Chapter 4

\( \gamma \text{-graph } G(\gamma) \) of some special graphs

4.1 Introduction

In this chapter we consider the family of all \( \gamma \)-sets in a graph \( G \) and we define the graph \( G(\gamma) = (V(\gamma), E(\gamma)) \) of \( G \) to be the graph whose vertices correspond 1 to 1 with the \( \gamma \)-sets of \( G \) and two \( \gamma \)-sets \( S_1 \) and \( S_2 \) are adjacent in \( G(\gamma) \) if there exist a vertex \( v \in S_1 \) and a vertex \( w \in S_2 \) such that \( v \) is adjacent to \( w \) and \( S_1 = S_2 - \{w\} \cup \{v\} \) or equivalently \( S_2 = S_1 - \{v\} \cup \{w\} \). We obtain \( \gamma \)-graph of caterpillars, comb and Lobster.

**Definition 4.1.1.** [11] We consider the family of all \( \gamma \)-sets in a graph \( G \) and we define \( G(\gamma) = (V(\gamma), E(\gamma)) \) to be the graph whose vertices correspond 1 to 1 with the \( \gamma \)-sets of \( G \) and two \( \gamma \)-sets say \( S_1 \) and \( S_2 \) are adjacent in \( G(\gamma) \) if there exist a vertex \( v \in S_1 \) and a vertex \( w \in S_2 \)
such that \( v \) is adjacent to \( w \) and \( S_1 = S_2 - \{w\} \cup \{v\} \) or equivalently \( S_2 = S_1 - \{v\} \cup \{w\} \).

**Example 4.1.2.**

\[ S_1 = \{v_2, v_5, v_8\} \text{ and } S_2 = \{v_2, v_6, v_8\} \text{ are the } \gamma\text{-sets of } G. \]

Let \( v = v_5 \) and \( w = v_6 \).

Then

\[
S_2 - \{w\} \cup \{v\} = \{v_2, v_6, v_8\} - \{v_6\} \cup \{v_5\} = \{v_2, v_5, v_8\} = S_1
\]

\[
S_1 - \{v\} \cup \{w\} = \{v_2, v_5, v_8\} - \{v_5\} \cup \{v_6\} = \{v_2, v_6, v_8\} = S_2
\]
**Definition 4.1.3.** Caterpillar is a tree in which the removal of pendent vertices leaves a path.

**Definition 4.1.4.** A vertex \( v \) in a graph \( G = (V, E) \) is a \( \gamma \)-required vertex if it is an element of every \( \gamma \)-set of \( G \).

In a caterpillar, every vertex of degree \( \geq 4 \) is a \( \gamma \)-required vertex.

### 4.2 Main Results

**Theorem 4.2.1.** Let \( T \) be a caterpillar with exactly 2 supports that are \( \gamma \)-required vertices \( v_1 \) and \( v_2 \). If the number of vertices in between \( v_1 \) and \( v_2 \) of \( T \) is \( 3k + 1 \) then \( T(\gamma) \) is a path of length \( k \).

![Fig 4.3](image-url)

**Proof.** Let \( T \) be a caterpillar as shown in Fig(4.3). Then the \( \gamma \)-sets of \( T \) can be listed in order as follows.

\[
S_1 = \{v_1, u_3, u_6, u_9, \ldots, u_{3k}, v_2\}
\]

\[
S_2 = \{v_1, u_3, u_6, u_9, \ldots, u_{3k-6}, u_{3k-3}, u_{3k-1}, v_2\}
\]

\[
S_3 = \{v_1, u_3, u_6, u_9, \ldots, u_{3k-9}, u_{3k-6}, u_{3k-4}, u_{3k-1}, v_2\}
\]

\[
S_4 = \{v_1, u_3, u_6, u_9, \ldots, u_{3k-9}, u_{3k-7}, u_{3k-4}, u_{3k-1}, v_2\}
\]
Here the $\gamma$-sets $S_2, S_3, S_4, \ldots, S_k$ are adjacent to both the preceding and succeeding $\gamma$-sets and hence get deg 2. The $\gamma$-set $S_1$ is adjacent to $S_2$ alone and $S_{k+1}$ is adjacent to $S_k$ alone. So both $S_1$ and $S_{k+1}$ get deg 1. Thus we get a path $S_1, S_2, S_3, \ldots, S_{k+1}$ of length $k$. □

**Theorem 4.2.2.** Let $T$ be a caterpillar as shown in Fig(4.3). If the number of vertices in between $v_1$ and $v_2$ of $T$ is $3k + 2$ then $T(\gamma)$ is $K_1$.

**Proof.** Let $T$ be a caterpillar with exactly 2 supports $v_1$ and $v_2$ which are $\gamma$-required vertices and $u_1, u_2, u_3, \ldots, u_{3k+2}$ be the vertices between $v_1$ and $v_2$. Then $S = \{v_1, v_2, u_3, u_6, u_9, \ldots, u_{3k}\}$ is the only $\gamma$-set of $T$ and hence $T(\gamma)$ is $K_1$. □

**Theorem 4.2.3.** Let $T$ be a caterpillar as shown in Fig(4.3). If the number of vertices in between $v_1$ and $v_2$ of $T$ is $3k$ then $T(\gamma)$ is a step grid of order $k - 1$ with 2 pendent edges attached to it.

**Proof.** Let $T$ be a caterpillar as shown in Fig(4.3). Then the $\gamma$-sets of $T$ can be listed in order as follows.
\[ S_1 = \{ v_1, u_3, u_6, u_9, \ldots, u_{3k}, v_2 \} \]
\[ S_2 = \{ v_1, u_3, u_6, u_9, \ldots, u_{3k-6}, u_{3k-4}, u_{3k-1}, v_2 \} \]
\[ S_3 = \{ v_2, u_3, u_6, u_9, \ldots, u_{3k-9}, u_{3k-7}, u_{3k-4}, u_{3k-1}, v_2 \} \]
\[ \vdots \]
\[ S_{\frac{(k+1)(k+2)}{2}} = \{ v_1, u_1, u_4, u_7, \ldots, u_{3k-8}, u_{3k-5}, u_{3k-2}, v_2 \} \].

Of the \( \frac{(k+1)(k+2)}{2} \gamma \)-sets of \( T \), 2 \( \gamma \)-sets get deg 1, \( k \) \( \gamma \)-sets get deg 2, 2\( (k - 1) \) \( \gamma \)-sets get deg 3 and remaining \( \frac{(k+1)(k+2)}{2} - 3k \) \( \gamma \)-sets get deg 4 in \( T(\gamma) \) and they form a step grid of order \( k - 1 \). \( \square \)

Now let us consider a caterpillar with exactly 3 supports \( v_1, v_2, v_3 \) taken in order which are \( \gamma \)-required vertices.

**Theorem 4.2.4.** Let \( T \) be a caterpillar with exactly 3 supports \( v_1, v_2, v_3 \) taken in order and suppose that they are \( \gamma \)-required vertices. If the number of vertices in between \( v_1, v_2 \) and \( v_2, v_3 \) are \( 3m + 1 \) and \( 3n + 1 \) for \( m, n = 1, 2, 3, \ldots \) then \( T(\gamma) = P_{m+1} \times P_{n+1} \).

**Proof.** Let \( u_1, u_2, u_3, \ldots, u_{3m+1} \) and \( w_1, w_2, w_3, \ldots, w_{3n+1} \) be the vertices in between \( v_1, v_2 \) and \( v_2, v_3 \) respectively. Then the number of vertices in between \( v_1 \) and \( v_2 \) that occur in a \( \gamma \)-set of \( T \) is \( m \) and the number of such sets of vertices is \( m + 1 \). The possible sets are \( \{ u_2, u_5, u_8, \ldots, u_{3m-1} \}, \{ u_3, u_5, u_8, u_{11}, \ldots, u_{3m-1} \}, \{ u_3, u_6, u_8, u_{11}, \ldots, u_{3m-1} \}, \{ u_3, u_6, u_9, \ldots, u_{3m-1} \}, \ldots, \{ u_3, u_6, u_9, \ldots, u_{3m} \} \).
Similarly number of vertices in between $v_2$ and $v_3$ that occur in a 
$\gamma$-set of $T$ is $n$ and the number of possible sets of vertices is $n + 1$. The 
possible sets are $\{w_2, w_5, w_8, \ldots, w_{3n-1}\}, \{w_3, w_5, w_8, w_{11}, \ldots, w_{3n-1}\}$,
$\{w_3, w_6, w_8, w_{11}, \ldots, w_{3n-1}\}, \{w_3, w_6, w_9, w_{11}, \ldots w_{3n-1}\}, \ldots, \{w_3, w_6,$
$w_9, \ldots, w_{3n}\}$. Since $v_1, v_2, v_3$ are $\gamma$-required vertices, these 3 vertices 
occur in all the $\gamma$-sets of $T$. Every set of vertices between $v_1$ and $v_2$
together with a set of vertices in between $v_2$ and $v_3$ and $v_1, v_2, v_3$ form
a $\gamma$-set of $T$. Thus each $\gamma$-set of $T$ has $m + n + 3$ vertices and the
number of $\gamma$-sets of $T$ is $(m + 1) \times (n + 1)$. The $\gamma$-sets of $T$ are as
follows:

1. $S_1 = \{v_1, u_2, u_5, u_8, \ldots, u_{3m-1}, v_2, w_2, w_5, w_8, \ldots, w_{3n-1}, v_3\}$
2. $S_2 = \{v_1, u_2, u_5, u_8, \ldots, u_{3m-1}, v_2, w_3, w_5, w_8, \ldots, w_{3n-1}, v_3\}$
3. $S_3 = \{v_1, u_2, u_5, u_8, \ldots, u_{3m-1}, v_2, w_3, w_6, \ldots, w_{3n-1}, v_3\}$
4. $\vdots$
5. $S_{n+1} = \{v_1, u_2, u_5, u_8, \ldots, u_{3m-1}, v_2, w_3, w_6, w_9, \ldots, w_{3n}, v_3\}$

1. $S_1 = \{v_1, u_3, u_5, u_8, \ldots, u_{3m-1}, v_2, w_2, w_5, w_8, \ldots, w_{3n-1}, v_3\}$
2. $S_2 = \{v_1, u_3, u_5, u_8, \ldots, u_{3m-1}, v_2, w_3, w_5, w_8, \ldots, w_{3n-1}, v_3\}$
3. $S_3 = \{v_1, u_3, u_5, u_8, \ldots, u_{3m-1}, v_2, w_3, w_6, w_8, \ldots, w_{3n-1}, v_3\}$
4. $\vdots$
\[ S_{n+} = \{v_1, u_3, u_8, \ldots, u_{3m-1}, v_2, w_3, w_6, w_9, \ldots, w_{3n-1}, v_3\} \]

\[ \vdots \]

\[ S_1 = \{v_1, u_3, u_5, u_9, \ldots, u_{3m}, v_2, w_3, w_5, w_8, \ldots, w_{3n-1}, v_3\} \]

\[ S_2 = \{v_1, u_3, u_6, u_9, \ldots, u_{3m}, v_2, w_3, w_5, w_8, \ldots, w_{3n-1}, v_3\} \]

\[ S_3 = \{v_1, u_3, u_6, u_9, \ldots, u_{3m}, v_2, w_3, w_6, w_8, \ldots, w_{3n-1}, v_3\} \quad (m+1) \]

\[ \vdots \]

\[ S_{n+1} = \{v_1, u_3, u_6, u_9, \ldots, u_{3m}, v_2, w_3, w_6, w_9, \ldots, w_{3n}, v_3\} \]

The \( \gamma \)-sets \( S_i \) for \( i = 2, 3, 4, \ldots, n \) of the collections \( (2), (3), \ldots, (m) \) are adjacent to the preceding and succeeding \( \gamma \)-sets of the same collection and \( S_i \) of the preceding and succeeding collections. Thus the number of \( \gamma \)-sets of \( T \) getting deg 4 is \( (n-1)(m-1) \).

The \( \gamma \)-set \( S_1 \) of \( (1) \) is adjacent to \( S_2 \) of \( (1) \) and \( S_1 \) of \( (2) \).

The \( \gamma \)-set \( S_1 \) of \( (m+1) \) is adjacent to \( S_2 \) of \( (m+1) \) and \( S_{n+1} \) of \( (2) \).

The \( \gamma \)-set \( S_{n+1} \) of \( (1) \) is adjacent to \( S_n \) of \( (1) \) and \( S_{n+1} \) of \( (2) \).

The \( \gamma \)-set \( S_{n+1} \) of \( (m+1) \) is adjacent to \( S_n \) of \( (m+1) \) and \( S_{n+1} \).

Thus the number of \( \gamma \)-sets getting deg 2 is 4.

The \( \gamma \)-sets \( S_2, S_3, S_4, \ldots, S_n \) of \( (1) \) are adjacent to the preceding and succeeding \( \gamma \)-sets of same collection and respective \( \gamma \)-sets of \( (2) \).

Similarly the \( \gamma \)-sets \( S_2, S_3, S_4, \ldots, S_n \) of \( (m+1) \) are adjacent to preceding and succeeding \( \gamma \)-sets of the same collection and succeeding \( \gamma \)-sets of collection \( (m) \). Thus these \( (2n-1) \) \( \gamma \)-sets get deg 3. Also
$S_1$ of the collections 2, 3, 4, . . . , $m$ get deg 3 by the adjacency of $S_2$, of the same collection and $S_1$ of the preceding and succeeding collections. Similarly $S_{n+1}$ of the collections 2, 3, 4, . . . , $m$ get deg 3 by the adjacency of $S_n$ of the same collections and $S_{n+1}$ of the preceding and succeeding collections. Thus number of $\gamma$-sets of $T$ getting deg 3 is $2(n - 1) + 2(m - 1)$. As the $\gamma$-graph of a tree is connected, these vertices form a grid graph $P_{m+1} \times P_{n+1}$.

**Note 4.2.5.** Every grid graph $P_m \times P_n$ is a $\gamma$-graph of some caterpillar for $m, n \geq 2$.

**Theorem 4.2.6.** Let $T$ be a caterpillar with exactly 3 supports $v_1, v_2, v_3$ taken in order and suppose that they are $\gamma$-required vertices. If the number of vertices in between $v_1, v_2$ and $v_2, v_3$ are $3m + 1$ and $3n + 2$ for $m, n = 1, 2, 3, \ldots$, then $T(\gamma)$ is a path of length $m$.

**Proof.** As in the above theorem the number of possible sets of vertices in between $v_1$ and $v_2$ is $m + 1$ and the number of vertices in each such set is $m$. Since in between $v_1$ and $v_3$ there are $n + 2$ vertices, the only set that dominates these $n + 2$ vertices is $(3, 6, 9, \ldots, 3k)$ and hence the number of $\gamma$-sets of $T$ are $(m + 1) \times 1 = m + 1$ and the number of vertices in each $\gamma$-set is $m + n - 3$. The adjacency among
these $m+1$ $\gamma$-sets is due to the adjacency among the $m+1$ vertices in between $v_1$ and $v_2$. Hence the $\gamma$-graph $T(\gamma)$ is a path of length $m$. \hfill \Box

**Theorem 4.2.7.** Let $T$ be a caterpillar with exactly 3 supports $v_1, v_2, v_3$ taken in order and suppose that they are $\gamma$-required vertices. If the number of vertices in between $v_1, v_2$ and $v_2, v_3$, are $3m+2$ and $3n+2$ for $m, n = 1, 2, 3, \ldots$ then $T(\gamma)$ is $K_1$.

**Proof.** We know that if there are $3m+2$ vertices in between $v_1$ and $v_2$ then only one $\gamma$-set exist. Here there is only one dominating set and hence the number of $\gamma$-sets of $T$ is 1. So the $\gamma$-graph $T(\gamma)$ is $K_1$. \hfill \Box

**Theorem 4.2.8.** Let $T$ be a caterpillar with exactly 3 supports $v_1, v_2, v_3$ taken in order and suppose that they are $\gamma$-required vertices. If the number of vertices in between $v_1, v_2$ and $v_2, v_3$ are $3m+2$ and $3n$ for $m, n = 1, 2, 3, \ldots$ then $T(\gamma)$ is a step grid of order $m-1$ with 2 pendent edges.

**Proof.** By theorem 4.2.3 in between $v_1$ and $v_2$ we get a step grid of order $m-1$ with 2 pendent edges attached to it and by theorem 4.2.2
there is only one dominating set between $v_2$ and $v_3$. Hence the $\gamma$-graph of $T$ is a step grid of order $m - 1$ with 2 pendent edges attached to it.

\[ \square \]

4.3 $\gamma$-graph of a comb

**Definition 4.3.1.** The comb denoted by $Cb_n$ is a graph obtained from the path $P_n$ by attaching an edge at each vertex of $P_n$.

**Notation 4.3.2.** Let the vertices of the comb whose edges are subdivided $3k$ times be labeled as in the figure given below.

**Theorem 4.3.3.** If $G$ is a graph obtained by subdividing the pendent edges of $Cb_n$ $l$ times then we get the following.

1. If $l = 3k + 1$ then $G(\gamma) \cong K_1$. 72
(2) If \( l = 3k + 2 \) then \( G(\gamma) \cong P_n(\gamma) \).

(3) If \( l = 3k \) then \( G(\gamma) \) is a graph of order \((k + 2) + (k + 1)(nc_1 + nc_2 + \ldots + nc_{n-1})\).

Proof. Case 1. \( l = 3k + 1 \)

\( S = \{21, 22, 23, \ldots, 2n, 51, 52, 53, \ldots, 5n, 81, 82, 83, \ldots, 8n, \ldots, (3k + 2)1, (3k + 2)2, (3k + 2)3, \ldots, (3k + 2)n\} \) is the only \( \gamma \)-set of \( G \) and hence \( G(\gamma) \cong K_1 \).

Case 2. \( l = 3k + 2 \)

\( S = \{31, 32, 33, \ldots, 3n, 61, 62, \ldots, 6n, \ldots, (3k + 3)1, (3k + 3)2, (3k + 3)3, \ldots, (3k + 3)n\} \) together with a \( \gamma \)-set of \( <\{11, 12, 13, \ldots, 1n\}> \) form a \( \gamma \)-set of \( G \). As \( S \) is fixed and \( P_n \) has different \( \gamma \)-sets, the \( \gamma \)-graph of \( G \) is isomorphic to the \( \gamma \)-graph of \( P_n \),

(i.e) \( G(\gamma) \cong P_n(\gamma) \). Hence ,

(i) if \( n = 3m \) then \( G(\gamma) \cong K_1 \).

(ii) if \( n = 3m + 1 \) then \( G(\gamma) \cong SG(k + 1) \); a step grid of order \( k + 1 \)

(iii) if \( n = 3m + 2 \) then \( G(\gamma) \cong P_{m+2} \).

Case 3. \( l = 3k \)

\( S_1 = \{11, 12, 13, \ldots, 1n, 41, 42, 43, \ldots, 4n, 71, 72, 73, \ldots, 7n, \ldots, (3k + 1)1, (3k+1)2, (3k+1)3, \ldots, (3k+1)n\} \), \( S_2 = \{21, 22, 23, \ldots, 2n, 41, 42, \ldots \} \).
43, ..., 4n, 71, 72, 73, ..., 7n, ..., (3k+1)1, (3k+1)2, (3k+1)3, ..., (3k+1)n}, S_3 = \{21, 22, 23, ..., 2n, 51, 52, ..., 5n, 71, 72, ..., 7n, ..., (3k+1)1, (3k+1)2, ..., (3k+1)n\}, ..., S_{k+1} = \{21, 22, 23, ..., 2n, 51, 52, 53, ..., 5n, 81, 82, 83, ..., 8n, ..., (3k+2)1, (3k+2)2, (3k+2)3, ..., (3k+2)n\}, ... are \(k+2\) \(\gamma\)-sets of \(G\) with \(|S_i| = (k+1)n\) for \(i = 1, 2, 3, \ldots, k+2\). These are the \(k+2\) \(\gamma\)-sets in which all the vertices of \((k+1)\) rows occur.

Now consider 21, 22, 23, ..., 2n of \(S_2\). Let us replace one of the vertices of 21, 22, 23, ..., 2n by the corresponding vertex of 11, 12, 13, ..., 1n. Doing this for all the \(n\) vertices we get \(nC_1 = n\ \gamma\)-sets of \(G\). Let us denote the set of these \(n\ \gamma\)-sets by \(A_1\).

Similarly let us replace any 2 vertices of 21, 22, 23, ..., 2n of \(S_2\) by the corresponding 2 vertices of 11, 12, 13, ..., 1n. Doing this for all pairs of vertices we get \(nC_2\ \gamma\)-sets of \(G\). Let us denote these \(\gamma\)-sets by the set \(A_2\).

Proceeding like this, replacing \((n-1)\) vertices of 21, 22, 23, ..., 2n by the corresponding \((n-1)\) vertices of 11, 12, 13, ..., 1n we get \(nC_{n-1} = n\ \gamma\)-sets of \(G\). Let us denote these \(n\gamma\)-sets by \(A_{n-1}\). Thus, using 2 sets of \(n\) tuples we get \(nC_1 + nC_2 + nC_3 + \ldots + nC_{n-1}\) sets of \(G\). Let us denote these \(nC_1 + nC_2 + nC_3 + \ldots + nC_{n-1}\ \gamma\)-sets by \(U_1\).

Similar process can be done for 41, 42, 43, ..., 4n of \(S_2\) with 51, 52, 53,
\[ \ldots, 5n \text{ of } S_3, 71, 72, 73, \ldots, 7n \text{ of } S_2, 81, 82, 83, \ldots, 8n \text{ of } S_4, 101, 102, 103, \ldots, 10n \text{ of } S_2 \text{ with } 111, 112, 113, \ldots, 11n \text{ of } S_5, \ldots, (3k+1)1, (3k+1)2, (3k+1)3, \ldots, (3k+1)n \text{ of } S_2 \text{ with } (3k+2)1, (3k+2)2, (3k+2)3, \ldots, (3k+2)n \text{ of } S_{k+2}. \] Thus we get the set \( U_2, U_3, U_4, \ldots, U_{k+1} \). Hence the total number of \( \gamma \)-sets of \( G \) is \( (k+2) + (k+1)(nC_1 + nC_2 + \ldots + nCn - 1) \). \( \square \)

**Theorem 4.3.4.** If \( G \) is a graph obtained by subdividing the pendent edges of \( C_b_n \) \( 3k \) times then the degree of all the vertices of \( G(\gamma) \) is \( n \) except for the \( k \) vertices which have degree \( 2n \). Further \( G(\gamma) \) is bipartite.

**Proof.** First let us find the deg of vertices of \( G(\gamma) \). The \( \gamma \)-sets of \( G \) are found as in case (3) of theorem 4.3.3.

**Step (1).** By the construction of \( \gamma \)-sets, every \( \gamma \)-set in \( A_1 \) is adjacent to \( S_1 \). As \( S_1 \) is the only \( \gamma \)-set containing \( 11, 12, 13, \ldots, 1n \) it is not adjacent to any other \( \gamma \)-set of \( G \). Hence degree of \( S_1 \) is \( n \).

**Step (2).** The \( n \) \( \gamma \)-sets obtained by replacing the \( nC_{n-1} \) vertices of \( 11, 12, 13, \ldots, 1n \) of \( S_1 \) by the corresponding \( (n - 1) \) vertices of \( 21, 22, 23, \ldots, 2n \) of \( S_2 \) have only one vertex different from \( S_2 \). Also the \( n \) \( \gamma \)-sets obtained by replacing only one vertex of \( 41, 42, 43, \ldots, 4n \) of \( S_2 \) by the corresponding vertex of \( 51, 51, 53, \ldots, 5n \) of \( S_3 \) have only
one vertex different from $S_2$. Hence these $n$ $\gamma$-sets are adjacent to $S_2$. Thus degree of $S_2$ is $2n$.

**Step (3).** The $n$ $\gamma$-sets obtained by replacing the $nC_{n-1}$ vertices of $41, 42, 43, \ldots, 4n$ of $S_2$ by the corresponding vertices of $51, 52, 53, \ldots, 5n$ of $S_3$ have only one vertex different from $S_3$. Hence these $n$ $\gamma$-sets are adjacent to $S_3$. Also the $nC_1 = n$ $\gamma$-sets obtained by replacing only one vertex of $71, 72, 73, \ldots, 7n$ of $S_2$ by the corresponding vertex of $81, 82, 83, \ldots, 8n$ of $S_4$ have only one vertex different from $S_3$. Hence these $n$ $\gamma$-sets are also adjacent to $S_3$. Thus degree of $S_3$ is $2n$.

**Step (4).** Proceeding in this way, degree of $S_{k+1}$ is $2n$ and $S_{k+2}$ is adjacent to $n$ $\gamma$-sets which are obtained by replacing $nC_{n-1}$ vertices of $(3k + 1)1, (3k + 1)2, \ldots, (3k + 1)n$ of $S_2$ by the corresponding vertices of $(3k + 2)1, (3k + 2)2, \ldots, (3k + 2)n$ of $S_{k+2}$ and $S_{k+2}$ is the only $\gamma$-set containing $(3k + 2)1, (3k + 2)2, \ldots, (3k + 2)n$. Thus $S_{k+2}$ in $G(\gamma)$ is of degree $2n$.

**Step (5).** By step (1), every $\gamma$-set $A_1 \in U_1$ get deg 1. By the construction of $A_1$ and $A_2$, corresponding to every $\gamma$-set of $A_2 \in U_1$, there are $2$ $\gamma$-sets in $A_1$ which differ by one vertex. Hence every $\gamma$-set of $A_2$ is adjacent to $2$ $\gamma$-sets of $A_1$. Thus the $2$ $\gamma$-sets of $A_1$ get degree $nC_{n-2}$ and these $2 \times nC_2$ is divided equally among all the $\gamma$-sets of $A_1$. Hence the total degree of each $\gamma$-set of $A_1 = 2 \times \frac{nC_2}{nC_1} + 1 =$
\[
\frac{2n(n-1)}{1 \times 2} \times \frac{1}{n} + 1 = n - 1 + 1 = n.
\]

**Step (6).** By step (5), every \( \gamma \)-set of \( \mathcal{A}_2 \) get deg 2. By the construction of \( \mathcal{A}_3 \), corresponding to every \( \gamma \)-set of \( \mathcal{A}_3 \), there are 3 \( \gamma \)-sets on \( \mathcal{A}_2 \) which differ by one vertex. Hence every \( \gamma \)-set of \( \mathcal{A}_3 \) is adjacent to 3 \( \gamma \)-sets of \( \mathcal{A}_2 \). Thus the \( \gamma \)-sets of \( \mathcal{A}_2 \) get degree \( 3 \times nC_3 \) and this is distributed equally among all the \( \gamma \)-sets of \( \mathcal{A}_2 \). Thus every \( \gamma \)-set of \( \mathcal{A}_2 \) get deg \( 3 \times nC_3 + 2 = n \).

**Step (7).** This process continues up to \( \mathcal{A}_{n-2} \) and the degree of each \( \gamma \)-set of \( \mathcal{A}_{n-2} \in U = n - 1 \times \frac{nC_3}{nC_2} + n - 2 = n \).

**Step (8).** By step (4), each \( \gamma \)-set of \( \mathcal{A}_{n-1} \) is adjacent to \( S_{k+2} \) and hence get deg 1 and get deg \( n - 1 \) by the adjacency of \( \gamma \)-sets of \( \mathcal{A}_{n-2} \).

Thus deg of \( \gamma \)-sets of \( \mathcal{A}_{n-1} \in U_1 = n \times \frac{nC_n}{nC_{n-1}} + n - 1 = n \).

Thus each \( \gamma \)-set of \( U_1 \) is adjacent to \( n \) \( \gamma \)-sets of \( G \). Similarly we can prove that each \( \gamma \)-set of \( U_2, U_3, U_4, \ldots U_{k+1} \) are also adjacent to \( n \) \( \gamma \)-sets of \( G \). Thus each \( \gamma \)-set of \( G \) is adjacent to \( n \) \( \gamma \)-sets of \( G \) except the \( k \) \( \gamma \)-sets namely \( S_2, S_3, S_4, \ldots, S_{k+1} \).

Now let us prove that \( G(\gamma) \) is bipartite.

By the construction, the \( \gamma \)-set \( S_1 \) is adjacent to the \( \gamma \)-set of \( \mathcal{A}_1 \in U_1 \). Similarly the \( \gamma \)-sets of \( \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_{n-1} \) of \( U_1 \) are adjacent to the \( \gamma \)-set of \( \mathcal{A}_1 \) and \( \mathcal{A}_3, \mathcal{A}_2 \) and \( \mathcal{A}_4, \ldots, \mathcal{A}_{n-2} \) and \( S_2 \) respectively. Hence if we collect \( S_1, \mathcal{A}_2, \mathcal{A}_4, \ldots \) in one set \( C \) and \( \mathcal{A}_1, \mathcal{A}_3, \mathcal{A}_5, \ldots \) in another.
set $\mathcal{D}$ then every edge of $A_1, A_2, A_3, \ldots, A_{n-1} \in U_1, S_1, S_2$ has one end in the first collection and the other end in the second collection. As the $\gamma$-set $S_2$ is adjacent to $A_{n-1} \in U_1$ and $A_1 \in U_2$, it is contained in the collection which does not contain $A_{n-1} \in U_2$. This process can be continued for $S_2$ and $A_1, A_2, A_3, \ldots, A_{n-1} \in U_2, S_3$ and $A_1, A_2, A_3, \ldots, A_{n-1} \in U_3$ and so on.

**Case (1).** $n$ is even and $k$ is either odd or even.

Let $S_1$ and $A_2, A_4, A_6, \ldots, A_{n-2} \in U_1$ belong to $\mathcal{C}$ and $A_1, A_3, A_5, \ldots, A_{n-1} \in U_1, S_1, S_2$ has one end in the first collection and the other end in the second collection. As the $\gamma$-set $S_2$ is adjacent to $A_{n-1} \in U_1$ and $A_1 \in U_2$, it is contained in the collection which does not contain $A_{n-1} \in U_2$. This process can be continued for $S_2$ and $A_1, A_3, A_5, \ldots, A_{n-1} \in U_2, S_3$ and $A_1, A_2, A_3, \ldots, A_{n-1} \in U_3$ and so on.

**Case (2).** Both $n$ and $k$ are odd.

Let $A_2, A_4, A_6, \ldots, A_{n-1} \in U_1, S_1 \in \mathcal{C}$. Then $A_1, A_3, A_5, \ldots, A_{n-2} \in U_1, S_1, S_2, S_3, \ldots, S_{k+2}$ belong to $\mathcal{D}$. Then $S_2 \in \mathcal{D}$ and hence $A_1, A_3, A_5, \ldots, A_{n-2} \in U_2$ belong to $\mathcal{C}$ . Proceeding like this

**Case (3).** $n$ is odd and $k$ is even.
Obviously if \( S_1 \) and \( A_2, A_4, A_6, \ldots, A_{n-1} \) of \( U_1 \) belong to \( C \) then
\( A_1, A_3, A_5, \ldots, A_{n-2} \) belong to \( D \), \( S_2 \in D \) and \( A_1, A_3, A_5, \ldots, A_{n-2} \in U_2 \) belong to \( C \), \( A_2, A_4, A_6, \ldots, A_{n-1} \in U_2 \) belong to \( D \) and so on
\[ C = \{ A_2, A_4, A_6, \ldots, A_{n-1} \in U_1, U_3, U_5, \ldots U_{k+1}, S_1, S_3, S_{K+1} \} \]
\[ D = \{ A_2, A_4, A_6, \ldots, A_{n-1} \in U_2, U_4, U_6, \ldots U_k \} \]

In all the 3 cases \( C \cup D = V(G(\gamma)) \) and every edge of \( G(\gamma) \) has one end in \( C \) and other end in \( D \). Hence \( G(\gamma) \) is bipartite. \( \square \)

Let us find the number of \( \gamma \)-sets of a special type of spider.

**Definition 4.3.5.** A tree with exactly one vertex of deg \( \geq 3 \) is called a spider. It is a star graph \( K_{1,n} \) with some or all of the edges subdivided any number of times.

**Remark 4.3.6.** (1)If an edge of \( K_{1,n} \) is subdivided by \( m \) vertices in a spider \( T \) then we get a path \( P \) of length \( m + 1 \) from the centre of
$K_{1,n}$ to the pendent vertex of the subdivided edge. Hence

$$\gamma(P) = k + 1 \text{ if } m = 3k$$
$$= k + 1 \text{ if } m = 3k + 1$$
$$= k + 2 \text{ if } m = 3k + 2$$

(2) Let $T$ be a spider with a star graph $K_{1,4}$. Let $e_1, e_2, e_3, e_4$ be the edges of $K_{1,4}$ which are subdivided by $m_1, m_2, m_3, m_4$ vertices and let $P_1, P_2, P_3, P_4$ be the respective paths obtained by these subdivisions. Then we get the following.

(i) If $m_i = 3k_i$ for $i = 1, 2, 3, 4$ then paths obtained are dominated by $k_{i+1}$ vertices including the centre. In the $\gamma$-set of $T$ the centre of $K_{1,n}$ is counted in the $\gamma$-set of only one path. Hence $\gamma(T) = k_1 + k_2 + k_3 + k_4 + 1$ i.e. $\gamma(T) = \sum_{i=1}^{4} k_i + 1$.

(ii) If $m_i = 3k_i + 1$ for $i = 1, 2, 3, 4$ then also the paths obtained by subdivision are dominated by $k + 1$ vertices by excluding the centre and including at least one support. Hence $\gamma(T) = (k_1 + 1) + (k_2 + 1) + (k_3 + 1) + (k_4 + 1) = \sum_{i=1}^{4} (k_i + 1)$.

(iii) If $m_i = 3k_i + 2$ for $i = 1, 2, 3, 4$ then the path obtained are dominated by $k_{i+1} + 1$ vertices where the centre is included in the dominating set of each path. In the $\gamma$-set of $T$, the centre is
counted once. Hence $\gamma(T) = (k_1 + 1) + (k_2 + 1) + (k_3 + 1) + (k_4 + 1) + 1 = \sum_{i=1}^{4} (k_i + 1) + 1$.

(iv) If $m_1 = 3k_1, m_2 = 3k_2, m_3 = 3k_3 + 1, m_4 = 3k_4 + 1$ then $\gamma(T) = (k_1 + k_2 + 1) + (k_3 + 1) + (k_4 + 1) = \sum_{i=1}^{4} (k_i + 1) + 2$. [By (i) and (ii)].

(v) If $m_1 = 3k_1 + 1, m_2 = 3k_2 + 1, m_3 = 3k_3 + 2, m_4 = 3k_4 + 2$ then $\gamma(T) = (k_1 + 1) + (k_2 + 1) + [(k_3 + 1) + (k_4 + 1) + 1] = \sum_{i=1}^{4} (k_i + 1) + 1$. [By (ii) and (iii)].

(vi) If $m_1 = 3k_1 + 2, m_2 = 3k_2 + 2, m_3 = 3k_3, m_4 = 3k_4$ then $\gamma(P_1 \cup P_2) = (k_1 + 1) + (k_2 + 1) + 1$ and $\gamma(P_3 \cup P_4) = k_3 + k_4 + 1$. The $\gamma$-sets of both $P_1 \cup P_2$ and $P_3 \cup P_4$ contains the centre. Counting the centre once in the $\gamma$- set of $T$, $\gamma(T) = (k_1 + 1) + (k_2 + 2) + (k_3 + k_4 + 1) = \sum_{i=1}^{4} (k_i + 2) + 1$.

(vii) If $m_1 = 3k_1 + 1, m_2 = 3k_2, m_3 = 3k_3 + 1, m_4 = 3k_4 + 1$ then $\gamma(T) = (k_1 + k_2 + 1) + (k_3 + 1) + (k_4 + 1) = \sum_{i=1}^{4} (k_i + 3)$.

(viii) If $m_1 = 3k_1, m_2 = 3k_2 + 1, m_3 = 3k_3 + 1, m_4 = 3k_2$ then $\gamma(T) = (k_1 + 1) + (k_2 + 1) + (k_3 + 1) + (k_4 + 1) = \sum_{i=1}^{4} (k_i + 4)$.

(ix) If $m_1 = 3k_1, m_2 = 3k_2 + 1, m_3 = 3k_2 + 1, m_4 = 3k_4 + 2$ then $\gamma(T) = (k_1 + 1) + (k_2 + 1) + (k_3 + 1) + (k_4 + 1) = \sum_{i=1}^{4} (k_i + 4)$. 

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(x) If \( m_1 = 3k_1, m_2 = 3k_2, m_3 = 3k_3, m_4 = 3k_4 + 1 \), then \( \gamma(T) = (k_1 + k_2 + k_3 + 1) + (k_4 + 1) = \sum_{i=1}^{4}(k_i + 2) \).

(xi) If \( m_1 = 3k_1, m_2 = 3k_2 + 1, m_3 = 3k_3 + 1, m_4 = 3k_4 + 1 \) then
\[
\gamma(T) = k_1 + (k_2 + 1) + (k_3 + 1) + (k_4 + 1) = \sum_{i=1}^{4}(k_i + 3).
\]

(xii) If \( m_1 = 3k_1, m_2 = 3k_2, m_3 = 3k_3, m_4 = 3k_4 + 2 \) then
\[
\gamma(T) = (k_1 + k_2 + k_3 + 1) + (k_4 + 1) = \sum_{i=1}^{4}(k_i + 2).
\]

(xiii) If \( m_1 = 3k_1 + 1, m_2 = 3k_2 + 1, m_3 = 3k_3 + 1, m_4 = 3k_4 + 2 \) then
\[
\gamma(T) = (k_1 + k_2 + k_3 + 1) + (k_4 + 1) = \sum_{i=1}^{4}(k_i + 4).
\]

(xiv) If \( m_1 = 3k_1, m_2 = 3k_2 + 2, m_3 = 3k_3, m_4 = 3k_4 + 2 \) then
\[
\gamma(T) = (k_1 + 1) + (k_2 + 2) + (k_3 + 3) + (k_4 + 1) = \sum_{i=1}^{4}(k_i + 4).
\]

(xv) If \( m_1 = 3k_1 + 1, m_2 = 3k_2 + 2, m_3 = 3k_3 + 2, m_4 = 3k_4 + 2 \) then
\[
\gamma(T) = (k_1 + 1) + (k_2 + 1) + (k_3 + 1) + (k_4 + 1) = \sum_{i=1}^{4}(k_i + 4).
\]

**Theorem 4.3.7.** For a spider \( T \), \( \gamma(T) \leq \sum_{i=1}^{n}(k_i + 1) + 1 \) where \( k_i \)'s are given by \( m_i = 3k_i + r \), \( m_i \)'s are number of vertices subdividing the edges \( e_i (1 \leq i \leq n) \) and \( i = 0, 1, 2 \).

**Proof.** Suppose a spider \( T \) has a star graph \( K_{1,n} \) and each edge \( e_i \) of \( K_{1,n} \) be subdivided by \( m_i = 3k_i + r, r = 0, 1, 2 \).
Case (1) All the edges of the star graph are subdivided by equal number of vertices
i.e) $m_1 = m_2 = m_3 = \ldots = m_n$.

Case (1.1) $m_i = 3k \forall i$.

Then we get $k_1 = k_2 = k_3 = \ldots = k_n = k$.

$\gamma(T) = \sum_{i=1}^{n} k_i + 1 = nk + 1$.[as in (i)]

Case (1.2) $m_i = 3k_i + 1$.

Then as in (ii), $\gamma(T) = \sum_{i=1}^{n} k_i + 1 = n(k + 1)$.

Case (1.3) $m_i = 3k_i + 2$.

Then as in (iii), $\gamma(T) = \sum_{i=1}^{n} k_i + 1 = n(k + 1) + 1$.

Case (2) The edges of the star graph $K_{1,n}$ in $T$ are subdivided by different number of vertices

Case(2.1) All the edges of the star graph are subdivided by $m_i = 3k_i \forall i$ (or) $m_i = 3k_i + 1 \forall i$ (or) $m_i = 3k_i + 2 \forall i$ vertices.

When $m_i = 3k_i$, $\gamma(T) = \sum_{i=1}^{n} (k_i + 1)$. [as in (i)]

When $m_i = 3k_i + 1$, $\gamma(T) = \sum_{i=1}^{n} (k_i + 1)$ [as in (ii)].

When $m_i = 3k_i + 2$, $\gamma(T) = \sum_{i=1}^{n} (k_i + 1) + 1$ [as in (iii)]

Case(2.2) $m_i = 3k_i$ for $1 \leq i \leq l$ and $m_i = 3k_i + 1$ for $l + 1 \leq i \leq n$.

Without loss of generality, let $m_i = 3k_i$ for $i = 1, 2, 3, \ldots, l$ and $m_i = 3k_i + 1$ for $i = l + 1, l + 2, l + 3, \ldots, n$ where $1 \leq l < n$. Then as in
(iv) \( \gamma(T) = \sum_{i=1}^{n} (k_i + 1) + n - l \).

**Case (2.3)** \( m_i = 3k_i + 1 \) for \( 1 \leq i \leq l \) and \( m_i = 3k_i + 2 \) for \( l+1 \leq i \leq n \).

Without loss of generality, let \( m_i = 3k_i + 1 \) for \( i = 1, 2, 3, \ldots, l \) and \( m_i = 3k_i + 2 \) for \( i = l+1, l+2, \ldots, n \) where \( 1 \leq l \leq n \). Then as in (v) \( \gamma(T) = \sum_{i=1}^{n} (k_i + 1) + l \).

**Case (2.4)** \( m_i = 3k_i + 2 \) for \( 1 \leq i \leq l \) (or) \( m_i = 3k_i \) for \( l+1 \leq i \leq n \).

Let \( m_i = 3k_i + 2 \) for \( i = 1, 2, 3, \ldots l \) and \( m_i = 3k_i \) for \( i = l+1, l+2, \ldots, n \) where \( 1 \leq l < n \). Then by result (vi) \( \gamma(T) = \sum_{i=1}^{n} k_i + 1 + 1 \).

**Case (2.5)** Let \( 0 < r < s < n \), \( m_i = 3k_i \) for \( 1 \leq i \leq r \), \( m_i = 3k_i + 1 \) for \( r+1 \leq i \leq s \) and \( m_i = 3k_i + 2 \) for \( s+1 \leq i \leq n \). Then

\[
\gamma(T) = \sum_{i=1}^{r} k_i + 1 + \sum_{i=r+1}^{s} (k_i + 1) + \sum_{i=s+1}^{n} (k_i + 1) \\
= \sum_{i=1}^{n} k_i + 1 + s - r + n - s \\
= \sum_{i=1}^{n} k_i + n - r + 1
\]

Thus from all the above cases we get, \( \gamma(T) = \sum_{i=1}^{n} (k_i + 1) + 1 \). \( \square \)

Now let us find the number of \( \gamma \) - sets of a special type of Lobster.

**Definition 4.3.8.** Lobster is a tree in which the removal of pendent vertices leaves a path.

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Theorem 4.3.9. Let $G$ be a Lobster as given in the following figure. Then $G(\gamma)$ is of order $3 + 2 \left( nC_1 + nC_2 + nC_3 + \ldots + nC_{n-1} \right)$, where each $\gamma$-set of $G$ is of deg $n$ except one which is of deg $2n$.

Proof. Let $G$ be a Lobster such that each vertex of the path of length $n$ is adjacent to 2 paths of length 2. $v_{31}, v_{32}, v_{33}, \ldots, v_{3n}$ are the vertices of the path and $v_{21}, v_{22}, v_{23}, \ldots, v_{2n}$ and $v_{41}, v_{42}, v_{43}, \ldots, v_{4n}$ are vertices which are adjacent to the vertices of the path. $v_{11}, v_{12}, v_{13}, \ldots, v_{1n}$ and $v_{51}, v_{52}, v_{53}, \ldots, v_{5n}$ are the corresponding pendent vertices. Then $S_1 = \{v_{11}, v_{12}, v_{13}, \ldots, v_{1n}, v_{41}, v_{42}, v_{43}, \ldots, v_{4n}\}, S_2 = \{v_{21}, v_{22}, v_{23}, \ldots, v_{2n}, v_{41}, v_{42}, v_{43}, \ldots, v_{4n}\}, S_3 = \{v_{21}, v_{22}, v_{23}, \ldots, v_{2n}, v_{51}, v_{52}, v_{53}, \ldots, v_{5n}\}$ are the only 3 $\gamma$-sets of $G$ which contain all the $n$ vertices of 2 rows. Then by putting $k = 1$ in the theorem 4.3.4 we get order of $G$ is $3 + 2(nC_1 + nC_2 + nC_3 + \ldots, nC_{n-1})$ and each $\gamma$-set of $G$ is of deg $n$ except one which is of deg $2n$. □