frac{1}{a(1-t)x}\right)^{n+s}}{(r)!}$$

$$= (1-t)^{-x} e^{axt} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\left(-\frac{y}{v}\right)_{n+r+s} (x)_{n+s} \left(\frac{x}{v}\right)_s \left(\frac{vt}{1-t}\right)^n (vtax)^r \left(\frac{-v^2 t}{1-t}\right)^s}{(1+\beta)_{n+r+s} (1+\alpha)_s\, n!\, r!\, s!}$$

$$= (1-t)^{-x} e^{axt} F^{2:0;0;1}_{1:0;0;1} \left[\begin{matrix} [\frac{-y}{v}:1,1,1],[x:1,0,1]:-;-;\frac{x}{v} \quad ; \\ \qquad \left(\frac{vt}{1-t}\right), vtax, \left(\frac{-v^2 t}{1-t}\right) \\ [1+\beta:1,1,1],- \quad :-;-;1+\alpha; \end{matrix} \right]$$

--- (4.2.13)

Applications:

(i) Taking the limit $v \to 0$ in (4.2.12), we get

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(1+\alpha)_n (1+\beta)_n} L_n^{(\alpha,\beta)}(x,y)\, f_n(x;a) t^n$$

$$= (1-t)^{-x} e^{axt} F^{1:0;0;0}_{1:0;0;0} \left[\begin{matrix} [x:1,0,1] \quad :-;-;\ -; \\ \qquad -yt, (-axyt), xyt \\ [1+\beta:1,1,1]:-;-;\ -; \end{matrix} \right]$$

--- (4.2.14)

SECTION-II

4.3 BILINEAR GENERATING RELATION FOR $Z_n^{\alpha}(x;w,k)$.

In this article, bilinear generating relation for a class of generalized hypergeometric function $Z_n^{\alpha}(x;w,k)$ is obtained.

Theorem 4

$$\sum_{n=0}^{\infty}\frac{n!}{(1+\alpha)_{kn}(1+\beta)_{kn}}Z_n^{\alpha}(x;w,k)\,Z_n^{\beta}(\text{y};w,k)\,\text{t}^n$$

$$=F^{0:0:k+n+2}_{0:0:k+1}\left[\begin{matrix}-:-:-n,\frac{x}{w},\frac{x}{w}-1,---,\frac{x}{w}-k+1;\frac{y}{w},\frac{y}{w}-1,---,\frac{y}{w}-k+1; & n+1; \\ -:-:\quad \Delta(k,1+\alpha) \quad ; \quad - \quad ; \quad n-j+1 \quad ; & - \quad ;\end{matrix}\;\frac{w^k}{k^k},w^k t,t\right]$$

--- (4.3.1)

Proof:

$$\sum_{n=0}^{\infty}\frac{n!}{(1+\alpha)_{kn}(1+\beta)_{kn}}Z_n^{\alpha}(x;w,k)\,Z_n^{\beta}(\text{y};w,k)\,\text{t}^n$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma(1+\alpha)\;\Gamma(kn+\alpha+1)}{\Gamma(kn+\alpha+1)(1+\beta)_{kn}}\sum_{j=0}^{n}\frac{(-1)^j\,n!^{[jw]}\left\{x^{[kw]}\right\}}{(n-j)!\;j!\Gamma(kj+\alpha+1)}Z_n^{\beta}(\text{y};w,k)\,\text{t}^n$$

Replacing n '*n*' by '*n+j*', we obtain,

$$=\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j\,(n+j)!^{[jw]}\left\{x^{[kw]}\right\}\,Z_{n+j}^{\beta}(\text{y};w,k)t^{n+j}}{(1+\beta)_{k(n+j)}\;n!\;j!\,\Gamma(kj+\alpha+1)}$$

$$=\sum_{j=0}^{\infty} \frac{{}^{[jw]}\left\{x^{[kw]}\right\} \Gamma(1+\alpha)(-t)^j}{\Gamma(kj+\alpha+1)} \sum_{n=0}^{\infty} \frac{\binom{n+j}{n} Z_{n+j}^{\beta}(\mathrm{y};w,k)\, \mathrm{t}^n}{(1+\beta)_{k(n+j)}}$$

By using (2.6.1), we obtain

$$=\sum_{j=0}^{\infty} \frac{{}^{[jw]}\left\{x^{[kw]}\right\} (-t)^j}{(1+\alpha)_{kj}} \sum_{n=j}^{\infty} \frac{\binom{n}{j} {}^{[nw]}\left\{\mathrm{y}^{[kw]}\right\} t^{n-j}}{n!} {}_1F_1\left[\begin{matrix} n+1 \;; \\ n-j+1; \end{matrix} t\right]$$

$$=\sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-n)_j \left(\frac{x}{w}\right)_j \left(\frac{x}{w}-1\right)_j ---\left(\frac{x}{w}-k+1\right)_j w^{kj} (n+1)_r}{\Delta_j(k;1+\alpha)\, k^{kj}\, (n-j+1)_r\; n!\; j!}$$

$$\times \frac{\left(\frac{y}{w}\right)_n \left(\frac{y}{w}-1\right)_n ---\left(\frac{y}{w}-k+1\right)_n w^{kn}\, t^n\; t^r}{\mathrm{r}!}$$

$$=F_{0:0:k+1}^{0:0:k+n+2}\left[\begin{matrix} -:-:-n, \frac{x}{w}, \frac{x}{w}-1, ---, \frac{x}{w}-k+1; \\ -:-:\Delta(k,1+\alpha); \qquad - \qquad ; \end{matrix}\right.$$

$$\left.\begin{matrix} \frac{y}{w}, \frac{y}{w}-1, ---, \frac{y}{w}-k+1; n+1; \\ n-j+! \qquad ; \quad -; \end{matrix} \frac{w^k}{k^k}, w^k t, t\right]$$

--- (4.3.2)

Applications:

(i) By taking $w \to 0$ in (4.3.1), it leads to

$$\sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_{kn}(1+\beta)_{kn}} Z_n^{\alpha}(x;k)\, Z_n^{\beta}(\mathrm{y};k)\, t^n = F_{0:2}^{0:2}\left[\begin{matrix} -:-; \quad -n \quad ; \quad n+1 \;; \\ -:-;\Delta(k;1+\alpha); n-j+1; \end{matrix} \frac{x^k}{k^k}, y^k t, t\right]$$

--- (4.3.3)

(ii) By taking $w \to 0$ and k = 1 in (4.3.1), it reduces to

$$\sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_{kn}(1+\beta)_{kn}} L_n^{\alpha}(x) L_n^{\beta}(\mathrm{y}) t^n = F_{0:2}^{0:2}\left[\begin{matrix} -:-; & -n & ; & n+1 & ; \\ -:-; \Delta(1;1+\alpha); n-j+1; \end{matrix} \; yt, x, t\right]$$

--- (4.3.4)

4.4 BILATERAL GENERATING RELATIONS FOR $Z_n^{\alpha}(x;w,k)$

In this article, bilateral generating relation for a class of generalized hypergeometric function $Z_n^{\alpha}(x;w,k)$ with generalized hypergeometric function $I_{n;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w)$, Gottlieb polynomial set introduced by Gottlieb, M.J. $\phi_n(x;\lambda)$ and the Generalized Sylvester polynomial $f_n(x;a)$ are obtained.

Theorem 5

$$\sum_{n=0}^{\infty} \frac{n!(\eta)_n}{(1+\alpha)_n(1+\beta)_{nk}} Z_n^{\beta}(y;v,k) I_{n;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w) t^n$$

$$= (1-t)^{-\eta} F_{q+k+1:0;0;0;0}^{p+k+2:0;0;0;1}\left[\begin{matrix} [(a_p):0,1,1,1],[\eta:1,1,1,0]: \\ [(b_q):0,1,1,1]: \quad - \quad : \end{matrix}\right.$$

$$\begin{matrix} \left[\frac{y}{v}:1,0,1,0\right],\left[\frac{y}{v}-1:1,0,1,0\right],\ldots,\left[\frac{y}{v}-k+1:1,0,1,0\right]; \\ [\Delta(k;1+\beta):1,0,1,0] \quad ; \end{matrix}$$

$$\left.\begin{matrix} [\frac{x}{w}-\mu+1:0,1,1,0];-;-;-;-\frac{x}{w}+\lambda; \\ [1+\alpha:0,1,1,0];-;-; \; - \; ; \; - \; ; \end{matrix} \; E,F,G,H \right] \qquad \textbf{--- (4.4.1)}$$

where $E=\dfrac{t}{t-1}\left(\dfrac{v}{k}\right)^{k}$, $F=\dfrac{-wt}{1-t}$, $G=\dfrac{wt}{t-1}\left(\dfrac{v}{k}\right)^{k}$ *and* $H=w$

Proof:

Consider $\displaystyle\sum_{n=0}^{\infty}\frac{n!\,(\eta)_n\,(1+\beta)_{kn}}{(1+\alpha)_n(1+\beta)_{nk}\;n!}$

$$\times\,{}_{k+1}F_k\left[\begin{matrix}-n,\frac{y}{v},\frac{y}{v}-1,\ldots,\frac{y}{v}-k+1; \\ \Delta(k,1+\beta)\quad ;\end{matrix}\left(\frac{v}{k}\right)^{k}\right] I^{\alpha;\mu;(a_p)}_{n;\lambda;(b_q)}(x,w)t^n$$

$$=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{(\eta)_n\,(-1)^l\,n!\left(\frac{y}{v}\right)_l\left(\frac{y}{v}-1\right)_l\ldots\left(\frac{y}{v}-k+1\right)_l\left(\frac{v}{k}\right)^{lk} I^{\alpha;\mu;(a_p)}_{n;\lambda;(b_q)}(x,w)t^n}{(n-l)!\,(1+\alpha)_n\;\Delta_l(k;1+\beta)\;l!}$$

Replacing ‘n’ by ‘$n+l$’, we get

`

$$=\sum_{n,l=0}^{\infty}\frac{(\eta)_{n+l}(-1)^l(n+l)!\left(\frac{y}{v}\right)_l\left(\frac{y}{v}-1\right)_l\ldots\left(\frac{y}{v}-k+1\right)_l}{(1+\alpha)_{n+l}\;\Delta_l(k;1+\beta)\;n!}$$

$$\times\frac{\left(\frac{v}{k}\right)^{lk}}{l!}I^{\alpha;\mu;(a_p)}_{n+l;\lambda;(b_q)}(x,w)t^{n+l}$$

$$=\sum_{l=0}^{\infty}\frac{(\eta)_l\left(-t\left(\frac{v}{k}\right)^{k}\right)^{l}\left(\frac{y}{v}\right)_l\left(\frac{y}{v}-1\right)_l\ldots\left(\frac{y}{v}-k+1\right)_l}{(1+\alpha)_l\;\Delta_l(k;1+\beta)}$$

$$\times\sum_{n=0}^{\infty}\frac{\binom{n+l}{l}(\eta+l)_n}{(1+\alpha+l)_n}I^{\alpha;\mu;(a_p)}_{n+l;\lambda;(b_q)}(x,w)t^n$$

By using (1.3.31), we obtain

$$= \sum_{l=0}^{\infty} \frac{(\eta)_l \left(-t\left(\frac{v}{k}\right)^k\right)^l \left(\frac{y}{v}\right)_l \left(\frac{y}{v}-1\right)_l \cdots \left(\frac{y}{v}-k+1\right)_l (1-t)^{-\eta-l} \binom{\alpha+l}{l}}{(1+\alpha)_l \ \Delta_l(k;1+\beta)}$$

$$\times F^{(3)}\left[\begin{matrix}(a_p)::\frac{x}{w}-\mu+1;-;-;\eta+l;-l;-\frac{x}{w}+\lambda; \\ (b_q):: \quad 1+\alpha \ ;-;-; \ - \ ;- ; \quad - \quad ;\end{matrix} \ \frac{-wt}{1-t}, w, w\right]$$

$$= \sum_{l=0}^{\infty} \frac{(\eta)_l \left(\frac{t}{t-1}\left(\frac{v}{k}\right)^k\right)^l \left(\frac{y}{v}\right)_l \left(\frac{y}{v}-1\right)_l \cdots \left(\frac{y}{v}-k+1\right)_l (1-t)^{-\eta}}{l! \ \Delta_l(k;1+\beta)}$$

$$\times \sum_{m,j=0}^{\infty} \sum_{n=0}^{l} \frac{(a_p)_{m+j+n} \left(\frac{x}{w}-\lambda+1\right)_{m+n} (\eta+l)_m}{(b_q)_{m+n+j} (1+\alpha)_{m+n}}$$

$$\times \frac{(-l)_n \left(-\frac{x}{w}+\lambda\right)_j \left(\frac{-wt}{1-t}\right)^m w^n w^j}{m! n! j!}$$

$$= \sum_{l=0}^{\infty} \frac{\left(\frac{t}{t-1}\left(\frac{v}{k}\right)^k\right)^l \left(\frac{y}{v}\right)_l \left(\frac{y}{v}-1\right)_l \cdots \left(\frac{y}{v}-k+1\right)_l (1-t)^{-\eta}}{l! \ \Delta_l(k;1+\beta)}$$

$$\times \sum_{m,j=0}^{\infty} \sum_{n=0}^{l} \frac{(a_p)_{m+j+n} \left(\frac{x}{w}-\lambda+1\right)_{m+n} (\eta)_{l+m}}{(b_q)_{m+n+j} (1+\alpha)_{m+n}}$$

$$\times \frac{(-1)^n l! \left(-\frac{x}{w}+\lambda\right)_j \left(\frac{-wt}{1-t}\right)^m w^n w^j}{(l-n)! \ m! \ n! \ j!}$$

Replacing $'l'$ *by* $'l+n'$, we get

$$=(1-t)^{-\eta}\sum_{l,m,n,j=0}^{\infty}\frac{(a_p)_{m+j+n}(\eta)_{m+l+n}\left(\frac{y}{v}\right)_{l+n}}{(b_q)_{m+n+j}\Delta_{l+n}(k;1+\beta)}$$

$$\times\frac{\left(\frac{y}{v}-1\right)_{l+n}\cdots\left(\frac{y}{v}-k+1\right)_{l+n}\left(\frac{x}{w}-\mu+1\right)_{m+n}\left(-\frac{x}{w}+\lambda\right)_{j}}{(1+\alpha)_{m+n}}$$

$$\times\frac{\left(\frac{t}{t-1}\left(\frac{v}{k}\right)^{k}\right)^{l}\left(\frac{-wt}{1-t}\right)^{m}\left(\frac{wt}{t-1}\left(\frac{v}{k}\right)^{k}\right)^{n}w^{j}}{l!\,m!\,n!\,j!}$$

$$=(1-t)^{-\eta}F^{p+k+2:0;0;0;1}_{q+k+1:0;0;0;0}\left[\begin{matrix}[(a_p):0,1,1,1],[\eta:1,1,1,0]: \\ [(b_q):0,1,1,1]: \quad - \quad :\end{matrix}\right.$$

$$\begin{matrix}\left[\frac{y}{v}:1,0,1,0\right],\left[\frac{y}{v}-1:1,0,1,0\right],\ldots,\left[\frac{y}{v}-k+1:1,0,1,0\right]; \\ [\Delta(k;1+\beta):1,0,1,0] \qquad ;\end{matrix}$$

$$\left.\begin{matrix}[\frac{x}{w}-\mu+1:0,1,1,0];-;-;-;-\frac{x}{w}+\lambda; \\ [1+\alpha:0,1,1,0];-;-;\ -\ ;\ -\ ;\end{matrix}\quad E,F,G,H\right] \qquad \textbf{--- (4.4.2)}$$

where $E=\frac{t}{t-1}\left(\frac{v}{k}\right)^{k}$, $F=\frac{-wt}{1-t}$, $G=\frac{wt}{t-1}\left(\frac{v}{k}\right)^{k}$ *and* $H=w$

Applications:

(i) By setting $p=q$, $a_j=b_j$, $j=1,2,\ldots,p;$ $\mu=1$ *and* $\lambda=0$ in (4.4.1), it leads to

$$\sum_{n=0}^{\infty}\frac{n!(\eta)_n}{(1+\alpha)_n(1+\beta)_{nk}}Z_n^{\beta}(y;v,k)J_n^{\alpha}(x,w)t^n$$

$$=(1-t)^{-\eta}F_{k+1:0;0;0}^{k+2:0;0;0}\left[\begin{array}{l}[\eta:1,1,1],\left[\frac{y}{v}:1,0,1\right],\left[\frac{y}{v}-1:1,0,1\right],\ldots,\left[\frac{y}{v}-k+1:1,0,1\right],[\frac{x}{w}:0,1,1]:-;-;-; \\ \left[\Delta(k;1+\beta):1,0,1\right],[1+\alpha:0,1,1]:-;-;-;\end{array} E,F,G\right] \quad \text{--- (4.4.3)}$$

$$where\ \ E=\frac{t}{t-1}\left(\frac{v}{k}\right)^k,\ \ F=\frac{-wt}{1-t},\ \ G=\frac{wt}{t-1}\left(\frac{v}{k}\right)^k$$

(ii) By setting $p=q$, $a_j=b_j$, $j=1,2,...,p$; $\mu=1$ *and* $\lambda=0$ and taking $w\to 0$ in (4.4.1), it reduces to

$$\sum_{n=0}^{\infty}\frac{n!(\eta)_n}{(1+\alpha)_n(1+\beta)_{nk}}Z_n^{\beta}(y,v,k)L_n^{\alpha}(x)t^n$$

$$=(1-t)^{-\eta}F_{k+1:0;0;0}^{k+1:0;0;0}\left[\begin{array}{l}[\eta:1,1,1],\left[\frac{y}{v}:1,0,1\right],\left[\frac{y}{v}-1:1,0,1\right],\ldots,\left[\frac{y}{v}-k+1:1,0,1\right]:-;-;-; \\ \left[\Delta(k;1+\beta):1,0,1\right],[1+\alpha:1,0,1]:-;-;-;\end{array} \frac{t}{t-1}\left(\frac{v}{k}\right)^k,\frac{-xt}{1-t},\frac{xt}{t-1}\left(\frac{v}{k}\right)^k\right] \quad \text{--- (4.4.4)}$$

(iii) By taking $k = 1$ in (4.4.1), it reduces to

$$\sum_{n=0}^{\infty} \frac{n!(\eta)_n}{(1+\alpha)_n(1+\beta)_n} J_n^{\beta}(y,v)\, I_{n;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w)t^n$$

$$= (1-t)^{-\eta} F_{q+2:0;0;0;0}^{p+3:0;0;0;1}\left[\begin{matrix}[(a_p):0,1,1,1],[\eta:1,1,1,0],\left[\frac{y}{v}:1,0,1,0\right], \\ [(b_q):0,1,1,1],[1+\beta:1,0,1,0],[1+\alpha:0,1,1,0]\end{matrix}\right.$$

$$\left.\begin{matrix}[\frac{x}{w}-\mu+1:0,1,1,0]:-;-;-;-\frac{x}{w}+\lambda; \\ :-;-;-;\quad-\quad;\end{matrix}\;\frac{tv}{t-1},\frac{-wt}{1-t},\frac{wtv}{t-1},w\right] \quad \text{--- (4.4.5)}$$

(iv) By taking the limit as $v \to 0$ in (4.4.1), we arrive at the following result.

$$\sum_{n=0}^{\infty} \frac{n!\,(\eta)_n}{(1+\alpha)_n(1+\beta)_{nk}} Z_n^{\beta}(y,k)\, I_{n;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w)t^n$$

$$= (1-t)^{-\eta} F_{q+k+1:0;0;0;0}^{p+2:0;0;0;1}\left[\begin{matrix}[(a_p):0,1,1,1],[\eta:1,1,1,0],[\frac{x}{w}-\mu+1:0,1,1,0] \quad : \\ [(b_q):0,1,1,1],[\Delta(k;1+\beta):1,0,1,0],[1+\alpha:0,1,1,0]:\end{matrix}\right.$$

$$\left.\begin{matrix}-;-;-;-\frac{x}{w}+\lambda; \\ -;-;-;\quad-\quad;\end{matrix}\;\frac{t}{t-1}\left(\frac{y}{k}\right)^k,\frac{-wt}{1-t},\frac{wt}{t-1}\left(\frac{y}{k}\right)^k,w\right] \quad \text{--- (4.4.6)}$$

(v) By setting $p = q$, $a_j = b_j$, $j = 1, 2, ..., p$; $\mu = 1$, $\lambda = 0, k = 1$, $w \to 0$ and $v \to 0$ in (4.4.1), we arrive at the following result.

$$\sum_{n=0}^{\infty} \frac{n!\,(\eta)_n}{(1+\alpha)_n (1+\beta)_n} L_n^{\beta}(y)\; L_n^{\alpha}(x) t^n$$

$$= (1-t)^{-\eta} F_{2:0;0;0;0}^{1:0;0;0;0}\left[\begin{matrix} [\eta : 1,1,1,0] : -;-;-;-; \\ [1+\beta) : 1,0,1,0],[1+\alpha : 0,1,1,0] : -;-;-;-; \end{matrix} \; \frac{yt}{t-1}, \frac{-xt}{1-t}, \frac{xyt}{t-1}, x \right] \quad \textbf{--- (4.4.7)}$$

(vi) By taking the limit as $v \to 0$ *and* $k = 1$ in (4.4.1), we arrive at the following result.

$$\sum_{n=0}^{\infty} \frac{n!\,(\eta)_n}{(1+\alpha)_n (1+\beta)_n} L_n^{\beta}(y)\; I_{n;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x, w) t^n$$

$$= (1-t)^{-\eta} F_{q+2:0;0;0;0}^{p+2:0;0;0;1}\left[\begin{matrix} [(a_p) : 0,1,1,1],[\eta : 1,1,1,0],[\frac{x}{w} - \mu + 1 : 0,1,1,0] : \\ [(b_q) : 0,1,1,1],[1+\beta : 1,0,1,0],[1+\alpha : 0,1,1,0] \;\; : \end{matrix} \right.$$

$$\left. \begin{matrix} -;-;-;-\frac{x}{w} + \lambda; \\ -;-;-; \quad - \quad ; \end{matrix} \; \frac{yt}{t-1}, \frac{-wt}{1-t}, \frac{wyt}{t-1}, w \right] \quad \textbf{--- (4.4.8)}$$

Theorem 6

$$\sum_{n=0}^{\infty} \frac{n!}{(1+\beta)_{nk}} Z_n^{\beta}(y;v,k)\,\phi_n(x;\lambda)t^n$$

$$= (1-t)^x \left(1-te^{-\lambda}\right)^{-x-1} F_{k:1;0}^{k+1:1;0}\left[\begin{matrix}\left[\frac{y}{v}:1,1\right],\left[\frac{y}{v}-1:1,1\right],\ldots,\left[\frac{y}{v}-k+1:1,1\right], \\ [\Delta(k;1+\beta):1,1]\end{matrix}\right.$$

$$\left.\begin{matrix}[1:1,1]:-x;-; \\ :1\,;-\,;\end{matrix} -\frac{t}{1-t}\left(\frac{e^{\lambda}-1}{e^{\lambda}-t}\right)\left(\frac{v}{k}\right)^k, \left(\frac{-t}{e^{\lambda}-t}\right)\left(\frac{v}{k}\right)^k\right] \quad \textbf{--- (4.4.9)}$$

Proof:

Consider $\displaystyle\sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{(-n)_r \left(\frac{y}{v}\right)_r \left(\frac{y}{v}-1\right)_r \ldots \left(\frac{y}{v}-k+1\right)_r}{\Delta_r(k;1+\beta)} \frac{\left(\frac{v}{k}\right)^{rk}}{r!}\phi_n(x;\lambda)t^n$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{(-1)^r n! \left(\frac{y}{v}\right)_r \left(\frac{y}{v}-1\right)_r \ldots \left(\frac{y}{v}-k+1\right)_r}{(n-r)!\,\Delta_r(k;1+\beta)} \frac{\left(\frac{v}{k}\right)^{rk}}{r!}\phi_n(x;\lambda)t^n$$

Replacing '*n*' by '*n*+*r*', we get

$$= \sum_{r=0}^{\infty} \frac{\left(\frac{y}{v}\right)_r \left(\frac{y}{v}-1\right)_r \ldots \left(\frac{y}{v}-k+1\right)_r}{\Delta_r(k;1+\beta)} \left(-t\left(\frac{v}{k}\right)^k\right)^r \sum_{n=0}^{\infty} \binom{n+r}{r} \phi_{n+r}(x;\lambda)t^n$$

By using (1.3.15), we obtain

$$= \sum_{r=0}^{\infty} \frac{\left(\frac{y}{v}\right)_r \left(\frac{y}{v}-1\right)_r \ldots \left(\frac{y}{v}-k+1\right)_r \left(-t\left(\frac{v}{k}\right)^k\right)^r}{\Delta_r(k;1+\beta)}$$

$$\times (1-t)^{x-r} \left(1-te^{-\lambda}\right)^{-x-1} \phi_r\left(x; \log_e\left(\frac{e^{\lambda}-t}{1-t}\right)\right)$$

Using (1.3.16), we obtain

$$= \sum_{r=0}^{\infty} \frac{\left(\frac{y}{v}\right)_r \left(\frac{y}{v}-1\right)_r \ldots \left(\frac{y}{v}-k+1\right)_r \left(-\frac{t}{1-t}\left(\frac{v}{k}\right)^k\right)^r}{\Delta_r(k;1+\beta)}$$

$$\times (1-t)^{x} \left(1-te^{-\lambda}\right)^{-x-1} e^{-r\log\left(\frac{e^{\lambda}-t}{1-t}\right)} {}_2F_1\left[-r,-x;1;1-e^{\log\left(\frac{e^{\lambda}-t}{1-t}\right)}\right]$$

$$= (1-t)^{x} \left(1-te^{-\lambda}\right)^{-x-1} \sum_{r=0}^{\infty} \frac{\left(\frac{y}{v}\right)_r \left(\frac{y}{v}-1\right)_r \ldots \left(\frac{y}{v}-k+1\right)_r}{\Delta_r(k;1+\beta)}$$

$$\times \left(-\frac{t}{e^{\lambda}-t}\left(\frac{v}{k}\right)^k\right)^r \sum_{n=0}^{r} \frac{(-1)^n r!(-x)_n \left[1-\left(\frac{e^{\lambda}-t}{1-t}\right)\right]^n}{(r-n)!\ (1)_n\ n!}$$

Replacing 'r' by '$r+n$', we get

$$= (1-t)^{x} \left(1-te^{-\lambda}\right)^{-x-1} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{\left(\frac{y}{v}\right)_{r+n} \left(\frac{y}{v}-1\right)_{r+n} \ldots \left(\frac{y}{v}-k+1\right)_{r+n}}{\Delta_{r+n}(k;1+\beta)}$$

$$\times \frac{(1)_{r+n}(-x)_n \left[\left(\frac{e^{\lambda}-t}{1-t}\right)-1\right]^n \left(-\frac{t}{e^{\lambda}-t}\left(\frac{v}{k}\right)^k\right)^{r+n}}{(1)_n \;\; n! \;\; r!}$$

$$= (1-t)^x \left(1-te^{-\lambda}\right)^{-x-1} F^{k+1:1;0}_{k:1;0}\left[\begin{matrix}\left[\frac{y}{v}:1,1\right],\left[\frac{y}{v}-1:1,1\right],\ldots,\left[\frac{y}{v}-k+1:1,1\right], \\ [\Delta(k;1+\beta):1,1]\end{matrix}\right.$$

$$\left.\begin{matrix}[1:1,1]:-x;-; \\ :1\,;-\;\;;\end{matrix} -\frac{t}{1-t}\left(\frac{e^{\lambda}-1}{e^{\lambda}-t}\right)\left(\frac{v}{k}\right)^k, \left(\frac{-t}{e^{\lambda}-t}\right)\left(\frac{v}{k}\right)^k\right] \qquad \textbf{--- (4.4.10)}$$

Applications:

(i) By applying $v \to 0$ in (4.4.9), it leads to

$$\sum_{n=0}^{\infty} \frac{n!}{(1+\beta)_{nk}} Z_n^{\beta}(y;k)\phi_n(x;\lambda)t^n$$

$$=(1-t)^x (1-t\,e^{-\lambda})^{-x-1} F^{1:1;0}_{1:1;0}\left[\begin{matrix}[1:1,1] & :-x;-; \\ [\Delta(k;1+\beta):1,1]:1;-;\end{matrix} -\frac{(e^{\lambda}-1)y^k t}{(e^{\lambda}-t)(1-t)}, -\frac{y^k t}{(e^{\lambda}-t)}\right]$$

--- (4.4.11)

(ii) By taking k = 1 in(4.4.9), it reduces to

$$\sum_{n=0}^{\infty} \frac{n!}{(1+\beta)_{nk}} J_n^{\beta}(y;v)\phi_n(x;\lambda)t^n$$

$$= (1-t)^{x} (1-t\,e^{-\lambda})^{-x-1} F^{2:1;0}_{1:1;0}\left[\begin{matrix} [y/v:1,1],[1:1,1]: \\ [(1;1+\beta):1,1] \quad : \end{matrix}\right.$$

$$\left.\begin{matrix} -x;-; \\ 1\ ;-; \end{matrix} \; -\frac{v(e^{\lambda}-1)t}{(e^{\lambda}-t)(1-t)}, -\frac{vt}{(e^{\lambda}-t)}\right] \qquad \textbf{--- (4.4.12)}$$

(iii) By taking $v \to 0$ and k =1 in (4.4.9), it leads to

$$\sum_{n=0}^{\infty} \frac{n!}{(1+\beta)_{nk}} L_n^{\beta}(y)\phi_n(x;\lambda)t^n$$

$$=(1-t)^{x} (1-t\,e^{-\lambda})^{-x-1} F^{1:1;0}_{1:1;0}\left[\begin{matrix} [1:1,1]:-x;-; \\ [(1;1+\beta):1,1]:1\ ;-; \end{matrix} \; -\frac{y(e^{\lambda}-1)t}{(e^{\lambda}-t)(1-t)}, -\frac{yt}{(e^{\lambda}-t)}\right]$$

--- (4.4.13)

Theorem 7

$$\sum_{n=0}^{\infty} \frac{n!}{(1+\beta)_{nk}} Z_n^{\beta}(y;v,k)\, f_n(x;a)t^n$$

$$= (1-t)^{-x} e^{axt} F^{k:1;0}_{k:0;0}\left[\begin{matrix} \left[\frac{y}{v}:1,1\right],\left[\frac{y}{v}-1:1,1\right],\ldots,\left[\frac{y}{v}-k+1:1,1\right],[1:1,1]: \\ [\Delta(k;1+\beta):1,1] \qquad\qquad : \end{matrix}\right.$$

$$\left.\begin{matrix} x;-; \\ -;-; \end{matrix} \; -\frac{t}{1-t}\left(\frac{v}{k}\right)^{k},\ -axt\left(\frac{v}{k}\right)^{k}\right] \qquad \textbf{--- (4.4.14)}$$

Proof :

Consider $\displaystyle\sum_{n=0}^{\infty} \frac{n!\,(1+\beta)_{nk}}{(1+\beta)_{nk}\ n!}\ {}_{k+1}F_k\left[\begin{matrix} -n,\frac{y}{v},\frac{y}{v}-1,\ldots,\frac{y}{v}-k+1; \\ \Delta(k,1+\beta) \qquad ; \end{matrix} \left(\frac{v}{k}\right)^{k}\right] f_n(x;a)t^n$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(-n)_r \left(\frac{y}{v}\right)_r \left(\frac{y}{v}-1\right)_r \ldots \left(\frac{y}{v}-k+1\right)_r \left(\frac{v}{k}\right)^{rk}}{\Delta_r(k;1+\beta)\ \ r!} f_n(x;a)t^n$$

Replacing 'n' by '$n+r$', we get

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)! \left(\frac{y}{v}\right)_r \left(\frac{y}{v}-1\right)_r \ldots \left(\frac{y}{v}-k+1\right)_r \left(\frac{v}{k}\right)^{rk}}{n!\ \Delta_r(k;1+\beta)\ \ r!} f_{n+r}(x;a)t^{n+r}$$

$$= \sum_{r=0}^{\infty} \frac{\left(\frac{y}{v}\right)_r \left(\frac{y}{v}-1\right)_r \ldots \left(\frac{y}{v}-k+1\right)_r \left(-t\left(\frac{v}{k}\right)^{k}\right)^{r}}{\Delta_r(k;1+\beta)} \sum_{n=0}^{\infty} \binom{n+r}{r} f_{n+r}(x;a)t^n$$

Using (1.3.21), we obtain

$$= \sum_{r=0}^{\infty} \frac{\left(\frac{y}{v}\right)_r \left(\frac{y}{v}-1\right)_r \ldots \left(\frac{y}{v}-k+1\right)_r \left(-\frac{t}{1-t}\left(\frac{v}{k}\right)^{k}\right)^{r}}{\Delta_r(k;1+\beta)}$$

$$\times (1-t)^{-x} e^{axt} f_r(x;a(1-t))$$

Using (1.3.22), we obtain

$$= (1-t)^{-x} \sum_{r=0}^{\infty} \frac{\left(\frac{y}{v}\right)_r \left(\frac{y}{v}-1\right)_r \ldots \left(\frac{y}{v}-k+1\right)_r \left(-\frac{t}{1-t}\left(\frac{v}{k}\right)^{k}\right)^{r}}{\Delta_r(k;1+\beta)}$$

$$\times \frac{e^{axt}\left(ax(1-t)\right)^r}{r!}\ {}_2F_0\left[-r, x; -; -\frac{1}{ax(1-t)}\right]$$

$$= (1-t)^{-x} e^{axt} \sum_{r=0}^{\infty} \frac{\left(\frac{y}{v}\right)_r \left(\frac{y}{v}-1\right)_r \cdots \left(\frac{y}{v}-k+1\right)_r \left(-axt\left(\frac{v}{k}\right)^k\right)^r}{\Delta_r(k;1+\beta)}$$

$$\times \sum_{n=0}^{r} \frac{(x)_n \left[\frac{1}{ax(1-t)}\right]^n}{(r-n)!\ n!}$$

Replacing 'r' by '$r+n$', we get

$$= (1-t)^{-x} e^{axt} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{\left(\frac{y}{v}\right)_{r+n} \left(\frac{y}{v}-1\right)_{r+n} \cdots \left(\frac{y}{v}-k+1\right)_{r+n} (x)_n}{\Delta_{r+n}(k;1+\beta)}$$

$$\times \frac{\left[-\frac{t}{(1-t)}\left(\frac{v}{k}\right)^k\right]^n \left(-axt\left(\frac{v}{k}\right)^k\right)^r}{n!\ r!}$$

$$= (1-t)^{-x} e^{axt} F^{k:1;0}_{k:0;0}\left[\begin{matrix}\left[\frac{y}{v}:1,1\right],\left[\frac{y}{v}-1:1,1\right],\ldots,\left[\frac{y}{v}-k+1:1,1\right],[1:1,1]: \\ [\Delta(k;1+\beta):1,1] \qquad\qquad : \end{matrix}\right.$$

$$\left.\begin{matrix} x;-; \\ -;-; \end{matrix}\quad -\frac{t}{1-t}\left(\frac{v}{k}\right)^k,\ -axt\left(\frac{v}{k}\right)^k\right] \qquad \textbf{--- (4.4.15)}$$

Applications:

(i) By applying $v \to 0$ in (4.4.14), it reduces to

$$\sum_{n=0}^{\infty} \frac{n!}{(1+\beta)_{nk}} Z_n^{\beta}(y,k)\, f_n(x;a) t^n = (1-t)^{-x} e^{axt}\ \phi_3\left[\begin{matrix} x \\ \Delta(k;1+\beta) \end{matrix};\ \frac{-y^k t}{1-t}, -a\,x\,y^k t\right]$$

--- (4.4.16)

where ϕ_3 is the Confluent hypergeometric function of two variables, please see (1.2.3).

(ii) By taking k = 1 in (4.4.14), it reduces to

$$\sum_{n=0}^{\infty} \frac{n!}{(1+\beta)_{nk}} J_n^{\beta}(y,v) f_n(x;a) t^n = (1-t)^{-x} e^{axt} \phi_1 \left[\begin{array}{cc} \frac{y}{v}, x \quad ; & -\frac{vt}{1-t}, -a\,x\,v\,t \\ \Delta(1;1+\beta); & \end{array} \right]$$

--- (4.4.17)

where ϕ_1 is the Confluent Hypergeometric function of two variables, please see (1.2.1).

(iii) By applying $v \to 0$ and k = 1 in (4.4.14), it leads to

$$\sum_{n=0}^{\infty} \frac{n!}{(1+\beta)_{nk}} L_n^{\beta}(y) f_n(x;a) t^n = (1-t)^{-x} e^{axt} \phi_3 \left[\begin{array}{cc} x \quad ; & \frac{-yt}{1-t}, -a\,x\,y\,t \\ \Delta(1;1+\beta); & \end{array} \right]$$

--- (4.4.18)

where ϕ_3 is the Confluent Hypergeometric function of two variables, please see (1.2.3).