CHAPTER II

COMBINATORIAL REGULAR SEMIGROUPS
AND COMBINATORIAL FUNCTORS

In this chapter we describe the structure of all combinatorial regular semigroups with biordered set \( E \), in terms of a functor from the preorder \( (E, \omega) \) into the category of combinatorial Rees groupoids; satisfying axioms (C1 - C2) (cf. Definition 2.10, below). Such functors are called combinatorial functors and we prove that the category \( \text{CF} \) of combinatorial functors is equivalent to the category \( \text{CRS} \) of combinatorial regular semigroups. Section 1 is devoted to a discussion of some general properties of combinatorial semigroups. The material which appears in Sections 2, 3, and 4 of this chapter, is a revised version of the author's paper entitled "Structure of Combinatorial Regular Semigroups" [24] written in collaboration with Nambooripad.

1. COMBINATORIAL REGULAR SEMIGROUPS

A semigroup \( S \) is said to be combinatorial if every subgroup of it is of order one. In the case when \( S \) is regular this is equivalent to the assertion that the
Green's relation $H$ is the identity relation on $S$.

Subsemigroups of combinatorial semigroups are combinatorial. But homomorphic images of combinatorial semigroups need not be combinatorial. The following examples illustrate this.

**EXAMPLE 1.** The set of natural numbers $\mathbb{N}$ with the ordinary multiplication is a combinatorial semigroup, which is not regular. If $p$ is a prime, then $\mathbb{N}/p$ is a non-trivial group which is a homomorphic image of $\mathbb{N}$ and $\mathbb{N}/p$ is not combinatorial.

The next example shows that even for regular combinatorial semigroups, homomorphic images need not be combinatorial.

**EXAMPLE 2.** Let $S = C(p, q)$ be the bicyclic semigroup. 
$C(p, q) = \{(m, n) : m, n \in \mathbb{N}\}$, $\mathbb{N}$ the set of natural numbers, with product defined by

$$(m, n)(m', n') = (m - n + \max(n, m'), n' - m' + \max(n, m'))$$

Then $S$ is a combinatorial regular semigroup, which is also an inverse semigroup. Let $\mathcal{U}$ be the minimum group congruence on $S$. Then

$$\mathcal{U} = \{(x, y) \in S \times S : \text{there exists } e \in E(S) \text{ with } ex = ey\}$$
(cf. [10], p. 140). It follows from this, that $S/\sigma$ is isomorphic to $\mathbb{Z}$, the group of integers. Thus $S/\sigma$ is not combinatorial.

For convenience, in what follows, we shall adopt the following convention with regard to the terminology. Any concept associated with combinatorial semigroups will be indicated by using the adjective 'combinatorial'. Thus the biordered set of a combinatorial semigroup is a combinatorial biordered set. Similarly a congruence $\rho$ on a semigroup $S$ such that $S/\rho$ is combinatorial will be termed a combinatorial congruence and the Rees groupoid corresponding to a combinatorial regular semigroup will be called a combinatorial Rees groupoid. Often, we shall also abbreviate this as $C$-congruence, $C$-Rees groupoid etc.

**PROPOSITION 2.1.** Direct products and free products of combinatorial semigroups are combinatorial.

**Proof.** Let $S, T$ be combinatorial semigroups. Let $S \times T$ be the semigroup direct product of $S$ and $T$. If either $S$ or $T$ does not contain idempotents so is $S \times T$ and $S \times T$ is combinatorial. Suppose $E(S) \neq \emptyset$ and $E(T) \neq \emptyset$. By the definition of product in $S \times T$ an idempotent in $S \times T$ is $(e, f)$, where
e ∈ E(S) and f ∈ E(T). Let \((a, b) H (e, f)\) for some e ∈ E(S) and f ∈ E(T). Then a H e and b H f.

Since the subgroups \(H_e\) of \(S\) and \(H_f\) of \(T\) are trivial, \(a = e\) and \(b = f\) so that \((a, b) = (e, f)\). Thus all subgroups of \(S \times T\) are trivial.

Now let \(S \sqcup T\) be the semigroup free product of \(S\) and \(T\) with injections \(\Theta : S \rightarrow S \sqcup T\) and \(\wp : T \rightarrow S \sqcup T\). It is easy to see that if
\[
x = s_1 t_1 s_2 t_2 \cdots s_n t_n \quad \text{and} \quad y = s'_1 t'_1 s'_2 t'_2 \cdots s'_m t'_m
\]
with \(s_i, s'_i \in S\) and \(t_i, t'_i \in T\), are in \(S \sqcup T\), then
\[
x = y \iff m = n, s_i = s'_i \quad \text{and} \quad t_i = t'_i
\]
for every \(i\). Now we have, for \(x, y \in S \sqcup T\), \(x H y\) in \(S \sqcup T\) if and only if, one of the following holds:

(i) \(x = y\)

(ii) \(x, y \in \Theta (S)\) and \(\Theta^{-1} x H \Theta^{-1} y\) in \(S\)

(iii) \(x, y \in \wp (T)\) and \(\wp^{-1} x H \wp^{-1} y\) in \(T\).

Consequently \(S \sqcup T\) is combinatorial.

We call a regular biordered set \(E\) to be combinatorial if there exists some combinatorial regular semigroup \(S\) such
that \( E(S) \) is isomorphic to \( E \).

Bands, the bicyclic semigroup, the fundamental four-spiral semigroup (cf. [3]), locally testable semigroups [29] etc. are examples of combinatorial regular semigroups. A band considered as a biordered set is a combinatorial biordered set.

**PROPOSITION 2.2.** Every regular semigroup admits a minimum combinatorial congruence.

**Proof.** Let \( S \) be a regular semigroup. Let \( \rho_0 \) be the congruence on \( S \) generated by the Green's relation \( H \) on \( S \). Then \( S/\rho_0 \) is a regular semigroup which need not be combinatorial. Define

\[
\mathcal{H}_1 = \left\{ (x, y) \in S \times S : x \rho_0 \mathcal{H} y \rho_0 \text{ in } S/\rho_0 \right\}
\]

Then \( \rho_0 \subseteq \mathcal{H}_1 \) and \( \rho_1 \) be the congruence on \( S \) generated by \( \mathcal{H}_1 \). Then \( \rho_0 \subseteq \rho_1 \) Now define

\[
\mathcal{H}_2 = \left\{ (x, y) \in S \times S : x \rho_1 \mathcal{H} y \rho_1 \text{ in } S/\rho_1 \right\}
\]

and \( \rho_2 \) be the congruence generated by \( \mathcal{H}_2 \) and so on.

Continuing like this we get an ordered sequence of congruences \( \rho_0 \subseteq \rho_1 \subseteq \rho_2 \subseteq \cdots \) on \( S \). Suppose that for some ordinal \( \omega \), \( \rho_\omega = \rho_{\omega+1} \). Then \( S/\rho \), where \( \rho = \rho_\omega \), is combinatorial. For, let \( x \rho \mathcal{H} y \rho \) in \( S/\rho \).
Then \( x \rho_\omega \mathcal{H} y \rho_\omega \) and so \((x, y) \in \mathcal{H}_{\omega+1} \subseteq \rho_{\omega+1} = \rho_\omega\)

Hence \( x \rho_\omega = y \rho_\omega \). That is, \( x \rho = y \rho \). Therefore \( S/\rho \) is combinatorial. Now we prove that there exists an ordinal \( \omega \) such that \( \rho_\omega = \rho_{\omega+1} \). If not, all the \( \rho_i \)'s are different and so we get an ordinal \( \omega \) and congruences \( \rho_1, \rho_2, \ldots, \rho_\omega \), all of which are different from one another, and such that the cardinal number of is greater than the cardinal number of the set of all congruences on \( S \). This is a contradiction. Hence there exists \( \omega \) such that \( \rho_\omega = \rho_{\omega+1} \). Now we shall prove that \( \rho \) is the minimum \( C \)-congruence on \( S \). Let \( \sigma \) be a \( C \)-congruence on \( S \). Then \( S/\sigma \) is combinatorial. Therefore \( \mathcal{H} \subseteq \sigma \) and so \( \rho_\omega \subseteq \sigma \). Again since \( S/\sigma \) is combinatorial, \( \mathcal{H}_1 \subseteq \sigma \) and consequently \( \rho_1 \subseteq \sigma \).

By repeating this argument \( \rho_n \subseteq \sigma \) for every \( n \). Thus \( \rho \subseteq \sigma \). Therefore \( \rho \) is the minimum combinatorial congruence on \( S \).

2. STRUCTURE THEOREM USING INDUCTIVE GROUPOIDS

We define an inductive groupoid to be combinatorial if it is unicursal (cf. [9], p.9). That is, if there exists at most one morphism between any two vertices. If \( S \) is a combinatorial regular semigroup, then the inductive groupoid
$G(S)$ (cf. Proposition 1.19) determined by $S$, is clearly combinatorial.

From Theorem 1.22, we, therefore, obtain the following structure theorem for combinatorial regular semigroups.

**THEOREM 2.3.** Let $(G, \mathcal{E})$ be a combinatorial inductive groupoid over a biordered set $E$. Then $S(G)$ is a combinatorial regular semigroup. Conversely every combinatorial regular semigroup can be obtained in this way.

**Proof.** Let $S(G) = G/\mathcal{P}$ and let $\bar{\alpha}$ denote the $\mathcal{P}$-class of $G$ containing $\alpha$. Let $e, f \in E$ and $\alpha \in G(e, f)$. Then clearly $\bar{\alpha} \in R_{e} \cap L_{f}$. Further by Proposition 1.21 $\alpha, \beta \in G(e, f)$ with $\bar{\alpha} = \bar{\beta}$ implies $\alpha = \beta$. Now let $\bar{\beta} \in R_{e} \cap L_{f}$. Clearly $\bar{\beta} \in R_{e} \cap L_{f}$ so that $e R_{e} \beta$ and $f \not\perp f_{\beta}$. Let $\beta' = \mathcal{E}(e, e_{\beta}) \beta \mathcal{E}(f_{\beta}, f)$. Then $\bar{\beta} = \bar{\beta}'$ and $\beta' \in G(e, f)$. Hence the mapping $\alpha \mapsto \bar{\alpha}$ is a bijection of $G(e, f)$ onto $R_{e} \cap L_{f}$ and $R_{e} \cap L_{f}$ is an $H$-class in $S(G)$, if it is non-empty. Hence the theorem is proved $\square$

The foregoing theorem provides a characterization of combinatorial biordered sets as follows.
COROLLARY 2.4. A regular biordered set $E$ is combinatorial if and only if $\mathcal{C}^*(E)$ is a combinatorial inductive groupoid. That is, if and only if $\mathcal{C}(\gamma) = 1$ for all cycles $\gamma$ in $E$.

Proof. If $E$ is combinatorial, then there exists a combinatorial regular semigroup $S$ such that $E = E(S)$. Then $G(S)$ is a $C$-inductive groupoid and so $\mathcal{C}^*(E)$ is also combinatorial. Converse is clear.

A groupoid $G$ is said to be connected if for all $e, f \in V(G)$, $G(e, f)$ is non-empty. If $G(e, f)$ contains exactly one element for each $e, f \in V(G)$, $G$ is said to be simplicial (cf. [9], p.9). Thus a groupoid is simplicial if and only if it is both connected and unicursal. We have the following characterization of bisimple combinatorial regular semigroups.

COROLLARY 2.5. A regular semigroup $S$ is bisimple and combinatorial if and only if its inductive groupoid $G(S)$ is simplicial.

3. STRUCTURE THEOREM USING COMBINATORIAL FUNCTORS

Meakin [16] and Nambooripad [20] have introduced another approach to the study of structure of regular semigroups using structure mappings. If $S$ is a regular
semigroup, the trace of $S$ is the set $S$ together with a partial binary operation $*$ defined as follows.

$$x * y = \begin{cases} \ xy & \text{if } xy \in R_x \cap L_y \\ \text{undefined otherwise.} \end{cases}$$

By Theorem 3.4 of [6], $S(*)$ is a disjoint union of Rees groupoids. Since $E(S)$ is a biordered set, the set of idempotents of the partial algebra $S(*)$ is a biordered set. Given $S(*)$, Theorem 1.26 shows that the product in $S$ can be expressed in terms of the partial binary operation $*$ and products of the form $xh$ and $hy$, where $h \in \omega^+(e) \cap \omega^+(f)$, $e \in E(L_x)$ and $f \in E(R_y)$. Note that $x \mapsto xh$ for $x \in L_e$ is a mapping of $L_e \rightarrow L_h$ and $y \mapsto hy$ is a mapping of $R_f \rightarrow R_h$. Such mappings are called structure mappings by Meakin and Nambooripad.

Explicit axioms to characterise these mappings are given in [16] and [20]. In the case of combinatorial regular semigroups, these mappings determine a homomorphism of the trace of the corresponding $\mathcal{O}$-classes. In this case $S(*)$ becomes a small category, in which, objects are traces of $\mathcal{O}$-classes of $S$ and morphisms are homomorphisms of combinatorial Rees groupoids. In fact this becomes the image of a functor from the pre-order $(E(S), \omega)$ into
the category of C-Rees groupoids. Our objective in this section is to give a structure theorem for combinatorial regular semigroups using this functor.

Note that a Rees groupoid $G$ is said to be combinatorial if the corresponding completely $o$-simple semigroup $G^o$ is combinatorial. The class of C-Rees groupoids with homomorphisms forms a category which we denote by $\text{CRG}$.

In what follows $S$ denotes a combinatorial regular semigroup, $x, y$ etc. arbitrary elements of $S$; $e, f$ etc. idempotents and $D_e, D_f$ etc. the $\mathcal{D}$-classes of $S$ containing $e, f$ etc. respectively. For any subset $X$ of $S$, we denote by $E(X)$, the set of idempotents in $X$.

**Lemma 2.6.** Let $D_e, D_f$ be two $\mathcal{D}$-classes in $S$ with $f \omega e$. Define $\varphi(e, f) : E(D_e) \to E(D_f)$ as follows. If $e' \in E(D_e)$

$$(2.1) \quad e' \varphi(e, f) = f a(x, x')$$

where $x \in R_e \cap L_e$, $x' \in L_e \cap R_e$, and

$$a(x, x') : \omega(e) \to \omega(e')$$

is the $\omega$-isomorphism, defined by $f a(x, x') = x'fx$ (cf. Proposition 1.20).
Then $\varphi(e, f)$ can be extended to a morphism of Rees groupoids $D_e \rightarrow D_f$ such that $e \varphi(e, f) = f$.

Further, if $e \not\leq e'$, $f \not\leq f'$ and if

$\varphi(e', f') : D_{e'} \rightarrow D_{f'}$ is such that $e \varphi(e', f') = f$ then $\varphi(e, f) = \varphi(e', f')$.

Proof. Since $S$ is combinatorial, $x \in R_e \cap L_e$, and $x' \in R_{e'} \cap L_{e'}$ implies that $x' \in V(x)$ and so $\alpha(x, x')$ is defined. Note that Equation (2.1) implies that $e \varphi(e, f) = f$. First we show that, if $e', e'' \in E(D_e)$ with $e' \not\leq e''$, then $e' \varphi(e, f) \not\leq e'' \varphi(e, f)$. Now $e' \varphi(e, f) = f \alpha(x, x')$, where $x \in R_e \cap L_e$, and $x' \in R_{e'} \cap L_{e'}$. Since $x \in e'' R_e \cap L_e$ and $x' \in e'' R_{e''} \cap L_{e''}$ we have, by Proposition 1.18, that

$$e'' \varphi(e, f) = f \alpha(xe'', x')$$

$$= f \alpha(x, x') \alpha(e'', e')$$

Since $\alpha(e'', e') = \tau(e', e'')$ (cf. Proposition 1.20), it follows that

$$e'' \varphi(e, f) = f \alpha(x, x') \tau(e', e'') \not\leq e' \varphi(e, f).$$

Dually, if $e' \not\geq e''$, then $e' \varphi(e, f) \not\leq e'' \varphi(e, f)$.
Now we extend \( \varphi (e, f) \) to \( D_e \) as follows. Let \( y \in D_e \).

Then, there exists \( e', f' \in E(D_e) \) such that

\[ y \in R_{e'} \cap L_{f'} \]. Define \( y \varphi (e, f) \) to be the unique element in \( R_{e'} \cap L_{f'} \). It follows from the above observations that \( \varphi (e, f) : D_e \to D_f \) is well-defined.

To establish uniqueness, it is sufficient to prove that if \( e' \in E(D_e) \) and \( e' \varphi (e, f) = f' \), then \( f' \omega e' \) and \( \varphi (e, f) = \varphi (e', f') \). Since \( e' \varphi (e, f) = f a(x, x') \)

where \( x \in R_e \cap L_e \), \( x' \in R_{e'} \cap L_{e'} \) we get

\[ f' \omega x' x = e' \]. Now let \( g \in E(D_e) \). Then

\[ g \varphi (e, f) = f a(y, y') \], where \( y \in R_{e} \cap L_{g} \), \( y' \in R_{g} \cap L_{e} \).

If \( z \in R_{e} \cap L_{g} \) and \( z' \in R_{g} \cap L_{e} \), then \( x' x = z z' \), and so

\[ g \varphi (e', f') = f' a(z, z') \]
\[ = f a(x, x') a(z, z') \]
\[ = f a(xz, z' x'). \]

Since \( S \) is combinatorial, \( xz = y \) and \( z' x' = y' \). Therefore

\[ g \varphi (e', f') = f a(y, y') = g \varphi (e, f). \]
Hence $\varphi(e, f) = \varphi(e', f')$ and the lemma follows.

**LEMMA 2.7.** Let $x \in S$ and $x' \in V(x)$, where $V(x)$ denotes the set of inverses of $x$. If $e \omega xx'$, then $x' \varphi(xx', e) \in V(x \varphi(xx', e))$ and

$$a(x, x') \mid \omega(e) = a(x \varphi(xx', e), x' \varphi(xx', e))$$

**Proof.** Since $e \omega xx'$ and $x'e \in V(xx)$, it is easily seen that, $a(x, x') \mid \omega(e) = a(ex, x'e)$. We know that $ex \in R_e \cap L_{xx}$.

Since $x \in R_{xx'} \cap L_{xx'}$, we have $xx' \varphi(xx', e) = e$ and $x'x \varphi(xx', e) = e a(x, x') = x'ex$; hence $xx' \varphi(xx', e) \in R_e \cap L_{xx'}$. Consequently $x \varphi(xx', e) = ex$, since $S$ is combinatorial. Similarly $x' \varphi(xx', e) = x'e$. This proves the lemma.

**LEMMA 2.8.** Let $x, y \in S$, $e \in E(L_x)$, $f \in E(R_y)$ and $h \in S(e, f)$. Then

$$xy = x \varphi(e, eh)y \varphi(f, hf).$$

**Proof.** Let $x \in R_e \cap L_e$ and $x' \in R_e \cap L_e$. Then

$$e' \varphi(e, eh) = (eh) a(x^h, x)$$

$$= x (eh) x'$$

$$= xhx'. $$
Since
\[ \forall h \in L e \cap R x h x' = L e \varphi (e, eh) \cap R e' \varphi (e, eh) \]
we have \( x h = x \varphi (e, eh) \). Similarly \( h y = y \varphi (f, hf) \).
Hence the result follows from Theorem 1.26 \( \square \)

Recall that if \( S \) is a regular semigroup, then \( (E(S), \omega) \) will denote the preorder on the set of idempotents of \( S \).

**Theorem 2.9.** Let \( S \) be a combinatorial regular semigroup. Define the functor \( \varphi_S : (E(S), \omega) \to \text{CRG} \) as follows. For every \( e \in E(S) \), \( \varphi_S(e) = D_e \), the trace of the \( \varnothing \) -class containing \( e \), and for \( e \in \omega f \), \( \varphi_S(f, e) \) is the morphism from \( D_f \) to \( D_e \) defined in Lemma 2.6. Then \( \varphi_S \) is a functor satisfying the following conditions:

(a) If \( e \in R f [e \in \varnothing f] \) in \( E(S) \) then \( \varphi_S(e) = \varphi_S(f) \) and \( e \in R f [e \in \varnothing f] \) in the Rees groupoid \( \varphi_S(e) \).

(b) For all \( f, g \in E(S) \) with \( \varphi_S(g) = \varphi_S(f) \), the mapping \( \alpha_{f, g} : \omega(f) \to \omega(g) \) defined by
\[ e \alpha_{f, g} = g \varphi_S(f, e) \]
is an \( \omega \)-isomorphism, such that
(i) \( \alpha_f, g \preceq g, h = \alpha_f, h \), if \( \varphi_S(f) = \varphi_S(g) = \varphi_S(h) \)
and

(ii) \( \alpha_f, g = \tau(f, g) \), if \( f \not\sim g \) or \( f \not\leq g \).

Proof. Suppose \( e, f, g \in E(S) \) with \( f \not\sim e \) and \( g \not\sim f \).

Then we shall show that \( \varphi_S(e, f) \varphi_S(f, g) = \varphi_S(e, g) \).

Let \( h \in E(\varphi_S(e)) \), \( x \in R_e \cap L_h \) and \( x' \in R_h \cap L_e \). Then

\[ x \varphi_S(e, f) \in R_f \cap L_h \varphi_S(e, f) \quad \text{and} \quad x' \varphi_S(e, f) \in R_h \varphi_S(e, f) \cap L_f. \]

Therefore

\[ h \varphi_S(e, f) \varphi_S(f, g) = g \alpha(x \varphi_S(e, f), x' \varphi_S(e, f)) \]

\[ = g \alpha(x, x') \quad \text{(by Lemma 2.7)} \]

\[ = h \varphi_S(e, g). \]

From the definition of \( \varphi_S(e, f) \), it is clear that \( \varphi_S(e, e) \)
is identity for every \( e \in E(S) \). Hence \( \varphi_S \) is a functor
and (a) follows.

By the definition of \( \varphi_S(f, e) \) for \( e \not\sim f \), we see
that \( \alpha_f, g = \alpha(x, x') \) where \( x \in R_f \cap L_g \) and
\( x' \in R_g \cap L_f \). Hence \( \alpha_f, g : \omega(f) \rightarrow \omega(g) \) is an
\( \omega \)-isomorphism. Let \( \alpha_f, g = \alpha(x, x') \) and 
\( \alpha_g, h = \alpha(y, y') \) where \( x \in R_f \cap L_g, x' \in R_g \cap L_f, \)
y \( \in R_g \cap L_h \) and y' \( \in R_h \cap L_g \). Then \( x'x = g = yy' \),
xy \( \in R_f \cap L_h \) and y'x' \( \in R_h \cap L_f \). Hence
\[
\alpha_f, g \alpha_g, h = \alpha(x, x') \alpha(y, y')
\]
\[
= \alpha(xy, y'x')
\]
\[
= \alpha_f, h
\]
Thus (b) (i) holds.

Now suppose that \( f \not\sim g \). Then
\[
\alpha_f, g = \alpha(g, f)
\]
\[
= \mathcal{T}(f, g) \quad (\text{by Proposition 1.20}).
\]
Similarly if \( f \not\triangleleft g \), then also \( \alpha_f, g = \mathcal{T}(f, g) \). This
completes the proof of the theorem \( \square \)

Now we proceed to define combinatorial functors. Let
\( S = \bigcup D_{\alpha} \) be a pseudo-groupoid (cf. Section I.4). We shall
say that a biorder structure on the set of idempotents
\( E(S) = \bigcup E(D_{\alpha}) \) of \( S \), is \textit{compatible} with the Green's
relations on \( S \), if \( E(S) \) has a biorder structure, such
that \( e \not\sim f \Leftrightarrow e \not\triangleleft f \) in the biordered set if and only if
e, f ∈ Dψ, for some Rees groupoid Dψ of S and eRf [e≤f] in Dψ.

DEFINITION 2.10. Let E be a biordered set and

φ : (E, ω) → CRG be a functor from the preorder (E, ω) into the category CRG of combinatorial Rees groupoids, such that if φ(e) ≠ φ(f), then φ(e) ∩ φ(f) = ∅. Let

S = Σ(φ) = U {φ(e) : e ∈ E} and
E(S) = U {E(φ(e)) : e ∈ E}.

Then φ is said to be a combinatorial functor (C-functor) if the following axioms hold.

(C1) There exists a bijection ψ : E → E(S), such that

ψ ∈ φ(e) and

eRf [e≤f] in E ⇔ φ(e) = φ(f) and

eψRfψ [eψ≤fψ] in φ(e).

(C2) For all f, g ∈ E, with, φ(f) = φ(g), the mapping

Lf, g : ω(f) → ω(g) defined by

eL f, g = (gψ φ(f, e))ψ⁻¹

is an ω -isomorphism in E, such that,

(i) Lf, g Lg, h = Lf, h if φ(f) = φ(g) = φ(h)

and

(ii) Lf, g = τ(f, g) if e R U L f
It may be noted that the image of \( \varphi \) is a small subcategory of \( \mathcal{CRG} \) which we denote by \( \text{im} \varphi \). Further the axioms given above are self-dual.

**DEFINITION 2.11.** Let \( \varphi, \varphi' \) be two C-functors on biordered sets \( E \) and \( E' \) respectively. A morphism of C-functors from \( \varphi \) to \( \varphi' \) is a pair \((\Theta, \sigma)\) where \( \Theta : E \to E' \) is a regular bimorphism and \( \sigma : \varphi \to \Theta \varphi' \) is a natural transformation such that for all \( e \in E \) and \( f \in E(\varphi(e)), (f \varphi) \sigma_e = (f \Theta) \psi' \), where \( \psi : E \to E(\Sigma(\varphi)) \) and \( \psi' : E' \to E(\Sigma(\varphi')) \) are the bijections given by axiom (C1).

Given morphisms \((\Theta, \sigma) : \varphi \to \varphi'\) and \((\Theta', \sigma') : \varphi' \to \varphi''\) of C-functors, their composition is defined by

\[
(\Theta, \sigma)(\Theta', \sigma') = (\Theta \Theta', \sigma \sigma')
\]

where \( \sigma \sigma' : \varphi \to \Theta \Theta' \varphi'' \) has components \( (\sigma \sigma')_e = \sigma_e \sigma'_{\Theta e} \) for \( e \) in the domain \( E \) of \( \varphi \). It is easily seen that \((\Theta \Theta', \sigma \sigma')\) is a morphism from \( \varphi \) to \( \varphi'' \). Clearly \((\Theta, \sigma)\) is an isomorphism, if and only if, both \( \Theta \) and \( \sigma \) are isomorphisms. We denote by \( \mathcal{CF} \) the category of combinatorial functors with the above defined morphisms.
The bijection $\psi$ in (C1) naturally induces a biorder structure on $E(S)$ such that $\psi : E \rightarrow E(S)$ is a biorder isomorphism. Further this biorder structure on $E(S)$ is compatible with the Green's relations on $S$.

REMARK. It may be noted that the axioms for combinatorial functors given above are different from those appeared in [24]. These axioms are somewhat simpler and the proof of the structure theorem given below is also simplified by the use of these new axioms. The original axioms in [24] appear in Proposition 2.14 below.

In what follows, whenever a combinatorial functor $\varphi : (E, \omega) \rightarrow \text{CGR}$ is considered $E$ is identified with $E(S)$, so that $\psi$ becomes the identity map on $E$.

The following theorem gives the structure of combinatorial regular semigroups.

**Theorem 2.12.** Let $\varphi : (E, \omega) \rightarrow \text{CGR}$ be a C-functor. On $S = \Sigma(\varphi)$, define, product by

$$(2.2) \ xy = x \varphi(e, eh)\varphi(f, hf)$$

where $e \in E(L_x)$, $f \in E(R_y)$ and $h \in \mathcal{S}(e, f)$. Then $S$, with this product, is a combinatorial regular semigroup whose biordered set is isomorphic to $E$, such that $\varphi_s = \varphi$ where $\varphi_s$ is the functor defined in Theorem 2.9.
Conversely if $S$ is any combinatorial regular semigroup, then $\mathfrak{C}_S$ (cf. Theorem 2.9) is a $C$-functor and $S = \Sigma (\mathfrak{C}_S)$.

We prove the theorem through a sequence of lemmas. In the following $\mathfrak{C}$ denotes a fixed $C$-functor on a biordered set $E$ and $S = \Sigma (\mathfrak{C})$.

**Lemma 2.13.** If $f \omega e$, then $e \mathfrak{C} (e, f) = f$ and if $f' \omega e'$ with $\mathfrak{C} (e) = \mathfrak{C} (e')$ and if there exists $g \in E (\mathfrak{C} (e))$ such that $g \mathfrak{C} (e, f) = g \mathfrak{C} (e', f')$, then, $\mathfrak{C} (e, f) = \mathfrak{C} (e', f')$. In particular, for any $g \in E (\mathfrak{C} (e))$, $\mathfrak{C} (e, f) = \mathfrak{C} (g, g \mathfrak{C} (e, f))$.

**Proof.** Let $h \in E (\mathfrak{C} (e))$. Then $\mathfrak{C} (e) = \mathfrak{C} (h)$, for $\mathfrak{C} (e) \neq \mathfrak{C} (h)$ implies $\mathfrak{C} (e) \cap \mathfrak{C} (f) = \emptyset$. Therefore,

$$h \mathfrak{C} (e, f) = f \mathfrak{C} e, h = f \mathfrak{C} e, g \mathfrak{C} g, h \quad \text{(by axiom C2 (i))}$$

$$= g \mathfrak{C} (e, f) \mathfrak{C} g, h$$

$$= g \mathfrak{C} (e', f') \mathfrak{C} g, h$$

$$= f' \mathfrak{C} e', g \mathfrak{C} g, h$$

$$= f' \mathfrak{C} e', h \quad \text{(by axiom C2 (i))}$$

$$= h \mathfrak{C} (e', f').$$
Hence \( \varphi(e, f) = \varphi(e', f') \). To see the last part, notice first that \( g \varphi(e, f) = f \varphi(gg(e, f)) \). Therefore, \( \varphi(g, g \varphi(e, f)) \) is defined. Now \( e \varphi(e, f) = f \varphi(e, e) = f \) since \( \varphi(e, e) \) is identity, by axiom (C2) (ii). Hence the last part follows.

Now we give another characterization of the axioms for a combinatorial functor.

**Proposition 2.14.** \( \varphi : (E, \omega) \rightarrow \mathbf{CRG} \) is a combinatorial functor, if and only if \( \varphi \) satisfies the following axioms and their duals.

1. **(CF1)** \( e R f [e \subseteq f] \) in \( E \iff \varphi(e) = \varphi(f) \) and \( e R f [e \subseteq f] \) in \( \varphi(e) \).
2. **(CF2)** If \( e \omega f \) in \( E \), then \( \varphi(f, e) : \varphi(f) \rightarrow \varphi(e) \) is the unique morphism in \( \mathrm{im} \varphi \) mapping \( f \) to \( e \).
3. **(CF3)** If \( e', e'' \in \omega(e) \) with \( e' \omega^+ e'' \), then for all \( f \in E \varphi(e) \)
   \[ f \varphi(e, e) \omega^+ f \varphi(e, e') \]
   and
   \[ \varphi(f \varphi(e, e''), f \varphi(e, e') \varphi(e, e'')) = \varphi(e'', e'e'') \]
4. **(CF4)** If \( g \omega^+ f \omega^+ e \), then \( \varphi(g, g \varphi(f, e)) \) is defined.
\varphi(f, gf) = \varphi(fe, gfe)

Proof. Suppose \varphi is a C-functor.

(CF1) is the same as (C1) and (CF2) follows from Lemma 2.13

To prove (CF3). Let \( e', e'' \in \omega(e) \) and \( e' \omega^* e'' \) and let \( f \in E(\varphi(e)) \). Let \( f' = f \varphi(e, e') \) and \( f'' = f \varphi(e, e'') \).

By (C2) \( f' = e' \alpha_e, f \) and \( f'' = e'' \alpha_e, f \).

Since \( \alpha_e, f : \omega(e) \to \omega(f) \) is an \( \omega \)-isomorphism \( f' \omega^* f'' \). Further

\[
\begin{align*}
  f'' \varphi(e'', e'e'') &= (e'' \alpha_e, f) \varphi(e'', e'e'') \\
  &= f \varphi(e, e'') \varphi(e'', e'e'') \\
  &= f \varphi(e, e'e'') \text{ (since } \varphi \text{ is a functor)} \\
  &= (e'e'') \alpha_e, f \\
  &= f'f''.
\end{align*}
\]

Now by Lemma 2.13, \( \varphi(e'', e'e'') = \varphi(f'', f'f'') \).

Hence (CF3) holds.

To prove (CF4). Suppose that \( g \omega^* f \omega^* e \). Then \( f \varphi (f \varphi (f, gf) = (gf) \alpha_{f, fe} \\
(\text{by axiom C2 (ii)}) \\
= gfe.\)
Again, by Lemma 2.13, \( \varphi(f, gf) = \varphi(fe, gfe) \). Thus (CF4) holds. The dual statements will follow along similar lines.

Conversely, suppose that \( \varphi \) satisfies (CF1-CF4) and their duals. Then clearly (C1) holds. Let \( f, g, e \in E \) with \( \varphi(f) = \varphi(g) \) and \( e \omega f \). First we show that \( \alpha_f, g : \omega(f) \to \omega(g) \) defined by \( e \alpha_f, g = g \varphi(f, e) \) is an \( \omega \)-isomorphism. Let \( e, e' \in \omega(f) \) with \( e \omega^* e' \). Then by (CF3),

\[
g \varphi(f, e) \omega^* g \varphi(f, e'), \text{ so that } e \alpha_f, g \omega^* e' \alpha_f, g.
\]

Further

\[
(ee') \alpha_f, g = g \varphi(f, ee') = g \varphi(f, e') \varphi(e', ee') = g \varphi(f, e') \varphi(g \varphi(f, e'), g \varphi(f, e)g \varphi(f, e')) \quad \text{(by (CF3))}
\]

\[
= g \varphi(f, e)g \varphi(f, e') = (e \alpha_f, g)(e' \alpha_f, g)
\]

Dually, we can show that, if \( e \omega^* e' \), then \( e \alpha_f, g \omega^* e' \alpha_f, g \) and \( (e'e) \alpha_f, g = (e' \alpha_f, g)(e \alpha_f, g) \). Hence \( \alpha_f, g \) is a bimorphism. Also \( \alpha_f, f \) is identity by axiom (CF2).
Further if \( \varphi(f) = \varphi(g) = \varphi(h) \) and \( e \omega f \), then
\[
e \varpropto_f, g \varpropto_g, h = g \varphi(f, e) \varpropto_g, h
\]
\[
= h \varphi(g, g \varphi(f, e))
\]
\[
= h \varphi(f, e) \quad \text{(by Lemma 2.13)}
\]
\[
= e \varpropto_f, h.
\]

Hence \( \varpropto_f, g \) is an \( \omega \)-isomorphism. Also (i) of axiom (C2) holds. Now let \( f \vartriangleleft g \) and \( e \omega f \). Then
\[
e \varpropto_f, g = g \varphi(f, e)
\]
\[
= g \varphi(g, eg) \quad \text{(by axiom (CF4))}
\]
\[
= eg
\]
\[
= e \tau(f, g).
\]

Dually if \( f \vartriangleleft g \) then \( e \varpropto_f, g = e \tau(f, g) \). Thus axiom (C2) (ii) holds. This completes the proof of Proposition 2.14.}

**Lemma 2.15.** If \( e \vartriangleleft f \) then for all \( e' \in \omega(e) \),
\[
\varphi(e, e') = \varphi(f, e'f) \quad \text{[\( \varphi(e, e') = \varphi(f, fe') \)]}
\]

**Proof.** Let \( e \vartriangleleft f \). Then \( \varpropto_e, f : \omega(e) \to \omega(f) \) is an \( \omega \)-isomorphism, such that,
\[
e' \varpropto_e, f = f \varphi(e, e') = e' \tau(e, f) \quad \text{by axiom (C2).}
Hence $f \varphi(e, e') = e'f$. Therefore by Lemma 2.13

$\varphi(e, e') = \varphi(f, e'f)$. The statement within brackets follows dually $\Box$

**Lemma 2.16.** Let $G(S) = \{(x, x') : x \in S, x' \in V(x)\}$

Define a product by

$$(x, x')(y, y') = (xy, y'x')$$

whenever $x'x = yy'$. Then $G(S)$ is a groupoid.

Further define $\alpha : G(S) \to T^*(E)$ by

$$(2.3) \quad \alpha(x, x') = x'x : \omega(xx') \to \omega(x'x)$$

Then $\alpha$ is an isomorphism of $G(S)$ into $T^*(E)$. Also if $e \not\in \{e \notin f\}$, then $\alpha(e, f) = \sigma(f, e)$ $[\sigma(e, f)]$

**Proof.** It is easily verified that $G(S)$ is a groupoid with identities $\{(e, e) : e \in E\}$. Let

$(x, x'), (y, y') \in G(S)$ with $x'x = yy'$ and let $e \omega(xx')$. Then

$$e \alpha(x, x') \alpha(y, y') = e \omega(xx', x'x \omega(yy', y'y) = e \omega(xx', y'y)$$

since $x'x = yy'$. Also $e \alpha(xy, y'x') = e \omega(xx', y'y)$

Hence $\alpha(x, x') \alpha(y, y') = \alpha(xy, y'x')$ so that $\alpha$ is a homomorphism. To show that $\alpha$ is one-to-one, assume that

$\alpha(x, x') = \alpha(y, y')$. Then $xx' = yy'$ and $x'x = y'y$. 
Therefore \( x R xx' = yy'R y \), and \( x L x'x = y'yL y \).

Hence \( x H y \). Since \( \varphi(xx') \) is combinatorial, we conclude that \( x = y \). Similarly \( x' = y' \).

Finally, let \( e R f \) and \( h \in \omega(f) \). Then
\[
 h \alpha(e, f) = h \alpha_f, \quad e = h \Upsilon(f, e)
\]

The statement within brackets follows dually \( \square \)

**LEMMA 2.17.** If \( e \in \omega xx' \), then
\[
\alpha(x \varphi(xx', e), x' \varphi(xx', e)) = \alpha(x, x') \mid \omega(e).
\]

and \( \mathcal{T}_S = \text{im} \alpha \) is an inductive groupoid over the biordered set \( E \), such that, for all \( e, f \in E \), \( \mathcal{T}_S(e, f) \) contains atmost one element.

**Proof.** Suppose that \( h \in \omega e \omega xx' \). Then
\[
 h \alpha(x \varphi(xx', e), x' \varphi(xx', e)) = x'x \varphi(xx', e) \varphi(e, h) = x'x \varphi(xx', h) = h \alpha(x, x').
\]

Now from Lemma 2.15 it follows that \( \mathcal{T}_S \) is an inductive subgroupoid of \( (\mathcal{T}(E), \Upsilon_E) \) (cf. Proposition 1.17).

Suppose now that \( \alpha(x, x'), \alpha(y, y') \in \mathcal{T}_S(e, f) \).
Then \( xx' = e = yy' \) and \( x'x = f = y'y \). Hence \( x \mathcal{H} y \) and \( x' \mathcal{H} y' \). Since \( \varrho \) is combinatorial, this implies, \( x = y \) and \( x' = y' \). Therefore \( \mathcal{F}_S(e, f) \) contains at most one element. 

**Lemma 2.18.** For \( x \in S \) define \( x \Psi = \overline{\alpha(x, x')} \), where \( \overline{\alpha(x, x')} \) denotes the \( p \)-class of \( \alpha(x, x') \) (cf. Equation (1.19)). Then \( \Psi \) is a bijection of \( S \) onto \( \mathcal{F}_S = \mathcal{F}_S / \rho \) such that, for \( e \in E(x) \), \( f \in E(y) \), and \( h \in S(e, f) \)
\[
(x \varphi(e, eh) y \varphi(f, hf)) \Psi = x \Psi y \Psi
\]

**Proof.** First we show that \( \Psi \) is well-defined. Let \( x', x'' \in V(x) \). Then \( xx' \not\sim xx'' \) and \( x'x \not\sim x''x \).

Further for \( h \in \omega(xx') \), by Lemma 2.16,
\[
h \alpha(x, x') \mathcal{H} (x'x, x''x) = h \alpha(x, x') a(x'x, x''x) = h \alpha(x, x''xx')
\]
and
\[
h \mathcal{H} (xx', xx'') \alpha(x, x'') = h \alpha(xx'', xx') a(x, x'') = h \alpha(x, x''xx')
\]

This proves that \( \overline{\alpha(x, x')} \parallel \overline{\alpha(x, x'')} \). Suppose that \( \overline{\alpha(x, x')} = \overline{\alpha(y, y')} \). Then \( \overline{\alpha(x, x')} \parallel \overline{\alpha(y, y')} \) and
so by the definition of \( f, x'x \mathcal{R} y'y' \) and \( x'x \perp y'y \).

This implies that \( x \mathcal{R} x'x \mathcal{R} y'y \mathcal{R} y \) and
\[ x \mathcal{L} x'x \mathcal{L} y'y \mathcal{L} y. \]
Thus \( x \mathcal{H} y \) and so \( x = y \). Hence \( \psi \) is one-to-one. Clearly \( \psi \) is also onto. Next we show

that for \( e \in \mathcal{E}(L_x), f \in \mathcal{E}(R_y) \) and \( h \in \mathcal{S}(e, f) \)
\[ (x \varphi (e, eh)y \varphi (f, hf)) \psi = x \psi y \psi \]

By Equation (1.20) we have
\[
\alpha(x, x') \alpha(y, y') = (\alpha(x, x') \star h)(h \star \alpha(y, y'))
\]
where
\[
\alpha(x, x') \star h = (\alpha(x, x') \omega (xx' \varphi (x'x, (x'x)h)))\mathcal{T}((x'x)h, h)
\]
\[ = \alpha(x \varphi (xx', e'), x' \varphi (xx', e')) \alpha((x'x)h, h)
\]
(\text{where } e' = xx' \varphi (x'x, (x'x)h))
\[ = \alpha(x \varphi (xx', e'), h(x' \varphi (xx', e')))
\]
\[ = \alpha(x \varphi (x'x, (x'x)h), h(x' \varphi (x'x, (x'x)h)))
\]
and
\[
h \star \alpha(y, y') = \mathcal{T}(h, h(yy'))(\alpha(y, y'); \omega(h(yy')))
\]
\[ = \alpha(h(yy'), h) \alpha(y \varphi(yy', h(yy')), yy' \varphi(yy', h(yy')))
\]
\[ = \alpha(y \varphi(yy', h(yy')), (yy' \varphi(yy', h(yy')))h)
\]

Since \( e \mathcal{L} x'x \) and \( f \mathcal{R} yy' \), by Lemma 2.15
\( \varphi(y'y', h(y'y')) = \varphi(f, hf) \) and \( \varphi(x'x, (x'x)h) = \varphi(e, eh) \)

Therefore

\[
(a(x, x') \ast h)(h \ast a(y, y')) = a(x \varphi(e, eh), h(x' \varphi(e, eh)))
\]
\[
\ast (y \varphi(f, hf), (y' \varphi(f, hf)) h)
\]
\[= a(x \varphi(e, eh) \varphi(f, hf), (y' \varphi(f, hf)) h(x' \varphi(e, eh)))
\]

Thus

\[x \, \Psi \, y \, \Psi = (x \varphi(e, eh) y \varphi(f, hf)) \Psi \quad \square\]

**Proof of Theorem 2.12.** Since \( eh \in h \, \mathcal{R} \, hf \), the product on the right side of equation (2.2) is defined in the Rees groupoid \( \varphi(h) \). By Proposition 1.10 and Lemma 2.18 we see that Equation (2.2) defines a binary operation on \( S \) and that \( \Psi \) is an isomorphism of \( S \) onto the regular semigroup \( \overline{\mathcal{S}} \). By Lemma 2.17 and Theorem 2.3, \( \overline{\mathcal{S}} \) is a combinatorial regular semigroup and so \( S \) also is a combinatorial regular semigroup whose biordered set \( E(S) \) is isomorphic to \( E \).

If \( x \mathcal{L} y \) in a Rees groupoid \( \varphi(e) \), then they are clearly \( \mathcal{L} \) -related in \( S \). Conversely, assume that, \( x \) and \( y \) are \( \mathcal{L} \) -related in the semigroup \( S \). Let \( e, f \) be idempotents such that \( e \mathcal{L} x \) and \( f \mathcal{L} y \) in the
respective Rees groupoids. Then $e \mathcal{L} x$ and $f \mathcal{L} y$
in the semigroup $S$. Hence $e \mathcal{L} f$ in the biordered
set $E$. By axiom (C1), we conclude that $e$ and $f$
belong to the same Rees groupoid and that they are
$\mathcal{L}$-related. Dually, if $x \mathcal{R} y$ in $S$, then they
belong to the same Rees groupoid and are $\mathcal{R}$-related.
Therefore $x$ and $y$ belong to the same Rees groupoid
if and only if they are $\mathcal{G}$-related in $S$. It now follows
that the product $xy$ exists in a Rees groupoid $\mathcal{P}(e)$
if and only if $xy$ exists in the trace of $S$ and
in this case they are equal. This proves that for all
$e \in E$, $\mathcal{P}(e) = \mathcal{P}_S(e)$. Next we show that
$\mathcal{P}(e, f) = \mathcal{P}_S(e, f)$ whenever $f \omega e$. It is clear
that $e \mathcal{P}(e, f) = f = e \mathcal{P}_S(e, f)$. Let
$e' \in E(\mathcal{P}_S(e)) = E(\mathcal{P}(e))$. We shall prove that
$e' \mathcal{P}_S(e, f) = e' \mathcal{P}(e, f)$. If $x \in R_e \cap L_e$, and
$x' \in R_{e'} \cap L_{e'}$ then by Equation (2.1) we have
\[
e' \mathcal{P}_S(e, f) = f A(x, x')
= x'fx = (x'f)(fx)
\]
Now by the definition of product in Equation (2.2), we get
$x'f = x' \mathcal{P}(e, ef)f \mathcal{P}(f, f)$
Let $\text{THEOREM 2.19.}$ of $C$-functions.

$$= (x' \varphi (e, f))f = x' \varphi (e, f)$$

and

$$fx = f \varphi (f, f)x \varphi (e, fe)$$
$$= f(x \varphi (e, f))$$
$$= x \varphi (e, f).$$

Therefore

$$e' \varphi_S (e, f) = x' \varphi (e, f)x \varphi (e, f)$$
$$= e' \varphi (e, f).$$

Hence $\varphi_S (e, f) = \varphi (e, f)$. This proves that $\varphi = \varphi_S$.

If $S$ is a combinatorial regular semigroup, then Theorem 2.9 shows that $\varphi_S$ is a $C$-functor and from Lemma 2.8, it follows that $S$ is equal to $\Sigma (\varphi_S)$.

4. HOMOMORPHISM THEOREMS

In this section we prove that homomorphisms of combinatorial regular semigroups are determined by morphisms of combinatorial functors and vice versa. Further we establish the equivalence of the categories $\text{CRS}$ of combinatorial regular semigroups and $\text{CF}$ of $C$-functors.

THEOREM 2.19. Let $(\theta, \sigma) : \varphi \longrightarrow \varphi'$ be a morphism of $C$-functors. Define $\Sigma (\theta, \sigma) : \Sigma (\varphi) \longrightarrow \Sigma (\varphi')$ by
(2.4) \( x \sum (\Theta, \Sigma) = x \sigma_e \)

for all \( e \in E \) and \( x \in \varphi(e) \). Then \( \sum (\Theta, \Sigma) \) is a homomorphism from \( \sum (\varphi) \) to \( \sum (\varphi') \). Further if

\[ (\Theta', \Sigma') : \varphi' \to \varphi'' \]

is another morphism in \( \mathcal{Q} \) then

\[ \sum (\Theta \Theta', \Sigma \Sigma') = \sum (\Theta, \Sigma) \sum (\Theta', \Sigma') \].

Conversely, given a homomorphism \( \beta : S \to S' \) of combinatorial regular semigroups, there exists a unique morphism

\( (\Theta, \Sigma) : \varphi_S \to \varphi_{S'} \)

such that, \( \beta = \sum (\Theta, \Sigma) \).

**Proof.** Let \( x, y \in \sum (\varphi), e \in x, f \in y \) and \( h \in \mathcal{S}(e, f) \). Then \( xy = x \varphi(e, eh) y \varphi(f, hf) \) and \( xy \in \varphi(h) \).

Also \( x \in \varphi(e) \) and \( y \in \varphi(f) \). Now

\[
(xy) \sum (\Theta, \Sigma) = (xy) \sigma_h \\
= x \varphi(e, eh) \sigma_h y \varphi(f, hf) \sigma_h \\
= x \sigma_e \varphi'(e \Theta, e \Theta h \Theta) y \sigma_f \varphi'(f \Theta, h \Theta f \Theta) \\
= (x \sigma_e)(y \sigma_f) \\
= x \sum (\Theta, \Sigma) y \sum (\Theta, \Sigma).
\]

Hence \( \sum (\Theta, \Sigma) \) is a homomorphism. Further

\[
x \sum (\Theta \Theta', \Sigma \Sigma') = x(\sigma \Sigma') e \\
= x \sigma_e \Sigma' e \theta.
\]
Thus $\Sigma(\theta'') = \Sigma(\theta, \sigma')$. 

Conversely, let $\beta : S \to S'$ be a homomorphism of combinatorial regular semigroups. Then 

$\Theta = \beta \mid_{E(S)} : E(S) \to E(S')$ is a regular bimorphism. 

Define $\sigma' : \mathcal{P}_S(e) \to \mathcal{P}_{S'}(e)$ by $x \mapsto x\beta$ 

Then, if $f \in e$, for $g \in E(\mathcal{P}(e))$, we have 

$$g \sigma' \mathcal{P}_{S'}(e \theta, f \theta) = g \theta \mathcal{P}_{S'}(e \theta, f \theta)$$

$$= (f \theta) \alpha(z\beta, z'\beta)$$

(where $z \in R_e \cap L_g$, $z' \in R_g \cap L_e$)

$$= (z'fz)\beta$$

$$= g \mathcal{P}_{S}(e, f) \sigma_f.$$ 

Thus $\sigma : \mathcal{P}_S \to \mathcal{P}_{S'}$ is a natural transformation and $\beta = \Sigma(\theta, \sigma)$. From the construction of $(\theta, \sigma)$ it follows that $(\theta, \sigma)$ is unique. 

In Theorem 2.12, we get the equality of functors $\mathcal{P}_S = \mathcal{P}$ by assuming that $\text{dom } \mathcal{P}$ is equal to $E(S)$. 

If we drop this assumption, the bijection $\psi$ of axiom (C1)
will be just an isomorphism of the biordered sets $E$ and $E(S)$. In this case $(\psi, 1): \varphi \to \varphi_S$ will be an isomorphism in $\text{CF}$.

By Theorem 2.19 $\Sigma: \text{CF} \to \text{CRS}$ is a functor. Now define $\hat{\Phi}: \text{CRS} \to \text{CF}$ as follows. If $S \in \text{CRS}$,

\[ \hat{\Phi}(S) = \varphi_S \]

as defined in Theorem 2.9 and if $\beta: S \to S'$ in $\text{CRS}$, then $\hat{\Phi}(\beta)$ is the unique morphism $(\theta, \sigma): \varphi_S \to \varphi_{S'}$ given by Theorem 2.19.

It is routine to check that $\hat{\Phi}$ is a functor. Further by Theorem 2.12 $\hat{\Phi}\Sigma = 1_{\text{CRS}}$ and $\eta: 1_{\text{CF}} \to \Sigma \hat{\Phi}$ with components $\eta_\varphi = (\psi, 1): \varphi \to \varphi_{\Sigma(\varphi)}$ as described above, is a natural isomorphism. Further the compositions

\[ \Sigma \eta_\Sigma \to \Sigma \hat{\Phi}\Sigma \xrightarrow{1} \Sigma \text{ and } \hat{\Phi} \xrightarrow{\hat{\Phi}\eta} \hat{\Phi}\Sigma \hat{\Phi} \xrightarrow{1} \hat{\Phi} \]

are identities. Thus we have the following theorem.

**Theorem 2.20.** $\Sigma: \text{CF} \to \text{CRS}$ and $\hat{\Phi}: \text{CRS} \to \text{CF}$ are functors such that $\langle \Sigma, \hat{\Phi}, \eta, 1 \rangle$ is an adjoint equivalence of the categories $\text{CRS}$ and $\text{CF}$.

**Remark.** Henceforth we shall denote $\varphi_S$ given in Theorem 2.9 by $\hat{\Phi}(S)$. 
