Chapter 6

Modified $\gamma$ graph-$G(\gamma_m)$ of some special graphs

6.1 Introduction

Gerd H.Frickle et.al [11] introduced $\gamma$-graph of a graph. The concept of $\gamma$-graph inspires the following concept. Consider the family of all $\gamma$-sets of a graph $G$ and define the modified $\gamma$-graph $G(\gamma_m) = (V(\gamma_m), E(\gamma_m))$ of $G$ to be the graph whose vertices $V(\gamma_m)$ corresponds 1-1 with the $\gamma$-sets of $G$ and two $\gamma$-sets $S_1$ and $S_2$ form an edge in $G(\gamma_m)$ if there exists a vertex $v \in S_1$ and $w \in S_2$ such that $S_1 = S_2 - \{w\} \cup \{v\}$ and $S_2 = S_1 - \{v\} \cup \{w\}$. In this chapter we determine $G(\gamma_m)$ of some special graphs.
6.2 Main results

**Definition 6.2.1.** Consider the family of all \(\gamma\)-sets of a graph \(G\) and define the modified \(\gamma\)-graph \(G(\gamma_m) = (V(\gamma_m), E(\gamma_m))\) of \(G\) to be the graph whose vertices \(V(\gamma_m)\) corresponds 1-1 with the \(\gamma\)-sets of \(G\) and two \(\gamma\)-sets \(S_1\) and \(S_2\) form an edge in \(G(\gamma_m)\) if there exists a vertex \(v \in S_1\) and \(w \in S_2\) such that \(S_1 = S_2 - \{w\} \cup \{v\}\) and \(S_1 = S_2 - \{v\} \cup \{w\}\).

Thus two \(\gamma\)-sets are said to be adjacent if they differ by one vertex.

**Example 6.2.2.**

Graph

\[
S_1 = \{v_2, v_6\}, S_2 = \{v_2, v_7\}, S_3 = \{v_2, v_8\} \text{ are the } \gamma\text{-sets of } G. \text{ Then } G(\gamma_m):\]

Graph

**Proposition 6.2.3.** If a graph \(G = (V, E)\) has a unique \(\gamma\)-set then \(G(\gamma_m) = K_1\) and conversely.

**Corollary 6.2.4.** \(K_{1,n}(\gamma_m) \cong K_1.\)
Proof. Since the set with the central vertex of \( K_1 \) is the only \( \gamma \)-set of \( K_{1,n}, K_{1,n}(\gamma_m) \cong K_1 \).

\[ \square \]

Proposition 6.2.5. \( \overline{K_n}(\gamma_m) \cong K_n \) where as \( K_n(\gamma_m) \cong K_n \).

Proposition 6.2.6. For \( m, n \geq 2, K_{m,n}(\gamma_m) \) is isomorphic to an \((m + n - 2)\) regular graph of order \( mn \).

Proof. Let \( \{u_1, u_2, u_3, \ldots, u_m\} \) and \( \{v_1, v_2, v_3, \ldots, v_n\} \) be the partitions of the vertex set of the graph \( K_{m,n} \). Then \( \{u_i, v_j\}, 1 \leq i \leq m \) and \( 1 \leq j \leq n \) are the \( \gamma \)-sets of \( K_{m,n} \) with cardinality 2. Thus \( K_{m,n} \) has \( mn \) \( \gamma \)-sets and so \( G(\gamma_m) \) has \( mn \) vertices. Each \( \{u_i, v_j\} \) is adjacent to \( \{u_i, v_k\} \) for \( k \neq j \) and \( \{u_k, v_j\} \) for \( k \neq i \). Hence each \( \{u_i, v_j\} \) for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \) is of deg \( m + n - 2 \). Thus degree of each vertex of \( G(\gamma_m) \) is \( m + n - 2 \).

\[ \square \]

Proposition 6.2.7. \( C_{3k+2}(\gamma_m) \cong C_{3k+2} \)

Proposition 6.2.8. \( C_{3k}(\gamma_m) \cong \overline{K}_3 \).

Proposition 6.2.9. For \( k \geq 1, C_{3k+1}(\gamma_m) \) is isomorphic to a 4-regular graph and
Order of $C_{3k+1} = \begin{cases} 
6 & \text{if } k = 1 \\
14 & \text{if } k = 2 \\
\frac{3k^2 + 7k + 2}{2} & \text{if } k \geq 3 
\end{cases}$

**Proof.** We know that $\gamma(C_{3k+1}) = k + 1$. Let $v_1, v_2, v_3, \ldots, v_{3k+1}$ to be the vertices of $C_{3k+1}$. For convenience let us arrange the vertices of the $\gamma$-sets of $C_{3k+1}$ in the ascending order of the suffixes of the vertices.

**Case(i):** $k = 1$

The cycle is $C_4$ and $C_4$ has 6 $\gamma$-sets namely $S_1 = \{v_1, v_2\}$, $S_2 = \{v_1, v_4\}$, $S_3 = \{v_1, v_3\}$, $S_4 = \{v_2, v_3\}$, $S_5 = \{v_2, v_4\}$ and $S_6 = \{v_3, v_4\}$. Number of $\gamma$-sets of $C_4$ is 6 (i.e) $C_4(\gamma_m)$ is of order 6.

**Case(ii):** $k = 2$

The cycle obtained is $C_7$. The $\gamma$-sets of $C_7$ are $S_1 = \{v_1, v_4, v_6\}$, $S_2 = \{v, v_3, v_6\}$, $S_3 = \{v_1, v_4, v_7\}$, $S_4 = \{v_1, v_2, v_5\}$, $S_5 = \{v_1, v_3, v_5\}$, $S_6 = \{v_1, v_4, v_5\}$, $S_7 = \{v_2, v_5, v_7\}$, $S_8 = \{v_2, v_5, v_6\}$, $S_9 = \{v_2, v_3, v_6\}$, $S_{10} = \{v_2, v_4, v_6\}$, $S_{11} = \{v_2, v_4, v_7\}$, $S_{12} = \{v_3, v_6, v_7\}$, $S_{13} = \{v_3, v_5, v_7\}$, $S_{14} = \{v_3, v_4, v_7\}$. Thus the total number of $\gamma$-sets of $C_7$ is 14.

**Case(iii):** $k \geq 3$

**Step (1):** Consider the 6 $\gamma$-sets of $C_{3k+1}$ namely $S_1 = \{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k+1}\}$, $S_2 = \{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k}\}$, $S_3 =$
\{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k-1}\}, \ S_4 = \{v_2, v_5, v_8, \ldots, v_{3k-1}, v_{3k+1}\}, \ S_5 = \{v_2, v_5, v_8, \ldots, v_{3k-1}, v_{3k}\}, \ S_6 = \{v_3, v_6, v_9, \ldots, v_{3k}, v_{3k+1}\}, \ S_1 \text{ is the only } \gamma\text{-set with first vertex } v_1 \text{ and last vertex } v_{3k+1}.

**Step (2):** Now fixing the first and last vertices of \(S_2\) and changing from the 2\(^{nd}\) vertex we get \(\{v_1, v_3, v_6, v_9 \ldots, v_{3k-3}, v_{3k}\}\). Similarly changing from the third , fourth , fifth, \ldots , \(k^{th}\) vertex we get \((k-2)\) \(\gamma\)-sets. Thus step(2) contains \((k-1)\) \(\gamma\)-sets.

**Step (3):** Fixing the first and last vertices of \(S_3\) and changing from the second vertex we get \(\{v_1, v_3, v_6, v_9 \ldots, v_{3k-3}, v_{3k-1}\}\). Similarly changing from the third , fourth , fifth, \ldots , \(k^{th}\) vertex we get \((k-2)\) \(\gamma\)-sets. Thus step(3) contains \((k-1)\) \(\gamma\)-sets.

**Step (4):** Now consider all the \(\gamma\)-sets with two adjacent vertices. They are \(\{v_1, v_2, v_5, v_8, v_{11} \ldots, v_{3k-4}, v_{3k-1}\}\), \(\{v_1, v_4, v_5, v_8, v_{11} \ldots, v_{3k-4}, v_{3k-1}\}\), \(\ldots\), \(\{v_1, v_4, v_7, v_{10} \ldots, v_{3k-5}, v_{3k-4}, v_{3k-1}\}\). Thus this step has \((k-1)\) \(\gamma\)-sets.

**Step (5):** Now consider all the \(\gamma\)-sets which contains 3 alternate vertices. They are \(\{v_1, v_3, v_5, v_8, v_{11} \ldots, v_{3k-1}\}\), \(\{v_1, v_4, v_6, v_{8}, v_{11} v_{14} \ldots, v_{3k-1}\}\), \(\ldots\), \(\{v_1, v_4, v_7, \ldots, v_{3k-8}, v_{3k-6}, v_{3k-4}, v_{3k-1}\}\). Thus this step has \((k-2)\) \(\gamma\)-sets. The succeeding \(\gamma\)-sets \(\{v_1, v_4, v_7, v_{10} \ldots, v_{3k-5}, v_{3k-3}, v_{3k-1}\}\) is already occurred in step(3).

**Step (6):** Fixing the first vertex as \(v_1\) and last two vertices as \(v_{3k-4}, v_{3k+1}\),
we get \((k - 2)C_2\) \(\gamma\)-sets with 2 pairs of alternate vertices except the two \(\gamma\)-sets \(\{v_1, v_3, v_6, v_9, \ldots, v_{3k-3}, v_{3k-1}\}\) and \(\{v_1, v_4, v_7, v_{10}, \ldots\}\),
v\(3k-7, v_{3k-4}, v_{3k-3}, v_{3k-1}\}\) which occurred in step(3).

Step(2) to step(6) can be repeated with the \(\gamma\)-sets \(S_4\) and \(S_5\). Step(2), step(4), step(5) and step(6) can be repeated with the \(\gamma\)-sets \(S_6\): there are no \(\gamma\)-sets other than the \(\gamma\)-sets got by the above steps. Thus the total number of \(\gamma\)-sets of \(C_{3k+1}\) is

\[
= 6 + k - 1 + k - 1 + k - 1 + k - 2 + (k - 2)C_2 + k - 1 + \\
k - 1 + k - 1 + k - 2 + (k - 2)C_2 + k - 1 + k - 1 + \\
k - 2 + (k - 2)C_2 \\
= 11k - 8 + 3(k - 2)C_2 \\
= 11k - 8 + \frac{3(k - 2)(k - 3)}{2} \\
= \frac{22k - 16 + 3(k^2 - 5k + 6)}{2} \\
= \frac{3k^2 + 7k + 2}{2}
\]

\(\square\)

**Remark 6.2.10.** The number of \(\gamma\)-sets of \(C_{10}, C_{13}C_{16}, C_{19}, \ldots\) are 25, 39, 56, 76, 99, \ldots The difference between three numbers is 14, 17, 20, 23, \ldots (ie) the difference is in A.P.
Proposition 6.2.11. $P_{3k}(\gamma_m) \cong K_1$.

Proposition 6.2.12. $P_{3k+2}(\gamma_m) \cong P_{k+2}$.

Proposition 6.2.13. $P_4(\gamma_m) \cong C_4$.

Proof. Let $v_1, v_2, v_3, v_4$ be the vertices of the path $P_4$. Then it has 4 $\gamma$-sets namely $S_1 = \{v_1, v_3\}, S_2 = \{v_1, v_4\}, S_3 = \{v_2, v_3\}, S_4 = \{v_2, v_4\}$. For $i = 1, 2, 3, 4$, deg $S_i = 2$. Hence $P_4(\gamma_m)$ has 4 vertices and each vertex is of deg 2 so that $P_4(\gamma_m) \cong C_4$. $\square$

Proposition 6.2.14. $P_{3k+1}(\gamma_m)$ is isomorphic to the graph of order $\frac{k^2+5k+2}{2}$ for $k \geq 2$.

Proof. Case (1): $k = 2$

The path obtained is $P_7$ and it has 8 $\gamma$-sets namely $S_1 = \{v_2, v_5, v_7\}$, $S_2 = \{v_2, v_5, v_6\}, S_3 = \{v_2, v_4, v_6\}, S_4 = \{v_2, v_3, v_6\}, S_5 = \{v_2, v_4, v_7\}$, $S_6 = \{v_1, v_4, v_7\}, S_7 = \{v_1, v_4, v_6\}$ and $S_8 = \{v_1, v_3, v_6\}$. The total number of $\gamma$-sets of $P_7$ is 8. So the order of $P_7(\gamma_m)$ is 8.

Case (2): $k \geq 3$

Step (1): Let $v_1, v_2, v_3, \ldots, v_{3k+1}$ be the vertices of the path $P_{3k+1}$.

Consider the 4 $\gamma$-sets $S_1 = \{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k+1}\}, S_2 = \{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k}\}, S_3 = \{v_2, v_5, v_8, \ldots, v_{3k-1}, v_{3k+1}\}, S_4 = \{v_2, v_5, v_8, v_{3k-1}$,
$v_{3k}$} of $P_{3k+1}$. $S_1$ is the only $\gamma$-set with first the vertex $v_1$ and last vertex $v_{3k+1}$.

**Step (ii):** Now fixing the first and last vertices of $S_2$ and changing from the 2nd vertex we get $S_5 = \{v_1, v_3, v_6, \ldots, v_{3k+1}\}$. Similarly changing from the $3^{rd}, 4^{th}, 5^{th}, \ldots, k^{th}$ vertex we get $(k - 2)$ $\gamma$-sets. Thus in step (ii) we get $(k - 1)$ $\gamma$-sets.

**Step (iii):** Now fixing the first and last vertices of $S_3$ and changing from the 2nd vertex we get $\{v_2, v_4, v_7, v_{10}, v_{3k-2}, v_{3k+1}\}$. Similarly by changing from the third, fourth, fifth, $\ldots$, $k^{th}$ vertex we get $(k - 2)$ $\gamma$-sets. Thus step (iii) contains $(k - 1)$ $k$-sets.

**Step (iv):** $(k - 1)$ $\gamma$-sets have 2 adjacent vertices. They are $\{v_2, v_3, v_6, v_9, \ldots, v_{3k}\}$, $\{v_2, v_5, v_6, v_9, \ldots, v_{12}, \ldots, v_{3k}\}$, $\ldots$, $\{v_2, v_5, v_8, \ldots, v_{3k-4}, v_{3k-3}, v_{3k}\}$. Thus this step contains $(k - 1)$ $\gamma$-sets. [since $\{v_2, v_5, v_8, \ldots, v_{3k-4}, v_{3k-3}, v_{3k}\} = S_4$].

**Step (v):** The last $\gamma$-set of step (iv) is $\{v_2, v_5, v_8, \ldots, v_{3k-4}, v_{3k-3}, v_{3k-3}\} \ldots(1)$. Fixing the first vertex and last two vertices of (1) changing from the $2^{nd}$ vertex we get $\{v_2, v_4, v_7, v_{10}, \ldots, v_{3k-5}, v_{3k-3}, v_{3k}\}$. Then changing from the $3^{rd}, 4^{th}, 5^{th}, \ldots, (k - 1)^{th}$ vertex we get $(k - 3)$ $\gamma$-sets. Thus step (v) has $(k - 2)$ $\gamma$-sets. [Here the last $\gamma$-set is $\{v_2, v_5, v_8, \ldots, v_{3k-7}, v_{3k-5}, v_{3k-3}, v_{3k}\} \ldots(2)$.]

**Step(vi):** Now consider the $\gamma$-set $\{v_2, v_5, v_8, \ldots, v_{3k-7}, v_{3k-4}, v_{3k-2},$
Fixing the first vertex and last two vertices of (3) and changing from the 2nd vertex we get \( \{v_2, v_4, v_7, v_{10}, \ldots, v_{3k-2}, v_{3k}\} \).

Similarly changing from the 3\(^{rd}\), 4\(^{th}\), 5\(^{th}\), \ldots, (k - 1)\(^{th}\) vertex we get \((k - 2)\) \(\gamma\)-sets. Thus step (vi) has \((k - 1)\) \(\gamma\)-sets including (3).

**Step (vii):** Now consider all the \(\gamma\)-sets containing 3 alternate vertices. They are \(\{v_2, v_4, v_6, v_9, v_{12}, \ldots, v_{3k-6}, v_{3k-3}, v_{3k}\}\), \(\{v_2, v_5, v_7, v_9, v_{12}, v_{15}, \ldots, v_{3k-6}, v_{3k-3}, v_{3k}\}\), \ldots, \(\{v_2, v_5, v_8, \ldots, v_{3k-10}, v_{3k-8}, v_{3k-6}, v_{3k-3}, v_{3k}\}\).

Thus step (vii) has \((k - 3)\) \(\gamma\)-sets. [The last 2 \(\gamma\)-sets are (2) of step (v) and (3) of step (vi)].

**Step (viii):** Using the above \((k - 3)\) \(\gamma\)-sets we can write \((k - 3)C_2\) \(\gamma\)-sets with 2 pairs of alternate vertices with first vertex \(v_2\) and last 2 vertices \(v_{3k-3}, v_{3k}\).

There are no \(\gamma\)-sets other than the \(\gamma\)-sets got by the above 8 steps.

Thus total number of \(\gamma\)-sets

\[
= 4 + k - 1 + k - 1 + k - 1 + k - 2 + k - 1 + k - 3 + (k - 3)C_2 \\
= 6k - 5 + \frac{k^2 - 7k + 12}{2} \\
= \frac{12k - 10 + k^2 - 7k + 12}{2} \\
= \frac{k^2 + 5k + 2}{2}.
\]  

(6.1)
Note 6.2.15. (1) Vertices of steps \((v), (vi), (vii)\) and \((viii)\) are of deg 4. Vertices of step \((ii)\) except \(S_5\), Vertices of step \((iii)\) and vertices of step \((iv)\) are of deg 3 and \(S_1, S_3, S_5\), are the only 3 vertices of deg 2. (2) Each dominating set is some number of swaps from \(S_1, S_2, S_5, S_6, S_7, \ldots, S_{k+3}\) and hence \(P_{3k+1}(\gamma_m)\) is a connected graph and is isomorphic to the following graph.

Theorem 6.2.16. \((P_2 \square P_2)(\gamma_m)\) is a 4-regular graph of 6 vertices.

Proof.
Let \{u_1, u_2, v_1, v_2\} be the vertices of the grid \(P_2 \square P_2\) and \(S_1 = \{u_1, u_2\}\), \(S_2 = \{u_1, v_1\}\), \(S_3 = \{u_1, v_2\}\), \(S_4 = \{u_2, v_1\}\), \(S_5 = \{u_2, v_2\}\), \(S_6 = \{v_1, v_2\}\) are the 6\(\gamma\)-sets of \(P_2 \square P_2\). Here

\(S_1\) is adjacent to \(S_2, S_3, S_4, S_6\).
\(S_2\) is adjacent to \(S_1, S_3, S_4, S_6\).
\(S_3\) is adjacent to \(S_1, S_2, S_5, S_6\).
\(S_4\) is adjacent to \(S_1, S_2, S_5, S_6\).
\(S_5\) is adjacent to \(S_1, S_3, S_4, S_6\).
\(S_6\) is adjacent to \(S_1, S_3, S_4, S_6\).

\[\Box\]

**Theorem 6.2.17.** \((P_2 \square P_4)(\gamma_m)\) is a 3-regular graph with 12 vertices.

**Proof.**

Let \(u_1, u_2, u_3, u_4\) and \(v_1, v_2, v_3, v_4\) be the vertices of the first and second row of the grid \(P_2 \square P_4\). \(S_1 = \{u_1, v_3, v_4\}\), \(S_2 = \{u_1, v_3, u_4\}\), \(S_3 = \ldots\)
\{u_1, v_3, u_3\}, S_4 = \{v_1, v_3, u_3\}, S_5 = \{u_1, v_2, u_3\}, S_6 = \{u_1, u_2, v_4\}, S_7 = \{u_1, u_3, v_4\}, S_8 = \{v_1, u_3, u_4\}, S_9 = \{v_1, u_2, u_4\}, S_{10} = \{v_1, v_2, u_4\}, S_{11} = \{u_2, v_2, u_4\}, S_{12} = \{u_2, v_2, v_4\} are the \(\gamma\)-sets of \(P_2 \Box P_4\). Here

- \(S_1\) is adjacent to \(S_2, S_3, S_6\)
- \(S_2\) is adjacent to \(S_1, S_3, S_5\)
- \(S_3\) is adjacent to \(S_1, S_2, S_4\)
- \(S_4\) is adjacent to \(S_3, S_7, S_8\)
- \(S_5\) is adjacent to \(S_2, S_{10}, S_{11}\)
- \(S_6\) is adjacent to \(S_1, S_9, S_{12}\)
- \(S_7\) is adjacent to \(S_4, S_8, S_9\)
- \(S_8\) is adjacent to \(S_4, S_7, S_{10}\)
- \(S_9\) is adjacent to \(S_6, S_7, S_{12}\)
- \(S_{10}\) is adjacent to \(S_5, S_8, S_{11}\)
- \(S_{11}\) is adjacent to \(S_5, S_{10}, S_{12}\)
- \(S_{12}\) is adjacent to \(S_6, S_9, S_{11}\).

\((P_2 \Box P_4) (\gamma_m)\):

Thus \((P_2 \Box P_4) (\gamma_m)\) is a cubic graph with 12 vertices. \(\Box\)

**Theorem 6.2.18.** \((P_2 \Box P_6) (\gamma_m)\) is isomorphic to the graph \(G\):
Proof.

Let $u_1, u_2, u_3, u_4, u_5, u_6$ and $v_1, v_2, v_3, v_4, v_5, v_6$ be the vertices of the first and second rows of the grid $P_2 \square P_4$. $S_1 = \{u_1, v_3, u_5, v_6\}, S_2 = \{u_1, v_3, u_5, v_6\}, S_3 = \{u_1, v_3, u_5, v_5\}, S_4 = \{u_1, v_3, u_4, v_6\}, S_5 = \{v_1, u_3, v_5, u_6\}, S_6 = \{v_1, u_3, v_5, v_6\}, S_7 = \{v_1, u_3, v_5, v_5\}, S_8 = \{v_1, u_3, v_4, v_6\}, S_9 = \{v_1, u_3, v_4, u_6\}, S_{10} = \{u_1, v_2, u_4, v_6\}, S_{11} = \{v_1, v_2, u_4, v_6\}, S_{12} = \{u_2, v_2, u_4, v_6\}, S_{13} = \{u_2, v_2, u_5, v_5\}, S_{14} = \{u_1, u_2, v_4, u_6\}, S_{15} = \{u_1, v_3, v_4, u_6\}, S_{16} = \{u_2, v_2, v_4, u_6\}, S_{17} = \{v_1, u_3, u_4, v_6\}$ are the $\gamma$-sets of $P_2 \square P_6$. Here $S_1$ is adjacent to $S_2, S_3, S_4$. $S_2$ is adjacent to $S_1, S_3, S_{15}$. $S_3$ is adjacent to $S_1, S_2$. $S_4$ is adjacent to $S_1, S_{10}$. $S_5$ is adjacent to $S_6, S_7, S_8$. 131
$S_6$ is adjacent to $S_5, S_7, S_{17}$.
$S_7$ is adjacent to $S_5, S_6$.
$S_8$ is adjacent to $S_5, S_9$.
$S_9$ is adjacent to $S_8, S_{14}, S_{16}$
$S_{10}$ is adjacent to $S_4, S_{11}, S_{12}$
$S_{11}$ is adjacent to $S_{10}, S_{12}, S_{17}$
$S_{12}$ is adjacent to $S_{10}, S_{11}$
$S_{14}$ is adjacent to $S_9, S_{15}, S_{16}$
$S_{15}$ is adjacent to $S_2, S_{14}$
$S_{16}$ is adjacent to $S_9, S_{14}$
$S_{17}$ is adjacent to $S_6, S_{11}$
and $S_{13}$ is an isolated vertex. Thus we get the following graph.

\[ \begin{array}{c}
S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_{14} & S_{15} & S_{16} & S_{17} & S_9 & S_8 & S_7 & S_{10} & S_{11} & S_{12} & S_{13} \\
\end{array} \]

**Theorem 6.2.19.** $(P_2 \square P_n)(\gamma_m)$ where $n = 2k, k \geq 4$ is isomorphic to the graph $G$ with order $4 \left\lfloor \frac{n+1}{2} \right\rfloor$ of which 8 vertices have deg 3 and the
remaining vertices have deg 2. $G$ is given below.

Proof.

Let $u_1, u_2, u_3, \ldots, u_n$ and $v_1, v_2, v_3, \ldots, v_n$ be the vertices of the 1st and 2nd rows of the grid $P_2 \square P_n$ when $n = 2k$. We know that $P_2 \square P_n$ has domination number $\left\lceil \frac{n+1}{2} \right\rceil$. Consider the 6 $\gamma$-sets

$S_1 = \{u_1, v_3, u_5, v_7, \ldots, u_{n-3}, u_{n-1}, u_n\}$

$S_2 = \{u_1, v_3, u_5, v_7, \ldots, v_{n-3}, u_{n-1}, v_n\}$

$S_3 = \{u_1, v_3, u_5, v_7, \ldots, v_{n-3}, u_{n-1}, v_{n-1}\}$

$S_4 = \{v_1, u_3, v_5, v_7, \ldots, u_{n-3}, v_{n-1}, u_n\}$

$S_5 = \{v_1, u_3, v_5, u_7, \ldots, u_{n-3}, v_{n-1}, u_n\}$

$S_6 = \{v_1, u_3, v_5, u_7, \ldots, u_{n-3}, v_{n-1}, v_n\}$

of $P_2 \square P_n$. 

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**Case(1):** $k$ is odd

**step(i):** Fixing the first and last vertex of $S_1$ and changing from the $2^{nd}$ vertex we get $S_7 = \{u_1, u_2, v_4, u_6, u_8, u_{10}, \ldots, u_{n-4}, v_{n-2}, u_n\}$. Fixing the first 2 vertices and changing from the $3^{rd}$ vertex we get $S_8 = \{u_1, v_3, v_4, u_6, v_8, u_{10}, \ldots, v_{n-2}, u_n\}$. Proceeding like this, fixing the $(k-1)$ vertices and changing from the $k^{th}$ vertex we get $S_{k+5} = \{u_1, v_3, u_5, v_7, u_9, \ldots, v_{n-3}, v_{n-2}, u_n\}$. Thus we get $(k-1)$ $\gamma$-sets in step(i).

**step(ii):** Now fixing the first vertex of $S_2$ and changing the $2^{nd}$ vertex we get $S_{k+6} = \{u_1, v_2, u_4, v_6, u_8, v_{10}, \ldots, u_{n-2}, v_n\}$. Fixing first 2 vertices of $S_2$ and changing from the $3^{rd}$ vertex we get $\{u_1, v_3, u_4, v_6, u_8, v_{10}, \ldots, v_{n-4}, u_{n-2}, v_n\}$. Continuing up to the change in $k^{th}$ vertex of $S_2$ we get $(k-1)$ $\gamma$-sets. Here the last $\gamma$-set is $\{u_1, v_3, u_5, v_7, \ldots, u_{n-5}, v_{n-3}, u_{n-2}, v_k\} = S_{2k+4}$.

**step(iii):** $S_3$ is the only $\gamma$-set with first vertex $u_1$ and first 2 vertices $u_{n-1}, v_{n-1}$ and $S_4$ is the only $\gamma$-set with first vertex $v_1$ and last 2 vertices $v_{k-1}, u_{k-1}$.

**step(iv):** Fixing the first vertex of $S_5$ and changing from the $2^{nd}$ vertex we get $S_{2k+5} = \{v_1, u_2, v_4, u_6, v_8, u_{10}, \ldots, u_{n-4}, v_{n-2}, u_n\}$. Fixing the first 2 vertices and changing from the $3^{rd}$ vertex we get $\{v_1, u_3, v_4, u_6, v_8, u_{10}, \ldots, v_{n-2}, u_n\}$. Continuing upto the change in $k^{th}$ vertex we get
\[ S_{3k+3} = \{v_1, u_3, v_5, u_7, \ldots, u_{n-3}, v_{n-2}, u_n\} . \] Thus we get step (iv) has \((k - 1)\) \(\gamma\)-sets.

**Step (v):** Fixing the first vertex of \(S_6\) and changing from the 2\(^{nd}\) vertex we get \(S_{3k+4} = \{v_1, v_2, u_4, v_6, v_{10}, \ldots, v_{n-4}, u_{n-2}, u_n\}\). Fixing the first 2 vertices of \(S_6\) and changing from the 3\(^{rd}\) vertex we get \(\{v_1, u_3, u_4, v_6, u_8, v_{10}, \ldots, v_{n-4}, u_{n-2}, u_n\}\). Proceeding in a similar manner we arrive at the set \(S_{4k+2} = \{v_1, u_3, v_5, u_7, \ldots, u_{n-3}, u_{n-2}, v_n3e\}\). Thus we get \((k - 1)\) \(\gamma\)-sets.

**Step (vi):** \(S_{4k+3} = \{u_2, v_2, u_4, v_6, v_{10}, \ldots, u_{n-2}, v_n\}\) and \(S_{4k+4} = \{u_2, v_2, v_4, u_6, v_8, u_{10}, \ldots, v_{n-2}, u_n\}\) are the 2\(\gamma\)-sets with first 2 vertices \(u_2, v_2\) and last 2 vertices \(v_n, u_n\) respectively. Thus total number of \(\gamma\)-sets is

\[
= 6 + k - 1 + k - 1 + k - 1 + 2
= 4k - 4 + 8
= 4k + 4
= 4(k + 1)
= 4 \left\lfloor \frac{2k + 1}{2} \right\rfloor
\]

**Case (2):** \(k\) is even

**Step (i):** Fixing the first vertex of \(S_1\) and changing from the 2\(^{nd}\) vertex we get \(S_7 = \{u_1, v_2, u_4, v_6, \ldots, v_{n-2}, u_n\}\). Fixing the first 2 vertices of \(S_1\)
and changing from the 3rd vertex we get $S_8 = \{u_1, v_3, u_4, v_6, u_8, \ldots, v_{n-2}, u_n\}$. Proceeding like this we get (by changing from the $k^{th}$ vertex) $S_{k+5} = \{u_1, v_3, u_5, v_7, \ldots, u_{n-3}, v_{n-2}, u_n\}$. Thus step(i) has $(k - 1)$ $\gamma$-sets.

**step(ii):** Fixing the first vertex of $S_2$ and changing from the 2nd vertex we get $S_{k+6} = \{u_1, u_2, v_4, u_6, v_8, \ldots, v_{n-4}, v_{n-2}, v_n\}$. Fixing the first 2 vertices of $S_2$ and changing from the 3rd vertex we get $S_{k+7} = \{u_1, v_3, v_4, u_6, v_8, \ldots, u_{n-2}, v_n\}$. Proceeding like this we arrive at the set $\{u_1, v_3, u_5, u_7, \ldots, u_{n-3}, v_{n-2}, v_n\}$. Thus this step contains $(k - 1)$ $\gamma$-sets.

**step(iii):** $S_3$ and $S_4$ are the only 2 $\gamma$-sets with last 2 vertices $u_{n-1}, v_{n-1}$ and first vertex $u_1$ and $v_1$ repectively.

**step(iv):** Fixing the first vertex of $S_5$ and changing from the 2nd vertex we get $S_{2k+5} = \{v_1, v_2, v_4, v_6, v_8, \ldots, v_{n-2}, v_n\}$. Fixing the first 2 vertices of $S_5$ and changing from the 3rd vertex we get $S_{2k+6} = \{v_1, u_3, v_4, v_6, u_8, \ldots, v_{n-2}, u_n\}$. Proceeding like this we get $S_{3k+3} = \{v_1, u_3, v_5, u_7, \ldots, u_{n-3}, v_{n-2}, u_n\}$. Thus step(iv) has $(k - 1)$ $\gamma$-sets.

**step(v):** Fixing the first vertex of $S_6$ and changing from the 2nd vertex we get $S_{3k+4} = \{v_1, u_2, v_4, u_6, \ldots, u_{n-2}, v_n\}$. Fixing the first 2 vertices of $S_6$ and changing from the 3rd vertex we get $S_{3k+5} = \{v_1, u_3, v_4, u_6, \ldots, u_{n-2}, v_n\}$. Proceeding like this by changing $k^{th}$ vertex we get
Thus step(v) has \((k - 1)\) \(\gamma\)-sets.

**step(vi):** \(S_{4k+3} = \{u_2, v_2, u_4, v_6v_8, \ldots, v_{n-2}, u_n\}\) and \(S_{4k+4} = \{u_2, v_2, u_4, u_6, v_8, \ldots, v_{n-2}, v_n\}\) are only 2 \(\gamma\)-sets with first 2 vertices \(u_2, v_2\) and the last vertices \(u_n, v_n\) respectively. Thus the total number of \(\gamma\)-sets=6+k-1+k-1+k-1+2=4(k+1)=4 \(\left\lfloor \frac{2k+1}{2} \right\rfloor\). Thus in both cases we get the total number of \(\gamma\)-sets of \(P_2 \square P_n\) = 4 \(\left\lfloor \frac{n+1}{2} \right\rfloor\).

Here \(S_1, S_2, S_3; s_4, S_5, S_6; S_7, S_{2k+5}, S_{4k+4}; S_{k+6}, S_{3k+4}, S_{4k+3}\) form a triangle \(S_1, S_{k+5}, S_{k+4}, S_{k+3}; \ldots, S_7\) form a path ; \(S_{2k+5}, S_{2k+6}, S_{2k+7}; \ldots, S_{3k+2}, S_{3k+3}, S_5\) form a path ; \(S_2, S_{2k+4}, S_{2k+3}; \ldots, S_{k+6}\) form a path ; \(S_{3k+4}, S_{3k+5}; \ldots S_{4k+2}, S_6\) form a path in \((P_2 \square P_n)(\gamma_m)\).

Thus \((P_2 \square P_n)(\gamma_m), \text{ where } n = 2k, k \geq 4\) is connected and is given below.

**Theorem 6.2.20.** \((K_n \circ C_n)(\gamma_m)\) is isomorphic to a graph \(G\) of order \((n-1)k\), of which \(k\) vertices containing all the members from the cycle
$C_k$ are of deg $2(n-1)$ and the remaining $(n-2)k$ vertices containing exactly one member from $K_n$ are of deg $(n-1)$ where $k$ is odd.

**Proof.** Let $u_1, u_2, u_3, \ldots, u_n$ be the vertices of the inner cycle of $K_n \circ C_k$ and $a_i, s_i, a_{i+1}, \ldots, a_{i(n-2)}$ be the vertices of $K_n$ drawn on the edge $u_iu_{i+1}$ of the cycle $C_k$ for $1 \leq i \leq k-1$ and edge $u_iu_1$ for $i = k$. For any $K_n \circ C_n$, $S_1 = \{u_1, u_3, u_5, \ldots, u_k\}$, $S_2 = \{u_1, u_2, u_4, u_5, u_8, \ldots, u_{k-1}\}$, $S_3 = \{u_2, u_3, u_5, u_7, \ldots, u_k\}$, \ldots, $S_{k-1} = \{u_1, u_3, u_5, \ldots, u_{k-2}, u_{k-1}\}$, $S_k = \{u_2, u_4, u_6, \ldots, u_{k-1}, u_k\}$ are the $k$ $\gamma$-sets such that all the vertices of these $\gamma$-sets are vertices of the cycle of $K_n \circ C_n$ and $|S_i| = \left\lceil \frac{k}{2} \right\rceil$ $\forall i = 1, 2, 3, \ldots, k$.

Now replacing the vertex $u_1$ of $S_1$ by $a_{1,1}, a_{1,2}, a_{1,3}, \ldots, a_{1, n-2}$ successively we get the following $(n-2)$ $\gamma$-sets.

\begin{align*}
\{u_3, u_5, u_7, \ldots, u_k, a_{1,1}\} \\
\{u_3, u_5, u_7, \ldots, u_k, a_{1,2}\} \\
\{u_3, u_5, u_7, \ldots, u_k, a_{1,3}\} & \quad (1) \\
\vdots \\
\{u_3, u_5, u_7, \ldots, u_k, a_{1, n-2}\}
\end{align*}

Now replacing the vertex $u_2$ of $S_2$ by $a_{2,1}, a_{2,2}, a_{2,3}, \ldots, a_{2, n-2}$ respectively we get the following $(n-2)$ $\gamma$-sets
\{u_1, u_4, u_6 \ldots u_{k-1}, u_{2,1}\}
\{u_1, u_4, u_6 \ldots u_{k-1}, u_{2,2}\}
\{u_1, u_4, u_6 \ldots u_{k-1}, u_{2,3}\} (2)
\vdots
\{u_1, u_4, u_6 \ldots u_{k-1}, u_{2,n-2}\}

Now replacing the vertex \(u_3\) of \(S_3\) by \(a_{3,1}, a_{3,2}, a_{3,3}, \ldots a_{3,n-2}\) successively we get the following \((n - 2)\) \(\gamma\)-sets
\{u_2, u_5, u_7 \ldots u_k, u_{3,1}\}
\{u_2, u_5, u_7 \ldots u_k, u_{3,2}\}
\{u_2, u_5, u_7 \ldots u_k, u_{3,3}\} (3)
\vdots
\{u_2, u_5, u_7 \ldots u_k, u_{3,n-2}\}

Proceeding like this at the \((k - 1)th\) step replacing the vertex \(u_{k-1}\) of \(S_{k-1}\) by \(a_{k-1,1}, a_{k-1,2}, a_{k-1,3}, \ldots a_{k-1,n-2}\) successively we get
\{u_1, u_3, u_5 \ldots u_{k-2}, u_{k-1,1}\}
\{u_1, u_3, u_5 \ldots u_{k-2}, u_{k-1,2}\}
\{u_1, u_3, u_5 \ldots u_{k-2}, u_{k-1,3}\} (k - 1)
\vdots
\{u_1, u_3, u_5 \ldots u_{k-2}, u_{k-1,n-2}\} and replacing the vertex \(u_k\) of \(S_k\) by \(a_{k,1}, a_{k,2}, a_{k,3}, \ldots a_{k,n-2}\) successively we get
\{u_2, u_4, u_6 \ldots u_{k-1}, u_{k,1}\}
\{u_2, u_4, u_6 \ldots u_{k-1}, u_{k,2}\}
\{u_2, u_4, u_6 \ldots u_{k-1}, u_{k,3}\}  \quad (k)
\vdots
\{u_2, u_4, u_6 \ldots u_{k-1}, u_{k,n-2}\}

There are no $\gamma$-sets other than the $\gamma$-sets obtained above. Thus total number of $\gamma$-sets of $K_n \circ C_k = k + (n - 2)k = (n - 1)k$.

Here $S_1$ is adjacent to $S_3, S_{k-1}$, all the $\gamma$-sets of (1) and $(k - 1)$
$S_2$ is adjacent to $S_4, S_k$, all the $\gamma$-sets of (2) and (k).
$S_3$ is adjacent to $S_5, S_1$, all the $\gamma$-sets of (3) and (1).
$S_4$ is adjacent to $S_6, S_2$, all the $\gamma$-sets of (4) and (2)
\vdots
$S_k$ is adjacent to $S_2, S_{k-2}$, all the $\gamma$-sets of $k$ and $(k - 2)$.

Thus these $k$ $\gamma$-sets have deg $(n - 2) + (n - 2) + 2$
= $2(n - 1)$.

Each $\gamma$-set of (1) is adjacent to the remaining $\gamma$-sets of (1), $S_1$ and $S_2$.
Each $\gamma$-sets of (2) is adjacent to the remaining $\gamma$-sets of (2) ,$S_2$ and $S_4$
\vdots
Each $\gamma$-sets of (k) is adjacent to the remaining $\gamma$-sets of (k),$S_k$ and $S_2$.

Thus the remaining $(n - 2)k$ $\gamma$-sets have deg $n - 3 + 2 = n - 1$
Example 6.2.21. Consider the graph $K_6 \circ C_7$.

\[ S_1 = \{u_1, u_3, u_5, u_7\} \]
\[ S_2 = \{u_1, u_2, u_4, u_6\} \]
\[ S_3 = \{u_2, u_3, u_5, u_7\} \]
\[ S_4 = \{u_1, u_3, u_4, u_6\} \]
\[ S_5 = \{u_2, u_4, u_5, u_6\} \]
\[ S_6 = \{u_1, u_3, u_5, u_6\} \]
\[ S_7 = \{u_2, u_4, u_6, u_7\} \]

\[ S_8 = \{u_3, u_5, u_7, a_{1,1}\} \]
\[ S_9 = \{u_3, u_5, u_7, a_{1,2}\} \]
\[ S_{10} = \{u_3, u_5, u_7, a_{1,3}\} \quad (1) \]
\[ S_{11} = \{u_3, u_5, u_7, a_{1,4}\} \]

\[ S_{12} = \{u_1, u_4, u_6, a_{2,1}\} \]
\[ S_{13} = \{u_1, u_4, u_6, a_{2,2}\} \]
\[ S_{14} = \{u_1, u_4, u_6, a_{2,3}\} \quad (2) \]
\[ S_{15} = \{u_1, u_4, u_6, a_{2,4}\} \]

\[ S_{16} = \{u_2, u_5, u_7, a_{3,1}\} \]
\[ S_{17} = \{u_2, u_5, u_7, a_{3,2}\} \]
\[ S_{18} = \{u_2, u_5, u_7, a_{3,3}\} \quad (3) \]
\[ S_{19} = \{u_2, u_5, u_7, a_{3,4}\} \]

\[ S_{20} = \{u_1, u_3, u_6, a_{4,1}\} \]
\[ S_{21} = \{u_1, u_3, u_6, a_{4,2}\} \]
\[ S_{22} = \{u_1, u_3, u_6, a_{4,3}\} \quad (4) \]
\[ S_{23} = \{u_1, u_3, u_6, a_{4,4}\} \]

\[ S_{24} = \{u_1, u_4, u_7, a_{5,1}\} \]
\[ S_{25} = \{u_1, u_4, u_7, a_{5,2}\} \]
\[ S_{26} = \{u_1, u_4, u_7, a_{5,3}\} \quad (5) \]
\[ S_{27} = \{u_1, u_4, u_7, a_{5,4}\} \]

\[ S_{28} = \{u_1, u_3, u_5, a_{6,1}\} \]
\[ s_{29} = \{u_1, u_3, u_5, a_{6,2}\} \]
\[ S_{30} = \{u_1, u_3, u_5, a_{6,3}\} \quad (6) \]
Here $S_1$ is adjacent to $S_3, S_6$ all the $\gamma$-sets of 1 and 6 $S_2$ is adjacent to $S_4, S_7$ all the $\gamma$-sets of 2 and 7. $S_3$ is adjacent to $S_5, S_1$ all the $\gamma$-sets of 3 and 1. $S_4$ is adjacent to $S_6, S_2$ all the $\gamma$-sets of 4 and 2. $S_5$ is adjacent to $S_7, S_3$ all the $\gamma$-sets of 5 and 3. $S_6$ is adjacent to $S_1, S_4$ all the $\gamma$-sets of 6 and 4. $S_7$ is adjacent to $S_2, S_5$ all the $\gamma$-sets of 7 and 5. Thus these 7 $\gamma$-sets have deg $4+4+2=10$ ie) deg of these 7 $\gamma$-sets is $2(6-1)$. Each $\gamma$-sets of 1 is adjacent to the remaining $\gamma$-sets of 1,$S_1$ and $S_3$; Each $\gamma$-sets of 2 is adjacent to the remaining $\gamma$-sets of 2,$S_2$ and $S_4$; Continuing in this way, each $\gamma$-set of 7 is adjacent to the remaining $\gamma$-sets of 7, $S_7$ and $S_2$ Thus these $(6-2)7=28$ $\gamma$-sets have degree $6-3+2=5$. 

\begin{align*}
S_{31} &= \{u_1, u_3, u_5, a_{6,4}\} \\
S_{32} &= \{u_2, u_4, u_6, a_{7,2}\} \\
S_{33} &= \{u_2, u_4, u_6, a_{7,3}\} \\
S_{34} &= \{u_2, u_4, u_6, a_{7,4}\}
\end{align*}
Theorem 6.2.22. The modified $\gamma$-graph of the comb of order $n$ denoted by $Cb_n$ is a $n$-regular bipartite graph of order $2^n$, isomorphic to either $G_1$ or $G_2$. $G_1$ and $G_2$ are given in fig given below.
Proof. Let \( u_1, u_2, u_3, \ldots, u_n \) be the supports and \( v_1, v_2, v_3, \ldots, v_n \) be the corresponding pendants. Number of vertices of \( Cb_n \) is 2\( n \). Since every vertex of \( Cb_n \) is either a pendent or a support, \( \gamma(Cb_n) = n \).

To get the \( \gamma \)-set of \( Cb_n \) we have to choose \( n \) vertices among the 2\( n \) vertices (ie) from each pair \( \{u_i, v_i\}, 1 \leq i \leq n \) we have to choose one vertex. This can be done in 2\( n \) ways. So number of \( \gamma \)-sets of \( Cb_n \) is 2\( n \).

Let the collection of \( \gamma \)-sets with all pendants be \( \mathcal{V}_0 \), (\( n-1 \)) pendants be \( \mathcal{V}_1 \), (\( n-2 \)) pendants be \( \mathcal{V}_2 \), \ldots, no pendants be \( \mathcal{V}_n \). The \( \gamma \)-sets are as follows.
Case (i): $n$ is even

\[ V_0 = \{v_1, v_2, v_3, \ldots, v_n\}. \]

\[ S_1 = \{v_2, v_3, v_4, \ldots, v_n, u_1\} \]
\[ S_2 = \{v_1, v_3, v_4, \ldots, v_n, u_2\} \]

\[ V_1 = S_3 = \{v_1, v_2, v_4, \ldots, v_n, u_3\} \]

\[ S_n = \{v_1, v_2, v_3, \ldots, v_n, u_n\} \]

i.e \[ V_1 = \{S_i | S_i = \{v_1, v_2, v_3, \ldots, v_{i-1}, v_{i+1}, v_{i+2}, \ldots, v_n, u_i | 1 \leq i \leq n\} \]

\[ V_2 = \{S_i | S_i = \{v_1, v_2, v_3, \ldots, v_{i-1}, v_{i+2}, v_{i+3}, \ldots, v_n, u_i, u_{i+1} | 1 \leq i \leq n\} \]

Thus \[ V_2 \] contains \( nC_2 \) \( \gamma \)-sets.

\[ V_3 = \{S_i | S_i = \{v_1, v_2, v_3, \ldots, v_{i-1}, v_{i+3}, v_{i+4}, \ldots, v_n, u_i, u_{i+1}, u_{i+2} | 1 \leq i \leq n\} \}

Thus \[ V_3 \] contains \( nC_3 \) \( \gamma \)-sets.

\[ \ldots V_{\frac{n}{2}-1}, V_{\frac{n}{2}+1}, \ldots, V_{n-1} = \{S_i | S_i = \{u_1, u_2, u_3, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n, v_i | 1 \leq i \leq n\} \text{ and } V_n = \{u_1, u_2, u_3, \ldots, u_n\}. \]

Case (ii): $n$ is odd.

Here also we get the \((n + 1)\) collection of \( \gamma \)-sets namely \( V_0, V_1, V_2, \ldots, V_{\frac{n}{2}-1}, V_{\frac{n}{2}+1}, \ldots, V_{n-1}, V_n. \) Thus the number of \( \gamma \)-sets with \( n, n - 1, n - 2, n - 3, \ldots, n - n \) pendent vertices are \( nC_0, nC_1, nC_2, nC_3, \ldots, nC_n. \)

Hence the total number of \( \gamma \)-sets of \( Cb_n \) is \( 2^n \) (i.e) the number of \( \gamma \)-sets of \( Cb_n \) is \( 2^n. \)
As $\mathcal{V}_0$ contains $n$ pendent vertices and every $\gamma$-set of $\mathcal{V}_1$ contains $(n-1)$ pendent vertices and one support, every $\gamma$-set of $\mathcal{V}_1$ is adjacent to $\mathcal{V}_0$, the $\gamma$-sets of $\mathcal{V}_0$ get deg $n$ and the $\gamma$-sets of $\mathcal{V}_1$ get deg 1.

$\mathcal{V}_2$ contains $nC_2$ $\gamma$-sets, each $\gamma$-set contains 2 supports and every support occurs in $(n-1)$ $\gamma$-sets. Hence each $\gamma$-set of $\mathcal{V}_1$ is adjacent to $(n-1)$ $\gamma$-sets of $\mathcal{V}_2$, $\gamma$-sets of $\mathcal{V}_1$ get deg $n-1$ and the $\gamma$-sets of $\mathcal{V}_2$ get deg 2. The $\gamma$-sets of $\mathcal{V}_1$ are not adjacent to any other $\gamma$-set of other collections. Hence deg of the $\gamma$-sets of $\mathcal{V}_1$ is $n$.

$\mathcal{V}_3$ contains $nC_3 = \frac{n(n-1)(n-2)}{3}$ $\gamma$-sets, each $\gamma$-set contains 3 supports and every pair of supports of $\mathcal{V}_2$ occur in $(n-2)$ $\gamma$-sets of $\mathcal{V}_3$. Hence every $\gamma$-set of $\mathcal{V}_2$ gets deg $n-2$ by the adjacency of $\gamma$-sets of $\mathcal{V}_3$ and in turn the $\gamma$-sets of $\mathcal{V}_3$ gets deg 3. The $\gamma$-sets of $\mathcal{V}_2$ are not adjacent with the $\gamma$-sets of any other collection. Hence deg of each $\gamma$-set of $\mathcal{V}_2$ is $2 + n - 2 = n$.

**Subcase(i):** $n$ is even

This process continues up to the collection $\mathcal{V}_{\frac{n}{2}}$ (ie) every $\gamma$-set of $\mathcal{V}_{\frac{n}{2}}$ gets deg $\frac{n}{2}$ by the adjacency of $\gamma$-sets of $\mathcal{V}_{\frac{n}{2}-1}$ and deg $\frac{n}{2}$ by the adjacency of $\gamma$-sets of $\mathcal{V}_{\frac{n}{2}+1}$. The number of $\gamma$-sets of $\mathcal{V}_{\frac{n}{2}-1}$ is equal to the number of $\gamma$-sets of $\mathcal{V}_{\frac{n}{2}+1}$. The $\gamma$-sets of $\mathcal{V}_{\frac{n}{2}+1}$ gets deg $\frac{n}{2} + 1$. The $\gamma$-sets of $\mathcal{V}_{\frac{n}{2}}$ are not adjacent with the $\gamma$-sets of any other collection and hence deg of $\gamma$-sets of $\mathcal{V}_{\frac{n}{2}}$ is $n$. 

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The $\gamma$-sets of $V_{n^2 + 1}$ get $\deg \frac{n}{2} - 1$ by the adjacency of $\gamma$-sets of the collection $V_{n^2 + 2}$ and the $\gamma$-sets of $V_{n^2 + 2}$ get $\deg \frac{n}{2} + 2$. The $\gamma$-sets of $V_{n^2 + 1}$ are not adjacent with $\gamma$-sets of any other collection and hence $\deg$ of $\gamma$-sets of $V_{n^2 + 1}$ in $n$.

The process continues upto $V_{n^2}$ (ie) the $\gamma$-sets of $V_{n^2 - 1}$ get $\deg n - 1$ by the adjacency of $\gamma$-sets of $V_{n^2 - 2}$ and get $\deg n - 1$ by the adjacency of the $\gamma$-sets of $V_n$. Thus $\deg$ of $\gamma$-sets of $V_{n-1}$ and $V_n$ are $n$.

**Subcase(ii):** $n$ is odd

The process continues upto the collection $V_{n^2 - 1}$(ie) every $\gamma$-set of $V_{n^2 - 1}$ gets $\deg \frac{n-1}{2}$ by the adjacency of $\gamma$-sets of the collection $V_{n^2 - 3}$ and get the degree $\frac{n+1}{2}$ by the adjacency of $\gamma$-sets of $V_{n^2 + 1}$. The $\gamma$-sets of $V_{n^2 - 1}$ is not adjacent with the $\gamma$-sets of any other collection. Hence degree of $\gamma$-sets of $V_{n^2 - 1}$ is $n$. The $\gamma$-sets of $V_{n^2 + 1}$ get $\deg \frac{n+1}{2}$ by the adjacency of $\gamma$-sets of $V_{n^2 + 3}$. Hence degree of $\gamma$-sets of $V_{n^2 + 1}$ is $n$.

Continuing like this the $\gamma$-sets of $V_{n^2 - 1}$ gets $\deg n - 1$ by the adjacency of $\gamma$-sets of $V_{n^2 - 2}$ and gets $\deg 1$ by the adjacency of the $\gamma$-sets of $V_n$. Thus degree of $\gamma$-sets of $V_{n-1}$ and $V_n$ are $n$.

$\square$

**Note 6.2.23.** Let $A = \{\mathcal{V}_i\}, i = 1, 3, 5, \ldots, n$ and $B = \{\mathcal{V}_i\}, i = 0, 2, 4, \ldots, n$. Then obviously every edge of $Cb_n(\gamma_m)$ has are edge in $A$
and other end in $\mathcal{B}$. Thus $Cb_n(\gamma_m)$ is a bipartite graph.

**Example 6.2.24.** Consider the comb of order 5, $Cb_n$.

\[
\begin{align*}
V_0 &= \{v_1, v_2, v_3, v_4, v_5\} \\
S_1 &= \{u_1, v_2, v_3, v_4, v_5\} \\
S_2 &= \{v_1, u_2, v_3, v_4, v_5\} \\
V_1 &= S_3 = \{v_1, v_2, u_3, v_4, v_5\} \\
S_4 &= \{v_1, v_2, v_3, u_4, v_5\} \\
S_5 &= \{v_1, v_2, v_3, v_4, u_5\} \\
S_6 &= \{u_1, u_2, v_3, v_4, v_5\} \\
S_7 &= \{u_1, v_2, v_3, u_4, v_5\} \\
S_8 &= \{v_1, v_2, u_3, u_4, v_5\} \\
V_2 &= S_5 = \{v_1, u_2, u_3, v_4, v_5\} \\
S_6 &= \{v_1, u_2, v_3, u_4, v_5\} \\
S_7 &= \{v_1, u_2, v_3, v_4, u_5\} \\
S_8 &= \{v_1, v_2, u_3, u_4, v_5\}
\end{align*}
\]
\[ S_9 = \{v_1, v_2, u_3, v_4, u_5\} \]
\[ S_{10} = \{v_1, v_2, v_3, u_4, u_5\} \]

\[ S_1 = \{u_1, u_2, u_3, v_4, v_5\} \]
\[ S_2 = \{u_1, u_2, v_3, u_4, v_5\} \]
\[ S_3 = \{u_1, u_2, v_3, v_4, u_5\} \]
\[ S_4 = \{u_1, v_2, u_3, u_4, v_5\} \]
\[ V_3 = S_5 = \{u_1, v_2, u_3, v_4, u_5\} \]
\[ S_6 = \{u_1, v_2, v_3, u_4, u_5\} \]
\[ S_7 = \{v_1, u_2, u_3, u_4, v_5\} \]
\[ S_8 = \{v_1, u_2, u_3, v_4, u_5\} \]
\[ S_9 = \{v_1, u_2, v_3, u_4, u_5\} \]
\[ S_{10} = \{v_1, v_2, u_3, u_4, u_5\} \]

\[ S_1 = \{u_1, u_2, u_3, u_4, v_5\} \]
\[ S_2 = \{u_1, u_2, u_3, v_4, u_5\} \]
\[ V_4 = S_3 = \{u_1, u_2, v_3, u_4, u_5\} \]
\[ S_4 = \{u_1, v_2, u_3, u_4, u_5\} \]
\[ S_5 = \{v_1, u_2, u_3, u_4, u_5\} \]

\[ V_5 = S_1 = \{u_1, u_2, u_3, u_4, u_5\} \]
Let $\mathcal{A} = \{V_0, V_2, V_4\}$

$\mathcal{B} = \{V_1, V_3, V_5\}$

**Theorem 6.2.25.** For every $n$, there exists a graph $G$ such that $G(\gamma_m) \cong K_n$.

**Proof.** Let $K_n$ be the complete graph with vertex set $\{u_1, u_2, u_3, \ldots, u_n\}$. By construction let us show that there exists a graph $G$ such that $G(\gamma_m) \cong K_n$. Consider $K_n$ with above vertex labels. To form $G$, add a path $P_3$ of order 3 with vertices $v_1, v_2, v_3$ and add an edge of the form $(u_i, v_2)$ for $1 \leq i \leq n$. Further add two non-adjacent vertices $x$ and $y$. 

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with edges $(u_i, x)$ and $(u_i, y)$ for $1 \leq i \leq n$. Since no vertex is adjacent to every other vertex, $\gamma(G) \geq 2$.

It is clear that $X_i = \{v_2, u_i\}$, $1 \leq i \leq n$ is a $\gamma$-set of $G$ for $1 \leq i \leq n$, since $v_2$ dominates each vertex of added $P_3$ and each vertex $u_i$, for $1 \leq i \leq n$ in the original graph $K_n$. Further each vertex $u_i$ for $1 \leq i \leq n$ dominates both $x$ and $y$. Since no dominating set of $G$ can contain less than two vertices, each $X_i$, $1 \leq i \leq n$ is a $\gamma$-set. To show that there are no others, let $S$ be a $\gamma$-set of $G$. $S$ must contain $v_2$ since otherwise $S$ would have to contain both $v_1, v_2$ and some vertex $u_i \in V(K_n)$, thus making $|S| > 2$. Further if $x \in S$ (or $y$) then as $x$ and $v_2$ do not dominate $y$ (or $x$), there must exist another vertex in the dominating set, contradicting the minimality of $S$. This implies that $S = X_i$ for some $1 \leq i \leq n$. Hence the sets $X_i$, $1 \leq i \leq n$ are the only minimum dominating sets for $G$.

Each $\gamma$-set differ by only one vertex as $v_2$ appears in every $\gamma$-set of $G$. The number of $\gamma$-sets of $G$ is $n$ and by the definition of modified $\gamma$-graph, each $\gamma$-set is adjacent to every other $\gamma$-set, thus forming a complete $K_n$. □