Chapter 2

A Note on Disjoint Dominating Sets in Graphs

2.1 Introduction

In this chapter we study the existence of disjoint dominating sets and find partial solutions to some problems posed in [16]. In [16], S. M. Hedetniemi et al. introduced the concept of disjoint dominating sets in graphs. The disjoint domination number $\gamma\gamma(G) = \min\{|S_1| + |S_2| : S_1$ and $S_2$ are disjoint dominating sets of $G\}$. Two disjoint dominating sets whose union has cardinality $\gamma\gamma(G)$ is called a $\gamma\gamma$-pair of $G$.

2.2 Main results

Definition 2.2.1. The disjoint domination number $\gamma\gamma(G)$ is defined as $\gamma\gamma(G) = \min\{|S_1| + |S_2| : S_1$ and $S_2$ are disjoint dominating sets of $G\}$. 
We say that a graph $G$ is $\gamma\gamma$-minimum if it has two disjoint $\gamma$-sets, that is $\gamma\gamma(G) = 2\gamma(G)$. Similarly a graph $G$ is called $\gamma\gamma$-maximum if $\gamma\gamma(G) = n$.

**Theorem 2.2.2.** If $G$ is a graph with at least two universal vertices, then $\gamma\gamma(G) = 2$.

**Proof.** Let $u$ and $v$ be two universal vertices of the graph $G$. Then $\{u\}$ and $\{v\}$ are two disjoint dominating sets of $G$ and hence $\gamma\gamma(G) = 2$. □

**Theorem 2.2.3.** If $W_n$ is the wheel $C_n + K_1$, $\gamma\gamma(W_n) = 1 + \lceil \frac{n}{3} \rceil$.

**Proof.** The singleton set $S$ containing the center of the wheel is the $\gamma$-set of $W_n$. Hence $\gamma(W_n) = 1.\gamma(C_n) = 2 \lceil \frac{n}{3} \rceil$ by proposition 1.1.29 and a $\gamma$-set of the cycle of $W_n$ say $S'$ dominates center of $W_n$. Also $S$ and $S'$ are disjoint. Hence $\gamma\gamma(W_n) = 1 + \lceil \frac{n}{3} \rceil$. □

**Corollary 2.2.4.** A wheel $W_n$ is a $\gamma\gamma$-minimum graph $\Leftrightarrow n = 3$.

**Proof.** $W_n$ is complete if and only if $n = 3$ and so the proof follows. □
Definition 2.2.5. A graph obtained from a wheel by attaching a pendent edge at each vertex of the \( n \)-cycle is a Helm and is denoted by \( H_n \). Thus \( H_n \) is a graph of order \( 2n + 1 \).

Theorem 2.2.6. For a Helm \( H_n \), \( \gamma \gamma(H_n) = 2n \).

Proof. We know that the Helm \( H_n \) contains \( 2n + 1 \) vertices. Let \( u_1, u_2, \ldots, u_n \) be the vertices of the cycle, \( v_1, v_2, v_3, \ldots, v_n \) be corresponding pendent vertices and \( u \) be the center. Then \( S = \{u_1, v_2, u_3, v_4, \ldots, u_{n-3}, v_{n-2}, u_{n-1}, v_n\} \) and \( S' = \{v_1, u_2, v_3, u_4, \ldots, v_{n-3}, u_{n-2}, v_{n-1}, u_n\} \) are two disjoint \( \gamma \)-sets of \( H_n \). Hence \( \gamma \gamma(H_n) = 2n \). \( \square \)

We note that \( H_n \) is a \( \gamma \gamma \)-minimum graph when \( n \) is even.

Definition 2.2.7. The Web graph is a graph obtained by joining the pendent vertices of a Helm \( H_n \) to form a cycle and then adding a single pendent edge to each vertex of this outer cycle. It is a graph of order \( 3n + 1 \).

Theorem 2.2.8. For a web graph \( G \), \( \gamma \gamma(G) = 2n + 1 + \left\lceil \frac{n}{3} \right\rceil \).

Proof. The web graph contains 2 cycles of order \( n \), \( n \) pendent vertices and a center. Thus \( |V(G)| = 3n + 1 \).
Claim: $\gamma(G) = 2n + 1 + \left\lceil \frac{n}{3} \right\rceil$.

Let $S$ be the $\gamma$-set of $G$ obtained by taking the alternate vertices of the outer cycle, the alternate pendent vertices (not corresponding to the vertices taken in the outer cycle) and the center. Thus $|S| = n+1$. The other dominating set $S'$ of $G$ can be obtained by taking the remaining vertices of the outer cycle, the remaining pendent vertices and a $\gamma$-set of the inner cycle (not corresponding to the vertices in the outer cycle). Thus $|S'| = n + \left\lceil \frac{n}{3} \right\rceil$. Also $S \cap S' = \emptyset$. Thus $\gamma(G) = n+1 + n + \left\lceil \frac{n}{3} \right\rceil = 2n + 1 + \left\lceil \frac{n}{3} \right\rceil$. □

**Definition 2.2.9.** Grid graph is the Cartesian product of two paths.

**Theorem 2.2.10.** $\gamma(P_2 \times P_n) = 2\gamma(P_2 \times P_n) = 2 \left\lfloor \frac{k+2}{2} \right\rfloor$ for $k \geq 1$.

**Proof.** For $P_2 \times P_1$ and $P_2 \times P_2$, the result is obvious. Let $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ be the vertices of the two rows of the grid as shown in the figure.
Case (i). $n = 4k - 1, k \in \mathbb{N}$

$S = \{u_1, v_3, u_5, v_7, \ldots, u_{n-2}, u_{n-1}, v_n\}$ and $S' = \{v_1, u_3, v_5, u_7, \ldots, v_{n-2}, v_{n-1}, u_n\}$ are two disjoint $\gamma$-sets of $P_2 \times P_n$ with $|S| = |S'|$. Hence $\gamma\gamma(P_2 \times P_n) = 2\gamma(P_2 \times P_n)$.

Case (ii). $n = 4k, k \in \mathbb{N}$

$S = \{u_1, v_3, u_5, v_7, \ldots, v_{n-5}, u_{n-3}, v_{n-1}, u_n\}$ and $S' = \{v_1, u_3, v_5, u_7, \ldots, u_{n-5}, v_{n-3}, u_{n-1}, v_n\}$ are two disjoint $\gamma$-sets of $P_2 \times P_n$ with $|S| = |S'|$. Hence $\gamma\gamma(P_2 \times P_n) = 2\gamma(P_2 \times P_n)$.

Case (iii). $n = 4k + 1, k \in \mathbb{N}$

$S = \{u_1, v_3, u_5, v_7, \ldots, u_{n-4}, v_{n-2}, u_n\}$ and $S' = \{v_1, u_3, v_5, u_7, \ldots, v_{n-4}, u_{n-2}, v_n\}$ are two disjoint $\gamma$-sets of $P_2 \times P_n$ with $|S| = |S'|$. Hence $\gamma\gamma(P_2 \times P_n) = 2\gamma(P_2 \times P_n)$.

Case (iv). $n = 4k + 2, k \in \mathbb{N}$

$S = \{u_1, v_3, u_5, v_7, \ldots, u_{n-5}, v_{n-3}, u_{n-1}, v_n\}$ and $S' = \{v_1, u_3, v_5, u_7, \ldots, v_{n-5}, u_{n-3}, v_{n-1}, u_n\}$ are two disjoint $\gamma$-sets of $P_2 \times P_n$ with $|S| = |S'|$. Hence $\gamma\gamma(P_2 \times P_n) = 2\gamma(P_2 \times P_n)$.

From the above cases we get, $\gamma\gamma(P_2 \times P_n) = 2\gamma(P_2 \times P_n)$. □
Theorem 2.2.11.

\[
\gamma\gamma(P_3 \times P_n) = \begin{cases} 
3 & \text{if } n = 1 \\
4 & \text{if } n = 2 \\
8 & \text{if } n = 4 \\
6k & \text{if } n = 4k - 1 \\
2\gamma(P_3 \times P_n) + 1 & \text{otherwise}
\end{cases}
\]

**Proof.** When \( n = 1, 2 \) and \( 4 \) the result is obvious.

Let \( \{u_1, u_2, \ldots, u_n\}, \{v_1, v_2, \ldots, v_n\} \) and \( \{w_1, w_2, \ldots, w_n\} \) be the vertices of the three rows of the grid \( P_3 \times P_n \) as shown in the figure. In [20], it has been proved that \( \gamma(P_3 \times P_n) = \left\lfloor \frac{3n+4}{4} \right\rfloor \).

**Case (i) \( n = 4k - 1, k \in \mathbb{N} \)**

\( S = \{u_1, w_1, v_3, u_5, w_5, v_7, \ldots, u_{n-2}, w_{n-2}, v_n\} \) and \( S' = \{v_1, u_3, w_3, v_5, u_7, w_7, \ldots, v_{n-2}, u_{n-2}, u_n, w_n\} \) are two disjoint \( \gamma \)-sets of \( P_3 \times P_n \) with \( |S| = 

\(|S'| = 3k. \text{ Hence } \gamma \gamma (P_3 \times P_n) = 6k.\)

Case (ii) \(n = 4k, k \in \mathbb{N} - \{1\}\)

\(S = \{u_1, w_1, v_3, u_5, w_5, v_7, \ldots, u_{n-3}, w_{n-3}, v_{n-1}, u_n, w_n\}\) and \(S' = \{v_1, u_3, w_3, v_5, u_7, w_7, \ldots, u_{n-5}, w_{n-5}, v_{n-3}, u_{n-1}, w_{n-1}, v_n\}\) are two disjoint \(\gamma\)-sets of \(P_3 \times P_n\) with \(|S| = \left\lfloor \frac{3k+4}{4} \right\rfloor\) and \(|S'| = \left\lfloor \frac{3k+4}{4} \right\rfloor + 1\). Hence \(\gamma \gamma (P_3 \times P_n) = 2\gamma (P_3 \times P_n) + 1.\)

Case (iii) \(n = 4k + 1, k \in \mathbb{N}\)

\(S = \{u_1, w_1, v_3, u_5, w_5, v_7, \ldots, v_{n-2}, u_n, w_n\}\) and \(S' = \{v_1, u_3, w_3, v_5, u_7, w_7, \ldots, u_{n-2}, w_{n-2}, v_n\}\) are two adjacent disjoint \(\gamma\)-sets of \(P_3 \times P_n\) with \(|S| = \gamma (P_3 \times P_n)\) and \(|S'| = \gamma (P_3 \times P_n) + 1\). Hence \(\gamma \gamma (P_3 \times P_3) = 2\gamma (P_3 \times P_n) + 1.\)

Case (iv) \(n = 4k + 2, k \in \mathbb{N}\)

\(S = \{u_1, w_1, v_3, u_5, v_7, \ldots, u_{n-1}, w_{n-1}, v_n\}\) and \(S' = \{v_1, u_3, w_3, v_5, u_7, w_7, \ldots, u_{n-3}, w_{n-3}, v_{n-1}, v_n\}\) are two disjoint \(\gamma\)-sets of \(P_3 \times P_n\) with \(|S| = \gamma (P_3 \times P_n) + 1\) and \(|S'| = \gamma (P_3 \times P_n)\). Hence \(\gamma \gamma (P_3 \times P_n) = 2\gamma (P_3 \times P_n) + 1.\)

\(\square\)

Theorem 2.2.12. \(\gamma \gamma (P_4 \times P_n) = 2\gamma (P_4 \times P_n).\)
Proof. \{u_1, u_2, \ldots, u_n\}, \{v_1, v_2, \ldots, v_n\}, \{w_1, w_2, \ldots, w_n\} and \{x_1, x_2, x_3, \ldots, x_n\} are the vertices of the 1st, 2nd, 3rd and 4th rows of the grid $P_4 \times P_n$ as shown in the figure. We have

$$\gamma(P_4 \times P_n) = \begin{cases} n + 1 & \text{if } n = 1, 2, 3, 5, 6, 9 \\ n & \text{otherwise} \end{cases}$$

Case(1): $n = 3k$

When $n = 3$, $S = \{w_1, u_2, v_3, x_3\}$ and $S' = \{v_1, x_2, u_3, w_3\}$ are two disjoint $\gamma$-sets. Hence $\gamma\gamma(P_4 \times P_3) = 2\gamma(P_4 \times P_3)$.

When $n = 6$, $S = \{v_1, x_2, u_3, w_4, v_5, u_6, x_6\}$ and $S' = \{w_1, u_2, x_3, v_4, u_5, x_5, w_6\}$ are two disjoint $\gamma$-sets. Hence $\gamma\gamma(P_4 \times P_6) = 2\gamma(P_4 \times P_6)$.

When $n = 9$, $S = \{u_1, x_1, v_2, w_3, u_4, x_5, v_6, x_7, u_8, w_9\}$ and $S' = \{v_1, x_2, u_3, w_4, u_5, x_6, v_7, w_8, u_9, x_9\}$ are two disjoint $\gamma$-sets. Hence $\gamma\gamma(P_4 \times P_9) = 2\gamma(P_4 \times P_9)$.
\(P_9) = 2\gamma(P_4 \times P_9)\).

When \(n = 12\), \(S = \{v_1, x_2, u_3, w_4, v_5, x_6, v_7, w_8, v_9, x_{10}, u_{11}, w_{12}\}\) and \(S' = \{w_1, u_2, x_3, v_4, w_5, u_6, x_7, v_8, w_9, u_{10}, x_{11}, v_{12}\}\) are two disjoint \(\gamma\)-sets. Hence \(\gamma\gamma(P_4 \times P_{12}) = 2\gamma(P_4 \times P_{12})\).

**Subcase(1.i):** \(n\) is odd, \(k = 5, 7, 9, \ldots\)

\(S = \{v_1, x_2, u_3, w_4, u_5, x_6, v_7, \ldots, w_{n-7}, u_{n-6}, x_{n-5}, v_{n-4}, w_{n-3}, u_{n-2}, x_{n-1}, v_n\}\) and \(S' = \{w_1, u_2, x_3, v_4, w_5, u_6, x_7, v_8, x_9, \ldots, w_{n-8}, v_{n-7}, x_{n-6}, u_{n-5}, w_{n-4}, v_{n-3}, x_{n-2}, u_{n-1}, w_n\}\) are two disjoint \(\gamma\)-sets. Hence \(\gamma\gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n)\).

**Subcase(1.ii):** \(n\) is even, \(k = 6, 8, 10, \ldots\)

\(S = \{v_1, x_2, u_3, w_4, u_5, x_6, v_7, x_8, v_9, \ldots, w_{n-8}, v_{n-7}, x_{n-6}, u_{n-5}, w_{n-4}, v_{n-3}, x_{n-2}, u_{n-1}, w_n\}\) and \(S' = \{w_1, u_2, x_3, v_4, x_5, u_6, w_7, u_8, x_9, \ldots, v_{n-8}, w_{n-7}, u_{n-6}, x_{n-5}, v_{n-4}, w_{n-3}, u_{n-2}, x_{n-1}, v_n\}\) are two disjoint \(\gamma\)-sets. Hence \(\gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n)\).

**Case (2):** \(n = 3k + 1\)

When \(n = 1\), \(S = \{u_1, w_1\}\) and \(S' = \{v_1, x_1\}\) are two disjoint \(\gamma\)-sets. Hence \(\gamma\gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n)\).

**Subcase(2.i):** \(n\) is odd, \(k = 2, 4, 6, \ldots\)

\(S = \{v_1, x_2, u_3, w_4, u_5, x_6, v_7, \ldots, x_{n-7}, u_{n-6}, x_{n-5}, v_{n-4}, w_{n-3}, u_{n-2}, x_{n-1}, v_n\}\) and \(S' = \{w_1, u_2, x_3, v_4, x_5, u_6, w_7, u_8, x_9, \ldots, u_{n-7}, w_{n-6}, u_{n-5}, x_{n-4}, v_{n-3}, x_{n-2}, u_{n-1}, w_n\}\) are two disjoint \(\gamma\)-sets. Hence \(\gamma\gamma(P_4 \times P_n) =\)
2\gamma(P_4 \times P_n).

**Subcase(2.ii):** $n$ is even, $k = 1, 3, 5, \ldots$

$S = \{v_1, x_2, u_3, w_4, u_5, x_6, v_7, x_8, \ldots, u_{n-7}, w_{n-6}, u_{n-5}, x_{n-4}, v_{n-3}, x_{n-2}, u_{n-1}, w_n\}$ and $S' = \{w_1, u_2, x_3, v_4, x_5, u_6, w_7, u_8, \ldots, x_{n-7}, v_{n-6}, x_{n-5}, u_{n-4}, w_{n-3}, u_{n-2}, x_{n-1}, v_n\}$ are two disjoint $\gamma$-sets. Hence $\gamma\gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n)$.

**Case(3):** $n = 3k + 2$

When $n = 2$, $S = \{u_1, x_1, v_2\}$ and $S' = \{v_1, u_2, x_2\}$ are two disjoint $\gamma$-sets. Hence $\gamma\gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n)$.

When $n = 5$, $S = \{v_1, x_2, u_3, w_4, v_5, x_6\}$ and $S' = \{u_2, x_3, v_4, u_5, w_5\}$ are two disjoint $\gamma$-sets. Hence $\gamma\gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n)$.

When $n = 8$, $S = \{v_1, x_2, u_3, w_4, v_5, x_6, u_7, w_8\}$ and $S' = \{u_2, x_3, v_4, w_5, u_6, x_7, v_8\}$ are two disjoint $\gamma$-sets. Hence $\gamma\gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n)$.

**Subcase(3.1):** $n$ is odd, $k = 3, 5, 7, \ldots$

$S = \{v_1, x_2, u_3, w_4, u_5, x_6, v_7, x_8, \ldots, v_{n-7}, u_{n-6}, x_{n-5}, v_{n-4}, w_{n-3}, u_{n-2}, x_{n-1}, v_n\}$ and $S' = \{w_1, u_2, x_3, v_4, x_5, u_6, w_7, u_8, \ldots, x_{n-6}, u_{n-5}, w_{n-4}, v_{n-3}, x_{n-2}, u_{n-1}, w_n\}$ are two disjoint $\gamma$-sets. Hence $\gamma\gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n)$.

**Subcase(3.ii):** $n$ is even, $k = 4, 6, 8, \ldots$

$S = \{v_1, x_2, u_3, w_4, u_5, x_6, v_7, x_8, u_9, \ldots, x_{n-8}, v_{n-7}, x_{n-6}, u_{n-5}, w_{n-4}, v_{n-3}, x_{n-2}, u_{n-1}, w_n\}$ and $S' = \{w_1, u_2, x_3, v_4, x_5, u_6, w_7, u_8, x_9, \ldots, w_8, u_7, x_6, v_5, x_4, v_3, u_2, x_1, v_1\}$ are two disjoint $\gamma$-sets. Hence $\gamma\gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n)$. 

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\(u_{n-8}, w_{n-7}, u_{n-6}, x_{n-5}, v_{n-4}, w_{n-3}, u_{n-2}, x_{n-1}, v_n\) are two disjoint \(\gamma\)-sets. Hence \(\gamma\gamma(P_4 \times P_3) = 2\gamma(P_4 \times P_3)\). Thus from the above cases, \(\gamma\gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n)\). \(\square\)

**Theorem 2.2.13.** For any 2 integers \(m\) and \(n\) with \(n \geq 5, m \geq 2\) we can construct a tree \(T\) with \(\gamma\gamma(T) = n - m + 1\).

**Proof.** Let \(m\) and \(n\) be 2 integers with \(n \geq 4\) and \(m \geq 2\). Let \(u\) be the root of \(T\) which is a pendent vertex and \(m\) be the degree of the support \(v\) of \(u\). Let the neighbors of \(v\) other than \(u\) be \(u_1, u_2, u_3, \ldots, u_{m-1}\). Each \(u_i\) can have any number of neighbors say \(u_{ij}\) other than \(v\) where \(1 \leq i < m - 1\) and \(j \geq 1\) such that \(\sum_i \sum_j 1 = r\) and each \(u_{ijk}, k \geq 1\) such that \(\sum_i \sum_j \sum_k 1 = s\). Then \(\sum_i \sum_j 1 + \sum_i \sum_j \sum_k 1 + m + 1 = n\). We now construct two disjoint dominating sets of \(T, S\) and \(S'\). Without loss of generality let \(u \in S\). Then as \(v\) is adjacent to \(u, u \notin S', v \in S'\). Then \(u_{ij}, 1 \leq i \leq m - 1\) must be in \(S\). Since \(u_{ijk}\) are adjacent to \(u_{ij}, u_{ijk} \in S'\). Thus members of \(S\) are \(u\) and \(u'_{ijk}\). Therefore \(S\) is a dominating set of \(T\). As \(v \in S'\), it dominates \(u\) and \(u_1, u_2, u_3, \ldots, u_{m-1}\). Hence \(u, u_1, u_2, u_3, \ldots, u_{m-1} \notin S'\). Let \(u_{ijk} \in S'\). Then \(u'_{ijk}\) dominates \(u'_{ij}\). Thus the set \(S'\) consisting of the vertices \(v\) and \(u'_{ijk}\) is a dominating set of \(T\) disjoint from \(S\). There is no other dominating set disjoint
with $S$ or $S'$ with minimum cardinality as any dominating set should contain either $u$ or $v$. Hence $\gamma\gamma(T) = |S \cup S'| = n - m + 1$. □

**Theorem 2.2.14.** For a $n$-star graph $S_n$, $\gamma\gamma(S_n) = 2\gamma(S_n)$ and consequently $S_n$ is a $\gamma\gamma$-minimum graph.

**Proof.** We know that $S_n$ is $(n - 1)$ regular. By Theorem 1.1.18, $\gamma(S_n) = (n - 1)!$ and $S_n$ is domatically full. Also every class of domatic partition of $S_n$ is a $\gamma$-set of $S_n$. Hence any two members of the domatic partition form a pair of disjoint dominating sets and so $\gamma\gamma(S_n) = 2\gamma(S_n)$. □

In [16], Hypercubes are conjectured to be $\gamma\gamma$-minimum for all $n \geq 2$. We give a partial solution to this conjecture.

**Definition 2.2.15.** [5] The $n$-cube $Q_n$ is a graph whose vertex set is the set of all $n$-dimensional boolean vectors, two vertices being joined if and only if they differ in exactly one co-ordinate. We use the following notation. By $(0)$ we mean the boolean vector with all coordinate 0. If $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, we denote by $(i_1, i_2, \ldots, i_k)$ the $n$-tuple having 1 in the coordinate $i_1, i_2, \ldots, i_k$ and 0 elsewhere.
Example 2.2.16. \( \gamma \gamma(Q_n) = 2\gamma(Q_n) \) for \( 1 \leq n \leq 7 \). \( \gamma \gamma(Q_1) = 2 \) since \( S = \{(1)\} \) and \( S' = \{(2)\} \) are two disjoint dominating sets of \( Q_1 \). \( \gamma \gamma(Q_2) = 4 \) since \( S = \{(1),(2)\} \) and \( S' = \{(0),(1,2)\} \) are two disjoint dominating sets of \( Q_2 \). \( \gamma \gamma(Q_3) = 4 \) since \( S = \{(0),(1,2,3)\} \) and \( S' = \{(1),(2,3)\} \) are two disjoint dominating sets of \( Q_3 \). \( \gamma \gamma(Q_4) = 8 \) since \( S = \{(1),(9,2),(3,4),(1,2,3,4)\} \) and \( S' = \{(3),(1,2),(2,4),(1,3,4)\} \) are two disjoint dominating sets of \( Q_4 \). \( \gamma \gamma(Q_5) = 14 \) since \( S = \{(0),(1,2),(1,3),(1,4,5),(2,3,4),(2,3,5),(2,3,4,5)\} \) and \( S' = \{(1),(2),(3),(4,5),(1,2,3,4),(1,2,3,5),(1,2,3,4,5)\} \) are two disjoint dominating sets of \( Q_5 \). \( \gamma \gamma(Q_6) = 24 \) since \( S = \{(0),(1,3),(2,3),(1,2,4),(1,5,6),(2,5,6),(3,4,5),(3,4,6),(1,2,4,5),(1,2,4,6),(3,4,5,6),(1,2,3,5,6)\} \) and \( S' = \{(5),(1,4),(2,6),(3,4),(1,2,3),(1,3,6),(2,4,5),(4,5,6),(1,2,5,6),(1,3,4,5),(2,3,5,6),(1,2,3,4,6)\} \) are two disjoint dominating sets of \( Q_6 \). \( \gamma \gamma(Q_7) = 32 \) since \( S = \{(0),(1,2,7),(1,3,4),(1,5,6),(2,3,5),(2,4,6),(3,6,7),(4,5,7),(1,2,3,6),(1,2,4,5),(1,3,5,7),(1,4,6,7),(2,3,4,7),(2,5,6,7),(3,4,5,6),(1,2,3,4,5,6,7)\} \) and \( S' = \{(1),(2,6),(3,4),(5,7),(2,3,7),(2,4,5),(3,5,6),(4,6,7),(1,2,3,5),(1,2,4,7),(1,3,6,7),(1,4,5,6),(1,2,3,4,6),(1,2,5,6,7),(1,3,4,5,7),(2,3,4,5,6,7)\} \) are two disjoint dominating sets of \( Q_7 \). Hence by [5] \( \gamma \gamma(Q_n) = 2\gamma(Q_n) \) for \( 1 \leq n \leq 7 \).
In this connection, we propose the following conjecture.

**Conjecture 2.2.17.** Hypercubes $Q_n$ are $\gamma\gamma$-minimum for $n \geq 8$.

**Theorem 2.2.18.** Let $G$ be a graph without isolated vertices. Then $2 \leq \gamma\gamma(G) \leq p$. Lower bound is attained if and only if $G \cong K_n$ or $G$ has at least two vertices of full degree.

**Proof.** Obviously $2 \leq \gamma\gamma(G) \leq p$. Suppose $\gamma\gamma(G) = 2$. Then there exists two disjoint dominating sets $S$ and $S'$ such that both have cardinality one. This is possible if and only if $G \cong K_n$ or $G$ has at least two vertices of full degree. □

**Definition 2.2.19.** The trestled graph of index $k$ denoted by $T_k(G)$ is a graph obtained from $G$ adding $k$ copies of $K_2$ corresponding to each edge $uv$ of $G$ and joining $u$ and $v$ to the respective end vertices of each $K_2$.

**Theorem 2.2.20.** If $G$ is a trestled graph of index $k$ of a cycle $C_n$ then $\gamma\gamma(G) = (k + 1)n$ where $k \in \mathbb{N}$.

**Proof.** The trestled graph of a cycle $C_n$ of index $k$ contains $n + 2kn = (2k + 1)n$ vertices. The set of $n$ vertices of the cycle $C_n$ say $S$ is a $\gamma$-set
of \( G \). The set of any one of the vertices of each of the newly added edge say \( S' \) is another minimum dominating set of \( G \) containing \( nk \) vertices. Also \( S \cap S' = \phi \). Hence \( \gamma\gamma(G) = (k + 1)n \). \( \square \)

**Corollary 2.2.21.** If \( G \cong T_1(C_n) \) then \( \gamma\gamma(G) = 2\gamma(G) \).

**Theorem 2.2.22.** If \( G \cong T_k(P_n) \) then \( \gamma\gamma(G) = n + k(n - 1) \) for \( k = 2, 3, 4, \ldots \).

**Proof.** The set of \( n \) vertices of the path \( P_n \) say \( S \) is a \( \gamma \)-set of \( T_k(P_n) \). Hence \( |S| = n \). \( P_n \) has \( (n - 1) \) edges and corresponding to each edge there are \( k \) edges. The set of one of the vertices of these \( k(n - 1) \) edges say \( S' \) form a dominating set of \( (T_k(P_n)) \) and \( S \cap S' = \phi \). Hence \( |S'| = k(n - 1) \) and so \( \gamma\gamma(T_k(P_n)) = n + k(n - 1) \). Thus \( \gamma\gamma(T_k(P_n)) = n + k(n - 1) \). \( \square \)

**Theorem 2.2.23.** \( \gamma\gamma(T_1(P_n)) = 2n \).

**Proof.** Clearly \( |V(T_1(P_n))| = 3n-2 \). Suppose the vertices are labeled as in the diagram.
Obviously $S_1 = \{v_1, v_{21}, v_{31}, v_{41}, \ldots, v_{n1}\}$ and $S_2 = \{v_{11}, v_{22}, v_{32}, \ldots, v_{n-1,2}, v_n\}$ are 2 disjoint $\gamma$-sets with $|S_1| = |S_2| = n$. Hence $\gamma\gamma(T_1(P_n)) = 2n$. □

**Theorem 2.2.24.** If $G \cong T_m(K_1,n))$ then $\gamma\gamma(T_m(K_1,n))) = n+1+mn$.

**Proof.** The set of $(n+1)$ vertices of the star say $S$ dominates $G$ and hence $|S| = n + 1$. $K_{1,n}$ has $n$ edges and corresponding to each edge there are $m$ edges. The set consisting of one vertex from each of the $nk$ edges $S'$ form an independent dominating set of $G$. Hence $|S'| = mn$. Also $S \cap S' = \phi$. Therefore $\gamma\gamma(T_m(K_{1,n})) = n+1+mn$. □

**Corollary 2.2.25.** $\gamma\gamma(T_1(K_{1,n})) = 2\gamma(T_1(K_{1,n}))$.

**Proof.** The $(n+1)$ vertices of $K_{1,n}$ dominates $T_1(K_{1,n})$. Hence $|S| = n + 1$. The set consisting of one vertex from each of the newly added edge of $T_1(K_{1,n})$ together with the other vertex of the last edge say $S'$ form a dominating set of $T_1(K_{1,n})$ disjoint from $S$. Hence $|S'| = n + 1$ and so $\gamma\gamma(T_1(K_{1,n})) = 2(n + 1) = 2\gamma(T_1(K_{1,n}))$. □

**Definition 2.2.26.** The total graph $T(G)$ of a graph $G = (V, E)$ has vertices that correspond one to one with elements of $V \cup E$. Two
vertices in $T(G)$ are adjacent if and only if the corresponding elements are adjacent or incident in $G$.

**Theorem 2.2.27.**

$$
\gamma(T(P_n)) = \begin{cases} 
2\gamma(T(P_n)) + 1 & \text{if } n \equiv 3 \pmod{5} \\
2\gamma(T(P_n)) & \text{otherwise}
\end{cases}
$$

**Proof.** Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set of $P_n$. As $P_n$ contains $n$ vertices, $T(P_n)$ contains $(2n-1)$ vertices say $v_1, e_1, v_2, e_2, v_3, \ldots, e_{n-1}, v_n$. It is easy to observe that $\gamma(T(P_n)) = \lceil \frac{2n-1}{5} \rceil$.

**Case(i):** $n \equiv 0 \pmod{5}$

$S = \{v_2, e_4, v_7, e_9, v_{12}, \ldots, v_{n-8}, e_{n-6}, v_{n-3}, e_{n-1}\}$ and $S' = \{e_1, v_4, e_6, v_9, e_{11}, \ldots, e_{n-9}, v_{n-6}, v_{n-4}, v_{n-1}\}$ are two disjoint $\gamma$-sets of $T(P_n)$. Hence $\gamma(T(P_n)) = 2\gamma(T(P_n))$.

**Case(ii):** $n \equiv 1 \pmod{5}$

$S = \{v_2, e_4, v_7, e_9, v_{12}, \ldots, e_{n-9}, v_{n-6}, e_{n-4}, v_{n-1}\}$ and $S' = \{e_1, v_4, e_6, v_9, e_{11}, \ldots, e_{n-7}, v_{n-4}, e_{n-2}, v_n\}$ are two disjoint $\gamma$-sets of $T(P_n)$. Hence $\gamma(T(P_n)) = 2\gamma(T(P_n))$.

**Case(iii):** $n \equiv 2 \pmod{5}$

$S = \{v_2, e_4, v_7, e_9, v_{12}, \ldots, e_{n-8}, v_{n-5}, e_{n-3}, v_{n-1}\}$ and $S' = \{e_1, v_4, e_6, v_9, e_{11}, \ldots, v_{n-8}, e_{n-6}, v_{n-3}, e_{n-1}\}$ are two disjoint $\gamma$-sets of $T(P_n)$. Hence
\[\gamma\gamma(T(P_n)) = 2\gamma(T(P_n)).\]

**Case (iv):** \(n \equiv 3 \pmod{5}\)

\[S = \{v_2, e_4, v_7, e_9, v_{12}, \ldots, e_{n-9}, v_{n-6}, e_{n-4}, v_{n-1}\}\]

is the \(\gamma\)-set of \(T(P_n)\) and is unique. \(S' = \{e_1, v_4, e_6, v_9, e_{11}, \ldots, e_{n-7}, v_{n-4}, e_{n-2}, v_n\}\) is the minimal dominating set of \(T(P_n)\) disjoint from \(S\) and \(|S'| = |S| + 1\).

Hence \(\gamma\gamma(T(P_n)) = 2\gamma(T(P_n)) + 1\).

**Case (v):** \(n \equiv 4 \pmod{5}\)

\[S = \{v_2, e_4, v_7, e_9, v_{12}, \ldots, v_{n-7}, e_{n-5}, v_{n-2}, e_{n-1}\}\]

and \(S' = \{e_1, v_4, e_6, v_9, e_{11}, \ldots, e_{n-8}, v_{n-5}, e_{n-3}, v_n\}\) are two disjoint \(\gamma\)-sets of \(T(P_n)\). Hence

\[\gamma(T(P_n)) = 2\gamma(T(P_n)).\]

\[\square\]

**Theorem 2.2.28.** \(\gamma\gamma(T(C_n)) = 2 \left\lceil \frac{2n}{5} \right\rceil\).

**Proof.** \(T(C_n)\) contains \(2n\) vertices and each vertex is of degree 4 as each vertex of \(T(C_n)\) is incident with two vertices and two edges of \(C_n\). Since \(T(C_n)\) has \(2n\) vertices, we can construct two disjoint dominating sets \(S\) and \(S'\) of \(T(C_n)\) in such a way that no two vertices of either \(S\) or \(S'\) dominates the same vertex. Thus \(|S| = |S'| = \left\lceil \frac{2n}{5} \right\rceil\) and therefore

\[\gamma\gamma(T(C_n)) = 2 \left\lceil \frac{2n}{5} \right\rceil.\]

\[\square\]

**Theorem 2.2.29.** \(\gamma\gamma(T(K_{1,n})) = n + 1\).
Proof. Let $v$ be the center and $v_1, v_2, v_3, \ldots, v_{n-1}, v_n$ be the pendant vertices of $K_{1,n}$. Then $T(K_{1,n})$ contains $(2n + 1)$ vertices. $v$ is the universal vertex of $T(K_{1,n})$. Hence $S = \{v\}$ is the $\gamma$-set of $T(K_{1,n})$ and $S' = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$ is a minimal dominating set of $T(K_{1,n})$. Also $S \cap S' = \emptyset$ and $|S'| = n$. Any minimal dominating set other than $S$ contains $n$ vertices. Hence $\gamma(T(K_{1,n})) = n + 1$.

In [16] “when is $\gamma(G) + \gamma(\overline{G}) = n + 4$? ” was posed as an open problem. We observe that if $G \cong C_4$ or a connected graph in which every vertex is a leaf or a stem then $\gamma(G) + \gamma(\overline{G}) = n + 4$. □

Example 2.2.30. The path $P_n$ with $n = 3k + 2, k \in \mathbb{N}$ has $\gamma\gamma(P_n) = \frac{2(n+1)}{3}$.

$\gamma\gamma(P_n) = 2\gamma(P_{3k+2}) = 2(k + 1) = 2\left(\frac{(n-2)}{3} + 1\right) = \frac{2n+2}{3} = \frac{2(n+1)}{3}$.

This example gives a partial answer to the question “for which class of trees $T$ is $\gamma\gamma(T) = \frac{2(n+1)}{3}$.”

We also see that the cycle $C_n$ with $n = 3k + 2, k \in \mathbb{N}$ has $\gamma(C_n) = \frac{2(n+1)}{3}$. For, $\gamma\gamma(C_n) = \gamma\gamma(C_{3k+2}) = 2 \left[\frac{3k+2}{3}\right] = 2(k + 1) = \frac{2(n+1)}{3}$.